MATHEMATICA MONTISNIGRI
Vol XLIX (2020)

AMENABILITY OF $A \oplus_T X$ AS AN EXTENSION OF BANACH ALGEBRA

M. GHORBAI, D. E. BAGHA*

Department of Mathematics, Central Tehran Branch, Islamic Azad university, Tehran, Iran.
*Corresponding author. E-mail: e_bagha@yahoo.com

DOI: 10.20948/mathmontis-2020-49-3

Summary. Let $A, X, \mathfrak{U}$ be Banach algebras and A be a Banach $\mathfrak{U}$-bimodule also X be a Banach $A-\mathfrak{U}$-module. In this paper we study the relation between module amenability, weak module amenability and module approximate amenability of Banach algebra $A \oplus_T X$ and that of Banach algebras A, X. Where T: $A \times A \to X$ is a bounded bi-linear mapping with specific conditions.

1 INTRODUCTION

The notation of amenability of Banach algebras was introduced by B.Johnson in [9]. A Banach algebra A is amenable if every bounded derivation from A into any dual Banach A-bimodule is inner, equivalently if $H(A,\ X^*) = \{0\}$ for any Banach A-bimodule X, where $H(A,\ X^*)$ is the first Hochschild co- homology group of A with coefficient in X^*. Also, a Banach algebra A is weakly amenable if $H(A,\ A^*) = \{0\}$. Bade, Curtis and Dales introduced the notion of weak amenability on Banach algebras in [5]. They considered this concept only for commutative Banach algebras. After a while, Johnson defined the weak amenability for arbitrary Banach algebras [8].

For a morphism $T: B \to A$ from a Banach algebra B to a commutative Banach algebra A. The notion of module amenability of Banach algebras was introduced by Amini in [1]. Amini and Ebrahimi Bagha in [3] studied the concept of weak module amenability. In [10] the notation of module approximate amenability and contractibility as modules over of another Ba- nach algebra was introduced for the notion of Banach algebras.

M. Sangani-Monfared in [11] defined a product on $A \times B$ and obtained the Banach algebra $A \times_\theta B$ using a character $\theta \in \sigma(B)$, for Banach algebras in a fairly general setting.

Later, S.J. Bhatt and P.A. Dabhi in [6] defined a product on $A \times B$ and obtained a Banach algebra $A \times_T B$ for a morphism $T : B \to A$ from a Banach algebra B to a commutative Banach algebra A.

The first and the second authors generalized all these constructions, and de- fined the module Lau product $A \times_\alpha B$ for Banach algebras A and B such that A is a Banach B-bimodule. They studied the ideal amenability of $A \times_\alpha B$ in [4].

T.Yazdan panah in [12] studied the concept of expanded modular of Banach algebra denoted by $A \oplus_T X$. He showed that $A \oplus_T X$ is amenable if and only if A is amenable and $X = \{0\}$. In this paper, we define a new Banach algebra different from of all above Banach algebras, named $A \oplus_T X$ in section 2. Then, some required basic properties of the following part are studied. In section 3, as the main section of paper, we study the relationship between module amenability of $A \oplus_T X$ and module amenability of A and X. We show that If $T(A,\ 0) = X$ and $A^2 = A$, then the module amenability of A implies module amenability of $A \oplus_T X$. Furthermore, it's

2010 Mathematics Subject Classification: 46H20,46H25.
Key words and Phrases: Banach module, module amenability, weak module amenability,module approximate amenability, module derivation, Expanded Banach algebra.

conversly obtained that the module amenability of $A \oplus_T X$ implies module amenability of A and moreover if $T(A,\ 0) = X$, then X also is module amenable. In sectiones 4 and 5 respectively we study the relationship between weak mod- ule amenability (based as definition in [1] and [2]) and module approximte amenability of $A \oplus_T X$ and weak module amenability and module approximte amenability of A, X.

2 DEFINITIONS AND BASIC PROPERTIES

Throughout this paper it's assumed that $\mathfrak{U}$ be a Banach algebra, A be a Banach $\mathfrak{U}$-bimodule and X be a Banach A-U-bimodule. Module actions are assumed as follow too:

$$A \times \mathfrak{U} \to A; (a,\quad \alpha) \longmapsto a \circ \alpha, \mathfrak{U} \times A \to A; (\alpha,\quad a) \longmapsto \alpha \cdot a.$$
$$X \times \mathfrak{U} \to X; (x,\quad \alpha) \longmapsto x \vartriangle \alpha, \mathfrak{U} \times X \to X; (\alpha,\quad x) \longmapsto \alpha \nabla x.$$
$$X \times A \to X; (x,\quad a) \longmapsto x \circ a, A \times X \to X; (a,\quad x) \longmapsto a \cdot x.$$

Consider the bounded bilinear map $T : A \times A \to X$, which has the following properties:

$a \cdot T(a_1 a_2, 0) = T(aa_1, 0) \circ a_2, T(a_1 a_2, 0) = T(a_1, 0)T(a_2, 0)$,

$T(\alpha \cdot a,\quad \alpha \nabla x) = \alpha \cdot T(a,\quad x), T(a \circ \alpha,\quad x \vartriangle \alpha) = T(a,\quad x) \cdot \alpha,$

$\| T(a,\ 0) \| = \| a \|$, for all $a, a_1, a_2 \in A, x \in X, \alpha \in \mathfrak{U}$.

Module extension $A \oplus X$, with the product

$$(a,\ x)(a_1,\ x_1) = (aa_1,\ a \cdot x_1 + x \circ a_1 + T(aa_1, 0))$$

and the norm $\| (a,\ x) \| = \| a \| + \| x \|$ is a Banach algebra denoted by $A \oplus_T X$.

Definition 2.1 *The bounded map* $D: A \to X^*$ *with* $D(a + b) = D(a) + D(b), D(ab) = a \cdot D(b) + D(a) \cdot b$ *for all* $a, b \in A$, *and* $D(\alpha \cdot a) = \alpha \cdot D(a), D(a \cdot \alpha) = D(a) \cdot \alpha (\alpha \in \mathfrak{U},\ a \in A)$ *, is called module derivation.*

Note that X^* is also Banach module over A and $\mathfrak{U}$ with compatible actions under the canonical actions of A and $\mathfrak{U}$, $\alpha \cdot (a \cdot f) = (\alpha \cdot a) \cdot f, (a \in A,\ \alpha \in \mathfrak{U},\ f \in X^*)$, and the same for right action. Here the canonical actions of A and $\mathfrak{U}$ on X^* are defined by $(\alpha \cdot f)(x) = f(x \vartriangle \alpha), (a \cdot f)(x) = f(x \circ a), (\alpha \in \mathfrak{U},\ a \in A,\ f \in X^*,\ x \in X)$ and it's the same for right actions. As in [1] we call A- module X which have a compatible $\mathfrak{U}$-action as above, a $A - \mathfrak{U}$ modules, above assertion is to say that if X is an $A - \mathfrak{U}$- module, then so is X^*. Also we use the notation $Z_{\mathfrak{U}}(A,\ X^*)$ for the set of all module derivations $D: A \to X^*$, and $N_{\mathfrak{U}}(A,\ X^*)$ for those which are inner and $H_{\mathfrak{U}}(A,\ X^*)$ for the quotient group.

Proposition 2.2 $A \oplus_T X$ *is a Banach* $\mathfrak{U}$- *bimodule.*

Proof. Consider the module actions as follow:

$\mathfrak{U} \times (A \oplus_T X) \to A \oplus_T X; \alpha \cdot (a,\ x) = (\alpha \cdot a,\ \alpha \nabla x)$, and $(A \oplus_T X) \times \mathfrak{U} \to A \oplus_T X; (a,\ x) \bullet \alpha = (a \circ \alpha,\ \alpha \vartriangle x)$. It is easy to check the satification of the properties. ■

Proposition 2.3 *If* Y *is an A-U-module, then* $Y \oplus \{O\}$ *is a Banach* $A \oplus_T X - \mathfrak{U}$*-bimodule.*

Proof. Assume that the module actions on Y, are as follows:

$\mathfrak{U} \times Y \to Y; (\alpha,\ y) \longmapsto \alpha \vartriangle y, Y \times \mathfrak{U} \to Y; (y,\ \alpha) \longmapsto y \bullet \alpha.$ And $A \times Y \to Y; (a,\ y) \longmapsto a \cdot y, Y \times A \to Y; (y,\ a) \longmapsto y.a$. Define the module actions as: $(Y \oplus \{0\}) \times \mathfrak{U} \to Y \oplus$

$\{0\}; (y,\ 0) \bullet \alpha = (y \bullet \alpha,\ 0), \mathfrak{U} \times (Y \oplus \{0\}) \to Y \oplus \{0\};\quad \alpha \bullet (y,\ 0) = (\alpha \vartriangle y,\ 0)$. And $(A \oplus_T X) \times (Y \oplus \{0\}) \to Y \oplus \{0\}; (a,\ x).(y,\ 0) = (a.y,\ 0), (Y \oplus \{0\}) \times (A \oplus_T X) \to Y \oplus \{0\}; (y,\ 0) \circ (a,\ x) = (y.\ a,\ 0)$. We only need to show that the actions are compatible.

$$\begin{aligned} 1)\alpha.\ ((a,\ x).(y,\ 0)) &= \alpha.\ (a.y,\ 0) \\ &= (\alpha \vartriangle (a.y),\ 0) = ((\alpha \cdot a) \cdot y,\ 0) \\ &= ((\alpha \cdot a,\ \alpha \nabla x) \cdot (y,\ 0) = (\alpha \cdot (a,\ x)).(y,\ 0)\ . \end{aligned}$$

$$\begin{aligned} 2)((a,\ x) \cdot (y,\ 0)) \bullet \alpha &= (a \cdot y,\ 0) \bullet \alpha \\ &= ((a \cdot y) \bullet \alpha,\ 0)) = (a \cdot (y \bullet \alpha),\ 0) \\ &= (a,\ x) \cdot (y \bullet \alpha,\ 0) = (a,\ x) \cdot ((y,\ 0) \bullet \alpha)\ . \end{aligned}$$

$$\begin{aligned} 3)(\alpha.\ (y,\ 0)) \cdot (a,\ x) &= (\alpha \vartriangle y,\ 0) \cdot (a,\ x) \\ &= ((\alpha \vartriangle y).\ a,\ 0) = (\alpha \vartriangle (y.\ a),\ 0)) \\ &= \alpha.\ (y.\ a,\ 0) = \alpha.\ ((y,\ 0) \circ (a,\ x)) \end{aligned}$$

■

Proposition 2.4 *Let $M \oplus N$ be a Banach $A \oplus_T X - \mathfrak{U}$-bimodule, then M is a Banach $A - \mathfrak{U}$-bimodule.*

Proof. Consider the map $Q_M : M \oplus N \to M; (m,\ n) \longmapsto m$ and define the module actions as: $M \times \mathfrak{U} \to M; (m,\ n) \longmapsto m \cdot \alpha = Q_M((m,\ 0) \bullet \alpha), \mathfrak{U} \times M \to M; (\alpha,\ m) \longmapsto \alpha \circ m = Q_M(\alpha \cdot (m,\ 0\ \ M \times A \to M; (m,\ a) \longmapsto m.a = Q_M((m,\ 0) \circ (a,\ 0))$ and $A \times M \to A; (a,\ m) \longmapsto a \bullet m = Q_M((a,\ 0) \cdot (m, 0))$ ■

Proposition 2.5 *Let M be a Banach $A - \mathfrak{U}$-module and N be a Banach $X - \mathfrak{U}$-bimodule, $M \oplus N$ is a Banach $A \oplus_T X - \mathfrak{U}$-bimodule.*

Proof. Given module actions on $M \oplus N$ as follows:

$(M \oplus N) \times \mathfrak{U} \to M \oplus N; (m,\ n) \cdot \alpha = (m.\alpha,\ n\nabla\alpha), \mathfrak{U} \times (M \oplus N) \to M \oplus N;\quad \alpha \bullet (m,\ n) = (\alpha \bullet m,\ \alpha \vartriangle m), (M \oplus N) \times (A \oplus_T X) \to (M \oplus N); (m,\ n) \cdot (a,\ x) = (m\ \alpha,\ n.\ T(a,\ 0\ \ (A \oplus_T X) \times (M \oplus N) \to M \oplus N; (a,\ x) \bullet (m,\ n) = (a \bullet m,\ T(a,\ 0) \odot n)$.

■

Proposition 2.6 *For each $(f,\ g) \in M^* \oplus N^*, (a,\ x) \in A \oplus_T X, (m,\ n) \in M \oplus N$ we have $(f,\ g) \cdot (a,\ x) = (f \cdot a,\ g \cdot T(a,\ 0))$ and $(a,\ x).(f,\ g) = (a.f,\ T(a,\ 0).g)$.*

Proof.

$$\begin{aligned} \langle (f,\ g).(a,\ x),\ (m,\ n)\rangle &= \langle (f,\ g),\ (a,\ x) \bullet (m,\ n)\rangle \\ &= \langle (f,\ g),\ (a \circ m,\ T(a,\ 0) \odot n)\rangle \\ &= \langle f,\ a \circ m\rangle + \langle g,\ T(a,\ 0) \odot n\rangle \\ &= \langle f.a,\ m\rangle + \langle g.T(a,\ 0),\ n\rangle \\ &= \langle (f.a,\ g.T(a,\ 0\ \ (m,\ n)))\end{aligned}$$

■

Proposition 2.7 *If N is a Banach $X - \mathfrak{U}$- bimodule, then is a Banach $A - \mathfrak{U}$-bimodule.*

Proof. The module actions are defined as follow:

$A \times N \to N; a.n = T(a,\ 0) \odot n$ and $N \times A \to N; n \bullet a = n.T(a,\ \odot)$. ■

3 MODULE AMENABILITY

Lemma 3.1 *$D \in Z_{\mathfrak{U}}(A \oplus_T X,\ M^* \oplus N^*)$ if and only if there are $D_1 \in Z_{\mathfrak{U}}(A,\ M^*), D_3 \in Z_{\mathfrak{U}}(X,\ N^*), R \in Z_{\mathfrak{U}}(A,\ N^*)$ and linear map $D_2 : X \to M^*$ such that*

1) $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D_3(x))$,
2) $D_2(a \bullet x) = a \cdot D_2(x)$,
3) $D_2(x \circ a) = D_2(x) \cdot a,$
4) $R(bd) = R(b) \cdot T(d,\ 0) + T(b,\ 0) \cdot R(d) = R(b).d + b.R(d)$,
5) $D_2(T(ab,\ 0)) = 0,$
6) $D_3(a.x) = T(a,\ 0).D_3(x)$,
7) $D_3(x \circ a) = D_3(x).T(a,\ 0)$,
8) $D_3(T(ab,\ 0)) = 0.$

Proof. Suppose that $D \in Z_{\mathfrak{U}}(A \oplus_T X,\ M^* \oplus N^*)$ then there are $d_1 : A \oplus \tau X \to M^*$, $d_2 : A \oplus_T X \to N^*$ such that $\mathrm{D} = (d_1,\ d_2)$, Set

$$D_1 : A \to M^*; D_1(a) = d_1(a,\ 0),$$

$$D_2 : X \to N^*; D_2(x) = d_1(0,\ x),$$

$$D_3 : X \to N^*; D_3(x) = d_2(0,\ x), R: A \to N^*; R(a) = d_2(a,\ 0).$$

Now

$$\begin{aligned} D(a,\ x) &= (d_1,\ d_2)((a,\ 0) + (0,\ x)) = (d_1,\ d_2)(a,\ 0) + (d_1,\ d_2)(0,\ x) \\ &= (d_1(a,\ 0),\ d_2(a,\ 0)) + ((d_1(0,\ x),\ d_2(0,\ x)) \\ &= (d_1(a,\ 0) + d_1(0,\ x)) + (d_2(a,\ 0) + d_2(0,\ x)) \\ &= (D_1(a) + D_2(x),\ R(a) + D_3(x)), \end{aligned} \tag{1}$$

Now

$$\begin{aligned} D((a,\ x)(m,\ x')) &= D(am,\ a \cdot x' + x \circ m + T(am,\ 0)) \\ &= (D_1(am) + D_2(a \cdot x') + D_2(x \circ m) + D_2(T(am,\ 0)), R(am) + D_3(a.x') \\ &\quad + D_3(x \circ m) + D_3(T(am, 0))), \end{aligned} \tag{2}$$

since D is module derivation so

$$\begin{aligned} D((a,\ x)(m,\ x')) &= D(a,\ x) \cdot (m,\ x') + (a,\ x) \cdot D(m,\ x') \\ &= (D_1(a) + D_2(x),\ R(a) + D_3(x)).(m,\ x') \\ &\quad + (a,\ x).(D_1(m) + D_2(x),\ R(m) + D_3(x')) \\ &= ((D_1(a) \cdot m + D_2(x)) \cdot m + a \cdot D_2(x') + D_2(x).m,\ R(a) \cdot T(m,\ 0) \\ &\quad + T(a,\ 0).R(m) + D_3(x).T(m,\ 0) + T(a,\ 0).D_3(x')). \end{aligned} \tag{3}$$

In 3, 2 Take $x = x' = 0$ to get $D_1 \in Z_{\mathfrak{U}}(A,\ M^*)$, (5), (4) and (8). Take $\mathrm{a} = 0$ to get (3) and (6). Take $\mathrm{m} = 0$ to get (2), (7). And in a similar way we can get other parameters. Conversely is in a same way. ■

Corollary 3.2 *Let $X = \{0\}$ and D, D_1 and R be as in perivious lemma, then $D = \delta_{(f,g)}$ if and only if $D_1 = \delta_f$ and $g = \overline{\delta}_g$. Where $\overline{\delta}_g(a) = gT(a,\ 0) - T(a,\ 0) \cdot g$.*

Proof. Since $X = \{0\}$ and $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D(x))$ so $D(a,\ 0) = (D_1(a),\ R(a))$. If $D = \delta_{(f,g)}$ then

$$\begin{aligned} D(a,\ 0) &= \delta_{(f,g)}(a,\ 0) \\ &= (f,\ g)\cdot(a,\ 0) - (a,\ 0)\cdot(f,\ g) \\ &= (f\cdot a,\ g\cdot T(a,\ 0)) - (a\cdot f,\ T(a,\ 0)\cdot g) \\ &= (f\cdot a - a\cdot f,\ g\cdot T(a,\ 0) - T(a,\ 0)\cdot g) = (\delta_f(a),\ \overline{\delta}_g(a))\ . \end{aligned}$$

So $D_1 = \delta_f$ and $R = \overline{\delta}_g$. Conversely

$$\begin{aligned} D(a,\ 0) &= (D_1(a),\ R(a)) \\ &= (\delta_f(a),\ \overline{\delta}_g(a)) \\ &= (f\cdot a - a\cdot f,\ g\cdot T(a,\ 0) - T(a,\ 0)\cdot g) \\ &= (f,\ g)\cdot(a,\ 0) - (a,\ 0)\cdot(f,\ g) \\ &= \delta_{(f,g)}(a,\ 0)\ . \end{aligned}$$

■

Theorem 3.3 *The module amenability of $A \oplus_T X$ implies module amenability of A. Moreover if $T(A,\ 0) = X$, then X is also module amenable.*

Proof. Assume that M, N are Banach $A - \mathfrak{U}$-bimodule and Banach $X - \mathfrak{U}$-bimodule respectively. Let $D_1 : A \to M^*$ and $D_2 : X \to N^*$ be module derivations. By Proposition 2.5, $M \oplus N$ is a Banach $A \oplus_T X - \mathfrak{U}$- bimodule. Define $D' : A \oplus_T X \to M^* \oplus N^*; D'(a,\ x) = (D_1(a), D_2(T(a,0))$. Now

$$\begin{aligned} D'((a,\ x)(m,\ x')) &= D'\ (am,\ a.x' + x\circ m + T(am,\ 0)) \\ &= (D_1(am), D_2(T(am,0))) \\ &= (D_1(a).m + a.D_1(m),\ D_2(T(a,\ 0)).T(m,\ 0) + T(a,\ 0).D_2(T(m,\ 0))) \\ &= (D_1(a),\ D_2(T(a,\ 0))).(m,\ x') + (a,\ x).(D_1(m),\ D_2(T(m,\ 0))) \\ &= D'(a,\ x).(m,\ x') + (a,\ x).D'(m,\ x)). \end{aligned}$$

Also

$$\begin{aligned} D'(\alpha\cdot(a,\ x)) &= D'(\alpha\cdot a,\ \alpha\nabla x) \\ &= ((D_1(\alpha\cdot a),\ D_2(T(\alpha\cdot a,\ 0))) \\ &= ((\alpha\cdot D_1(a),\ \alpha\cdot D_2(T(a,\ 0))) \\ &= (\alpha\cdot D'(a,\ x)\ . \end{aligned}$$

And

$$\begin{aligned} D'((a,\ x) + (m,\ x') &= D'((a+m,\ x+x')) \\ &= (D_1(a+m),\ D_2(T(a+m),\ 0)) \\ &= (D_1(a) + D_1(m),\ D_2(T(a,\ 0)) + D_2(T(m,\ 0)) \\ &= (D_1(a),\ D_2(T(a,\ 0)) + (D_1(m),\ D_2(T(m,\ 0)) \\ &= D'(a,\ x) + D'(m,\ x'). \end{aligned}$$

So D' is a module derivation. Sice $A \oplus_T X$ is module amenable, there exists $(f,\ g) \in M^* \oplus N^*$ such that $D' = \delta_{(f,g)}$. Thus

$$\begin{aligned} D'(a,\ x) &= \delta_{(f,g)}(a,\ x) \\ &= (f,\ g)\cdot(a,\ x) - (a,\ x)\cdot(f,\ g) \\ &= (f\cdot a + g\cdot T(a,\ 0)) - (a\cdot f,\ T(a,\ 0)\cdot g) \\ &= (f\cdot a - a\cdot f,\ g\cdot T(a,\ 0) - T(a,\ 0)\cdot g)\ . \end{aligned}$$

Consequently $D_1(a) = f \cdot a - a \cdot f$ i.e. $D_1 = \delta_f$ and $D_2(T(a,\ 0)) = \delta_g(T(a, 0))$. Since $T(A,\ 0) = X$, $D_2(x) = \delta_g(x)$ for all $x \in X$. ■

Theorem 3.4 *The module amenability of A implies module amenability of $A \oplus_T \{0\}$.*

Proof. Let $M \oplus N$ be a Banch $A \oplus_T X - \mathfrak{U}$-bimodule and $D: A \oplus_T X \to M^* \oplus N^*$ be a module derivation. By lemma 3.1, there are D_1, D_2, D_3 and R such that $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D_3(x))$. Since here $X = \{0\}$ so $D(a,\ 0) = (D_1(a),\ R(a))$. Module amenability of A implies there exist $f \in M^*$ such that $D_1 = \delta_f$ and since N^* is an A-U- bimodule and R is module derivation, there exist $g \in N^*$ such that $R = \delta_g$. Thus $D = \delta_{(f,g)}$. ■

Theorem 3.5 *If $T(A,\ 0) = X$ and $A^2 = A$, then the module amenability of A implies module amenability of $A \oplus_T X$.*

Proof. Let $M \oplus N$ be a Banch $A \oplus_T X - \mathfrak{U}$-bimodule and $D: A \oplus_T X \to M^* \oplus N^*$ be a module derivation. By lemma 3.1, there are D_1, D_2, D_3 and R such that $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D_3(x))$. Since here $T(A,\ 0) = X$ and $A^2 = A$ so $D(a,\ x) = (D_1(a),\ R(a))$. Module amenability of A implies there exist $f \in M^*$ such that $D_1 = \delta_f$ and since N^* is an $A - \mathfrak{U}$-bimodule and R is module derivation, there exist $g \in N^*$ such that $R = \delta_g$. Thus $D = \delta_{(f,g)}$. ■

Example 3.6 *Let $\mathbb{N}$ be the set of positive integers. Consider $S = (\mathbb{N},\ \vee)$ with the maximum operation $m \vee n = max\{m,\ n\}$, then S is a amenable countable, abelian inverse semigroup with the identity 1. Clearly $E_S = S$. This semigroup is denoted by $\mathbb{N}_\vee$. $l^1(\mathbb{N}_\vee)$ is unital with unit δ_1. Since $\mathbb{N}_\vee$ is amenable and $l^1(\mathbb{N}_\vee)$ is unital so $l^1(\mathbb{N}_\vee)$ is module amenable (as an $l^1(\mathbb{N}_\vee) - l^1(\mathbb{N}_\vee)$)-bimodule. Define $T: l^1(S) \times l^1(S) \to l^1(S)$; $T(\delta_x,\ \delta_y) = \begin{cases} \delta_x, & \delta_y = 0 \\ \delta_{x \vee y}, & \delta_y \neq 0. \end{cases}$ Then $l^1(\mathbb{N}_\vee) \oplus_T l^1(\mathbb{N}_\vee)$ is module amenable.*

4 WEAK MODULE AMENABILITY

The Banach algebra A is called weak module amenable (as an $\mathfrak{U}$-bimodule), if $H_{\mathfrak{U}}(A,\ X) = \{0\}$, where X is a commutative $\mathfrak{U}$-submodule of A^*([2]).

Theorem 4.1 *The weak module amenability of $A \oplus_T X$ implies weak module amenability of A. In addition if $T(A,\ 0) = X$ then X is also weak module amenable.*

Proof. Assume that M, N are commutative $\mathfrak{U}$-submodule of A^* and X^*, respectively. we can show that $M \oplus N$ is a commutative $\mathfrak{U}$-submodule of $(A \oplus_T X)^*$. Let $D_1 \in Z_{\mathfrak{U}}(A,\ M)$ and $D_2 \in Z_{\mathfrak{U}}(X,\ N)$. Define $D : A \oplus_T X \to M \oplus N; D(a,\ x) = (D_1(a), D_2(T(a, 0)))$, it is easy to see that $D \in Z_{\mathfrak{U}}(A \oplus \tau X, M \oplus N)$. Since $A \oplus_T X$ is weak module amenable there is $(f,\ g) \in M \oplus N$ such that $D = \delta_{(f,g)}$ and

$$\begin{aligned}(D_1(a), D_2(T(a,\ 0))) &= D(a,\ x) \\ &= \delta_{(f,g)}(a,\ x)\end{aligned}$$

$$= (f,\ g) \cdot (a,\ x) - (a,\ x) \cdot (f,\ g)$$
$$= (f \cdot a - a \cdot f,\ g \cdot T(a,\ 0) - T(a,\ 0) \cdot g)$$
$$= (\delta_f(a),\ \delta_g(T(a,\ 0)))$$

Hence A, X are weak module amenable. ■

Theorem 4.2 *The weak module amenability of A implies the weak module amenability of* $A \oplus_T \{0\}$.

Proof. Suppose that $M \oplus N$ is a commutative Banach $\mathfrak{U}$-submodule of $(A \oplus \tau\{0\})^*$, and $D \in Z_{\mathfrak{U}}(A \oplus \tau\{0\},\ M \oplus N)$. Then M and N are commutative $\mathfrak{U}$-submodule of A^*. Since $D \in Z_{\mathfrak{U}}(A \oplus \tau\{0\},\ M \oplus N)$, by lemma 3.1 there are $D_1 \in Z_{\mathfrak{U}}(A,\ M)$, and $R \in Z_{\mathfrak{U}}(A,\ N)$, such that $D(a,\ 0) = (D_1(a),\ R(a))$. Since A is weak module amenable so there are $m \in M$ and $n \in N$ such that $D_1 = \delta_m, R = \delta_n$, where $\delta_n(a) = a \cdot n - n \bullet a = T(a,\ 0) \circledcirc n - n.T(a,\ 0)$.

Now

$$\begin{aligned} D(a,\quad x) &= (D_1(a),\quad R(a)) \\ &= (\delta_m(a),\ \delta_n(a)) \\ &= (a \cdot m - a \bullet m,\ T(a,\ 0) \circledcirc n - n.\ T(a,\ 0)) \\ &= (a,\ 0) \cdot (m,\ n) - (m,\ n) \circ (a,\ 0) \\ &= \delta_{(m,n)}(a,\ 0)\ . \end{aligned}$$

■

Theorem 4.3 *If* $T(A,\ 0) = X$ *and* $A^2 = A$, *then the weak module amenability of A implies the weak module amenability of* $A \oplus_T X$.

Proof. The proof is as above theorem. ■

Example 4.4 *Let* $S = \mathbb{N}_\vee$ *be as in Example 3.6, since* $l^1(S)$ *is* $l^1(S) - l^1(S) -$ *module and* $l^1(S)$ *is weak module amenable. Let* $T: l^1(S) \times l^1(S) \to l^1(S)$ *have the properties as above theorems, then* $l^1(S) \oplus_T l^1(S)$ *is weak module amenable.*

5 MODULE APPROXIMATE AMENABILITY

Let A be as above, then A is module approximately amenable (as an $\mathfrak{U}$- bimodule), if for any commutative Banach $A - \mathfrak{U}$-bimodule X, each module derivation $D: A \to X^*$ is approximately inner.

A derivation $D: A \to X$ is said to be approximately inner if there exists a net $(x_i)_i \subseteq X$ such that $D(a) = \lim_i (a \cdot x_i - x_i \cdot a), a \in A$.([10]) .

Lemma 5.1 *Let* D_1, R, D_3 *and* D_2 *are such as in the Lemma 3.1, and* $D(a,\ b) = (D_1(a) + D_2(b),\ R(a) + D_3(b))$. *If* $T(A,\ 0) = X$ *and* $A^2 = A$ *then: D is approximately inner if and only if* D_1 *and R are approximately inner.*

Proof. Assume that M is a commutative $A - \mathfrak{U}$-bimodule and also N is com- mutative $X - \mathfrak{U}$-bimodule, then $M \oplus N$ is a commutative $A \oplus_T X - \mathfrak{U}$-bimodule. Let D be approximately inner so there is $(f_i,\ g_i)_i \subseteq M^* \oplus N^*$ such that

$$
\begin{aligned}
D(a,\quad x) &= T(a',\quad 0)\\
&= \lim_i((a,\ x)\cdot(f_i,\ g_i)-(f_i,\ g_i)\cdot(a,\ x))\\
&= \lim_i((a\cdot f_i,\ T(a,\ 0)\cdot g_i)-(f_i\cdot a,\ g_i.T(a,\ 0)))\\
&= \lim_i(a\cdot f_i-f_i\cdot a,\ T(a,\ 0)\cdot g_i-g_i\cdot T(a,\ 0)),
\end{aligned}
$$

i.e. $D(a)=\lim_i(a\cdot f_i-f_i\cdot a)$ and $R(a)=\lim_i(T(a,\ 0)\cdot g_i-g_i\cdot T(a,0))$.

Conversely, let $D_1(a)=\lim_{i\in I}(a\cdot f_i-f_i\cdot a)$ and $R(a)=\lim_{j\in J}(T(a,\ 0)\cdot g_j-g_j\cdot T(a,0))$

Since the index sets $(I,\ \leq),(J,\ \leq)$ are ordered sets, so the set $\Lambda=I\times J=\{(i,\ j)\ :\ i\in I,\ j\in J\}$ is ordered as follows

$$(i,\ j)\leq(i',\ j')\Leftrightarrow(i\leq i',\ j\leq j).$$

For $\lambda=(i,\ j)\in\Lambda$ set $t_\lambda=(f_i,\ g_j)$. Let $\epsilon>0$ be given. Since $D_1(a)=\lim_i(a\cdot f_i-f_i\cdot a)$ and $R(a)=\lim_j(T(a,\ 0)\cdot g_j-g_j\cdot T(a,0))$ there are $i_0\in I,j_0\in J$ such that

1) For all $i\geq i_0$, $\| D_1(a)-(a\cdot f_i-f_i\cdot a)\|\leq\frac{\epsilon}{3}$.

2) For all $j\geq j_0$, $\| R(a)-(T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\leq\frac{\epsilon}{3}$.

Now set $\lambda_0=(i_0,\ j_0)$, then for all $\lambda\geq\lambda_0$, since $D(a,\ T(a',\ 0))=(D_1(a),\ R(a))$, we have

$$
\begin{aligned}
\| D(a,\quad x)-((a,\quad x)\cdot t_\lambda-t_\lambda\cdot(a,\quad x))\| &=\| D(a,\quad x)-((a,\quad x)\cdot(f_i,\ g_j)-(f_i,\quad g_j)\cdot(a,\quad x))\|\\
&=\| D(a,\ x)-(a\cdot f_i-f_i\cdot a,\ T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\\
&=\|(D_1(a),\ R(a))-(a\cdot f_i-f_i\cdot a,\ T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\\
&=\|((D_1(a)-(a\cdot f_i-f_i\cdot a)),R(a)-(T(a,\ 0)\cdot g_j-g_j\cdot T(a,0)))\|\\
&\leq\| D_1(a)-(a\cdot f_i-f_i\cdot a)\|+\| R(a)-(T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\\
&<\epsilon.
\end{aligned}
$$

Hence $D(a,\ x)=\lim_\lambda((a,\ x)\cdot t_\lambda-t_\lambda(a,\ x$ where $x=T(a',\ 0)$ i.e. D is approximately inner. ■

Theorem 5.2 *If $A\oplus\tau X$ is module approximately amenable then A is module approximately amenable. Furthermore, if $T(A,\ 0)=X$ also X is module approximately amenable.*

Proof. In an argument as in the proof of Theorem 3.3 and the application, the usage of above lemma. ■

Theorem 5.3 *If $T(A.O)=X$ and $A^2=A$ then the module approximate amenability of A implies the module approximate amenability of $A\oplus_T X$.*

Proof. Let $M\oplus N$ be a commutative $A\oplus_T X-\mathfrak{U}$-bimodule and $D\in Z_{\mathfrak{U}}(A\oplus\tau X,M^*\oplus N^*)$. There are $D_1\in Z_{\mathfrak{U}}(A,\ M^*),D_3\in Z_{\mathfrak{U}}(X,\ N^*),R\in Z_{\mathfrak{U}}(A,\ N^*)$ and $D_2:X\rightarrow N^*$ such that $D(a,\ x)=(D_1(a)+D_2(x),\ R(a)+D(x))$ and since $T(A,\ 0)=X$ and $A^2=A$ we have $D(a,\ x)=(D_1(a),\ R(a))$. Since A,X are module approximate amenable, so D_1 and R are approximatly inner. Thus by the above lemma, D is approximately inner. ■

Example 5.4 *Let S be an amenable inverse semigroup such that the set of idempotents E_S be equal to S and $l^1(S)$ has approximately unit. Since S is amenable, $l^1(S)$ is module approximately amenable, [10]. Also $l^1(S)$ is $l^1(S) - l^1(S)$-bimodule, thus $l^1(S) \oplus_T l^1(S)$ is module approximately amenable. Where $T: l^1(S) \times l^1(S) \to l^1(S)$ is defined by*

$$T(\delta_x, \delta_y) = \begin{cases} \delta_x, & \delta_y = 0 \\ \delta_{xy}, & \delta_y \neq 0. \end{cases}$$

6 CONCLUSIONS

The module amenability of $A \oplus_T X$ implies module amenability of A and The module amenability of A implies module amenability of $A \oplus_T \{0\}$. Also If $T(A, 0) = X$ and $A^2 = A$, then the module amenability of A implies module amenability of $A \oplus_T X$. meanwhile, The weak module amenability of $A \oplus_T X$ implies weak module amenability of A. On the contrary, if $T(A, 0) = X$ and $A^2 = A$, then the weak module amenability of A implies the weak module amenability of $A \oplus_T X$.

Considering approximately, if $A \oplus \tau X$ is module approximately amenable then A is module approximately amenable. On the contrary, if $T(A, 0) = X$ and $A^2 = A$ then the module approximate amenability of A implies the module approximate amenability of $\oplus_T X$. For example, we have S be an amenable inverse semigroup such that the set of idempotents E_S be equal to S and $l^1(S)$ has approximately unit. Since S is amenable, $l^1(S)$ is module approximately amenable.

Acknowledgements: The authors would like to thank the referees for their careful reading and constructive comments which improved the paper.

REFERENCES

[1] M. Amini, "Module amenability for semigroup algebras", *Semigroup Forum*, **69**, 243-254 (2004).
[2] M.Amini and A.Bodaghi, "Module amenability and weak module amenability for second dual of Banach algebras", *Chamchuri Journal of Math*, 57-71 (2010).
[3] M.Amini and D.Ebrahimi Bagha, "Weak module amenability for semigroup algebras", *Semigroup Forum*. **71**, 18-26 (2005).
[4] H.Azaraien and D.Ebrahimi Bagha, "Ideal amenability of module Lau Banach algebra", *Indagationes Mathematicae,* **29**(2), 738-745 (2018).
[5] W. G. Bade, P. C. Curtis and H. G. Dales, "Amenability and weak amenability for Beurling and Lipschits algebra", *Proc. London Math. Soc*., **55**(3), 359-377 (1987).
[6] S.J. Bhatt and P.A. Dabhi, "Arens regularity and amenability of Lau product of Banach algebras defined by a Banach algebra morphism", *Bull. Aust. Math. Soc*. **87**, 195-206 (2013).
[7] B. E. Johnson, "Weak amenability of group Algebras", *Bull. London Math. Soc*., **23**, 281-284 (1991).
[8] B. E. Johnson, "Derivation from $L^1(G)$ into $L^1(G)$ and $L^\infty(G)$", *Lecture Note in Math*, 191-198 (1988).
[9] B.E.Johnson," Cohomology in Banach algebras", *Memoirs Amer.Math.Soc*., **127** (1972) .
[10] H.Pourmahmood-Aghababa and A.Bodaghi, "Module approximate amenability of Banach algebras", *Bulletin of Iranian Mathematical Soc*., **39**, 1137-1158 (2013).
[11] M.Sangani-Monfared, "On certain products of Banach algebras with applications to

harmonic analysis", *Studia Math*., **178**(3) , 277 − 294 (2007).
[12] T.Yazdan Panah, "Amenability and Weak amenability of module extension of Banach algebra by 2-cocycles", *Far East.Math. Sci*., 357-374 (2004).

Received March 16, 2020