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SOME NEW INTEGRAL INEQUALITIES THROUGH THE STEKLOV OPERATOR

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Summary. Hardy and Copson type inequalities have been studied by a large number of authors during the twentieth century and has motivated some important lines of study which are currently active. A large number of papers have been appeared involving Copson and Hardy inequalities (see [2-16] for more details).

In this paper some Hardy-Steklov and Copson-Steklov type integral inequalities were established. Namely the integral inequalities were proved there.

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(\mathcal{F}_s f)^p(x)dx \le \left(\frac{p}{|p-\alpha-1|}\right)^p \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)}|K(x)|^q dx\right)^{\frac{p}{q}}, \quad (*)$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx \le \left(\frac{p}{|p-\alpha-1|}\right)^p \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)}|J(x)|^q dx\right)^{\frac{p}{q}}. \quad (**)$$

Where $(\mathcal{F}_s f)$ is the Hardy-Steklov type operator and $(\mathcal{C}_s f)$ is the Copson-Steklov type operator (see the main results for more details).

Several Hardy-Steklov type, Hardy-type and Hardy integral inequalities were derived from $(*)$. Similarly, some Copson-Steklov type and Copson type integral inequalities are deduced from $(**)$.

1. INTRODUCTION

In 1928, G.H. Hardy proved the following integral inequalities [6]. Let f non-negative measurable function on $(0, \infty)$

$$(\mathcal{F}f)(x) = \begin{cases} \int_0^x f(t)dt & \text{for } \alpha < p-1, \\ \int_x^\infty f(t)dt & \text{for } \alpha > p-1, \end{cases}$$

then

$$\int_0^\infty x^{\alpha-p}(\mathcal{F}f)^p(x)dx \le \left(\frac{p}{|p-\alpha-1|}\right)^p \int_0^\infty x^\alpha f^p(x)dx, \quad \text{for } p > 1. \quad (1)$$

In 1976, E.T. Copson proved the following integral inequalities (see [4], Theorem 1, Theorem 3). Let f, ϕ non-negative measurable functions on $(0, \infty)$

$$\Phi(x) = \int_0^x \phi(t)dt, \qquad (\mathcal{C}f)(x) = \begin{cases} \int_0^x f(t)\phi(t)dt, & \text{for } c > 1, \\ \int_x^\infty f(t)\phi(t)dt, & \text{for } c < 1, \end{cases}$$

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then

$$\int_0^b (\mathcal{C}f)^p(x)\Phi^{-c}(x)\phi(x)dx \le \left(\frac{p}{|c-1|}\right)^p \int_0^b f^p(x)\Phi^{p-c}(x)\phi(x)dx, \quad \text{for } p \ge 1. \tag{2}$$

Inequality (2) can be easily rewritten in the following form

$$(\mathcal{C}f)(x) = \begin{cases} \int_0^x f(t)\phi(t)dt, & \text{for } \alpha < p-1, \\ \int_x^\infty f(t)\phi(t)dt, & \text{for } \alpha > p-1, \end{cases}$$

then

$$\int_0^b (\mathcal{C}f)^p(x)\Phi^{\alpha-p}(x)\phi(x)dx \le \left(\frac{p}{|p-1-\alpha|}\right)^p \int_0^b f^p(x)\Phi^{\alpha}(x)\phi(x)dx, \quad \text{for } p \ge 1. \tag{3}$$

The Hardy-Steklov operator is defined by

$$(Tf)(x) = g(x)\int_{r(x)}^{h(x)} f(t)dt, \qquad f \ge 0,$$

where g is a positive measurable function and r, h are functions defined on an interval (a,b) such that $r(x) < h(x)$ for all $x \in (a,b)$.

Particular cases of this operator are Hardy operator $(\mathcal{F}f)(x) = \int_0^x f(t)dt$, the Hardy averaging operator $(F_\mu f)(x) = x^\mu \int_0^x f(t)dt$ and the Steklov operator $(Sf)(x) = \int_{x-1}^{x+1} f(t)dt$, which has been studied intensively (see [9] for example).

Let f, v, ϕ be non-negative measurable functions on $(0,\infty)$. Suppose that r and h are increasing differentiable functions on $[0,\infty)$, such that

$$\begin{cases} 0 < r(x) < h(x) < \infty & \text{for all } x \in (0,\infty), \\ r(0) = h(0) = 0 & \text{and } r(\infty) = h(\infty) = \infty. \end{cases} \tag{4}$$

The Hardy-Steklov and Copson-Steklov type operators are defined as follows,

$$(\mathcal{F}_s f)(x) = \int_{r(x)}^{h(x)} f(y)v(y)dy, \qquad x > 0, \tag{5}$$

$$(\mathcal{C}_s f) = \int_{r(x)}^{h(x)} \frac{f(y)\phi(y)}{\Phi(y)}dy, \qquad x > 0, \tag{6}$$

where

$$\Phi(x) = \int_0^x \phi(t)dt, \quad \text{for} \quad x \in (0,\infty)\,.$$

We adopt the usual convention: $\frac{0}{0} = \frac{\infty}{\infty} = 0$.

2. MAIN RESULTS

Let $0 < b \leq \infty$. Throughout the paper, we will assume that the integrals exist and are finite. The following lemma is needed in the proof of the main results (was proved in [1]).

Lemma 2.1. Let $1 < p \leq q < \infty$ and f, g, w be non-negative measurable functions on (a, b) such that $W(x) = \int_0^x w(t)dt$. If $m \in \mathbb{R}, m \neq 1$, then

$$\int_a^b \frac{w(x)}{W^m(x)} g^p(f(x))dx \leq \left(\int_a^b w(x)dx\right)^{1-\frac{p}{q}} \left(\int_a^b \frac{w(x)}{W^{\frac{mq}{p}}(x)} g^q(f(x))dx\right)^{\frac{p}{q}}. \tag{7}$$

Remark 2.1. Let $V(x) = \int_0^x v(t)dt$. By putting $m = p - \alpha$ in inequality (7), $w(x) = v(x)$, $W(x) = V(x)$ (respectively $w(x) = \phi(x)$, $W(x) = \Phi(x)$ and $f(x) = g(f(x))$), we obtain

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} f^p(x)dx \leq \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} f^q(x)dx\right)^{\frac{p}{q}}. \tag{8}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} f^p(x)dx \leq \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} f^q(x)dx\right)^{\frac{p}{q}}. \tag{9}$$

The main results are presented in the following Theorem and Corollaries**.**

Theorem 2.1. Let f, v, ϕ be non-negative measurable functions on $(0,\infty)$, $1 < p \leq q < \infty$ and $r(x)$, $h(x)$ satisfied the conditions (4). If $\alpha < p - 1$, then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_s f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} |K(x)|^q dx\right)^{\frac{p}{q}}, \tag{10}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} |J(x)|^q dx\right)^{\frac{p}{q}}, \tag{11}$$

where

$$K(x) = \frac{V(x)}{v(x)}\left\{[V(h(x))]' f(h(x)) - [V(r(x))]' f(r(x))\right\},$$

$$J(x) = \frac{\Phi(x)}{\phi(x)}\left\{\frac{[\Phi(h(x))]'}{\Phi(h(x))} f(h(x)) - \frac{[\Phi(r(x))]'}{\Phi(r(x))} f(r(x))\right\}.$$

Proof. We consider the inequality (11), then

$$(\mathcal{C}_s f)'(x) = h'(x)\frac{\phi\left(h(x)\right)}{\Phi\left(h(x)\right)}f\big(h(x)\big) - r'(x)\frac{\phi\left(r(x)\right)}{\Phi\left(r(x)\right)}f\big(r(x)\big)$$

$$= \frac{\phi(x)J(x)}{\Phi(x)},$$

integrating by part in the left-hand side of (11), we get

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx = \left[\frac{-(\mathcal{C}_s f)^p(x)}{(p-\alpha-1)\Phi^{p-\alpha-1}(x)}\right]_0^b + \frac{p}{p-\alpha-1}$$

$$\times \int_0^b \frac{\phi(x)J(x)(\mathcal{C}_s f)^{p-1}(x)}{\Phi^{p-\alpha}(x)}dx.$$

Since $\alpha < p-1$, we have

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx \leq \frac{p}{p-\alpha-1}\int_0^b \frac{\phi(x)J(x)(\mathcal{C}_s f)^{p-1}(x)}{\Phi^{p-\alpha}(x)}dx.$$

The Hölder integral inequality for $\frac{1}{p}+\frac{1}{p'} = 1$, gives

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx \leq \frac{p}{p-\alpha-1}\left(\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx\right)^{\frac{1}{p'}}$$

$$\times\left(\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}|J(x)|^p dx\right)^{\frac{1}{p}},$$

therefore

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}|J(x)|^p dx.$$

Finally, by using inequality (9), we get (11).

The proof of inequality (10) is similar. So, the proof of Theorem is complete.

Now let $r(x) = 0$ in (5) and (6), thus

$$\left(\mathcal{F}_{s,1}f\right)(x) = \int_0^{h(x)} f(y)v(y)dy, \qquad x > 0,$$

$$(\mathcal{C}_{s,1}f)(x) = \int_0^{h(x)} \frac{f(y)\Phi(y)}{\Phi(y)} dy, \qquad x > 0\,.$$

If we set q = p in (10) and (11), we obtain the following corollary.

Corollary 2.1. Let f, v, ϕ be non-negative measurable functions on $(0,\infty)$, $p > 1$, $\alpha < p-1$ and

$$\begin{cases} 0 < h(x) < \infty \quad \text{for all} \quad x \in (0,\infty), \\ h(0) = 0 \quad \text{and} \quad h(\infty) = \infty. \end{cases} \tag{12}$$

Then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_{s,1}f)^p(x)dx \le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{v(x)}{V^{p-\alpha}(x)} |K_1(x)|^p dx, \tag{13}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_{s,1}\, f)^p(x)dx \le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} |J_1(x)|^p dx, \tag{14}$$

where

$$K_1(x) = \frac{V(x)[V(h(x))]' f(h(x))}{v(x)},$$

$$J_1(x) = \frac{\Phi(x)}{\phi(x)} \frac{[\Phi(h(x))]'}{\Phi(h(x))} f(h(x))\,.$$

Remark 2.2. If $h(x) = x$ in Corollary 2.1, we obtain the following weighted Hardy inequality and Copson-type inequality

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_{s,2}f)^p(x)dx \le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{v(x)}{V^{-\alpha}(x)} f^p(x)dx, \tag{15}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_{s,2}\, f)^p(x)dx \le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} f^p(x)dx, \tag{16}$$

where

$$(\mathcal{F}_{s,2}f)(x) = \int_0^x f(y)v(y)dy, \qquad x > 0\,,$$

$$(\mathcal{C}_{s,2}f)(x) = \int_0^x \frac{f(y)\Phi(y)}{\Phi(y)} dy, \qquad x > 0\,.$$

If we put $h\,(x) = \lambda\, x$ and $r\,(x) = \beta\, x$ and $q = p$ in Theorem 2.1, we get following corollary.

Corollary 2.2. Let f, v, ϕ be non-negative measurable functions on $(0, \infty)$, $0 < \beta < \lambda < \infty$, $p > 1$ and

$$\big(\mathcal{F}_{s,3} f\big)(x) = \int_{\beta x}^{\lambda x} f(y) v(y) dy, \qquad x > 0\,,$$

$$\big(\mathcal{C}_{s,,3} f\big)(x) = \int_{\beta x}^{\lambda x} \frac{f(y)\phi(y)}{\Phi(y)} dy, \qquad x > 0\,.$$

If $\alpha < p - 1$, then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} \big(\mathcal{F}_{s,3} f\big)^p(x) dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{v(x)}{V^{p-\alpha}(x)} |K_2(x)|^p dx, \tag{17}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} \big(\mathcal{C}_{s,3}\, f\big)^p(x) dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} |J_2(x)|^p dx. \tag{18}$$

Where

$$K_3(x) = \frac{V(x)[\lambda\, v(\lambda x) f(\lambda x) - \beta\, v(\beta x) f(\beta x)]}{v(x)},$$

$$J_3(x) = \frac{\Phi(x)}{\phi(x)} \left\{ \frac{\lambda\, \phi(\lambda x)}{\Phi\,(\lambda x)} f(\alpha x) - \frac{\beta\, \phi(\beta x)}{\Phi\,(\beta x)} f(\beta x) \right\}.$$

Remark 2.3. One can prove the boundedness of the operator $\mathcal{F}_{s,3}$ from $L_p(0,\infty)$ to $L_p(0,\infty)$ by using the Minkowski integral inequality for $p > 1$, it means that $\left\| \big(\mathcal{F}_{s,3} f\big)(x) \right\|_{L_p(0,\infty)} \leq C_{(\lambda,\beta,p)} \|f(x)\|_{L_{p,v}(0,\infty)}$, where $L_p(0,\infty)$ is the classical Lebesgue space and $L_{p,v}(0,\infty)$ is the weighted Lebesgue space, with the following norm $\|f(x)\|_{L_{p,v}(0,\infty)} = \left(\int_0^\infty |f(x) v(x)|^p dx\right)^{\frac{1}{p}}$ and $C_{(\lambda,\beta,p)}$ is a positive constant depending only on λ, β and p.

Remark 2.4. For $\lambda = 1$ and $\beta = \frac{1}{2}$, we get a Pachpatte-type inequality.

Let

$$(\mathcal{F}_s^* f)(x) = \int_{r(x)}^{\infty} f(y) v(y) dy, \qquad (\mathcal{C}_s^* f)(x) = \int_{r(x)}^{\infty} \frac{f(y)\phi(y)}{\Phi(y)} dy, \qquad x > 0,$$

with

$$\begin{cases} 0 < r(x) < \infty \quad \text{for all} \quad x \in (0,\infty), \\ \quad r(0) = 0 \quad \text{and} \quad r(\infty) = \infty. \end{cases} \tag{19}$$

By setting $h(x) = \infty$ and reasoning a manner analogous to the proof of Theorem 2.1, we get the following corollary.

Corollary 2.3 Let f, v, ϕ be non-negative measurable functions on $(0,\infty)$, $1 < p \leq q < \infty$. If $\alpha > p - 1$, then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_s^* f)^p(x)dx \leq \left(\frac{p}{\alpha - p + 1}\right)^p \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} |K^*(x)|^q dx\right)^{\frac{p}{q}}, \tag{20}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s^* f)^p(x)dx \leq \left(\frac{p}{\alpha - p + 1}\right)^p \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} |J^*(x)|^q dx\right)^{\frac{p}{q}}, \tag{21}$$

where

$$K^*(x) = -\frac{V(x)\,[V(r(x))]' f(r(x))}{v(x)},$$

$$J^*(x) = -\frac{\Phi(x)}{\phi(x)} \frac{[\Phi(r(x))]'}{\Phi(r(x))} f(r(x)).$$

Remark 2.5. The following particular case of Corollary 2.3 can be derived by taking $\mathrm{r}(x) = x$ and $q = p$.

$$\int_0^b \frac{v(x)}{\mathrm{V}^{p-\alpha}(x)} \left(\widetilde{\mathcal{F}_s^*} f\right)^p(x)dx \leq \left(\frac{p}{\alpha - p + 1}\right)^p \int_0^b \frac{v(x)}{\mathrm{V}^{-\alpha}(x)} f^p(x)dx, \tag{22}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} \left(\widetilde{\mathcal{C}_s^*} f\right)^p(x)dx \leq \left(\frac{p}{\alpha - p + 1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} f^p(x)dx, \tag{23}$$

where

$$\left(\widetilde{\mathcal{F}_s^*} f\right)(x) = \int_x^\infty f(y)v(y)dy, \qquad x > 0, \quad \left(\widetilde{\mathcal{C}_s^*} f\right)(x) = \int_x^\infty \frac{f(y)\phi(y)}{\Phi(y)} dy, \qquad x > 0.$$

Remark 2.6 We note that if $v(x) = 1$ in the inequalities (15) and (22), we get the Hardy inequalities (1).

3. CONCLUSION

By using Hardy-Steklov and Copson-Steklov type operators and by introducing a second parameter of integrability q, some new integral inequalities were established and proved. These integral inequalities generalize certain classical inequalities like those of Hardy Copson and Pachpatte. As a perspective, we propose to extended these results to $\mathbb{R}^n$ or subsets of $\mathbb{R}^n$ for $n \geq 2$. Also it would of interest to try apply some of this integral inequalities in the study of deferent fields of mathematics (partial deferential equations, functional spaces, mathematical modeling, …).

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