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INVERSE PROBLEMS FOR STURM – LIOUVILLE OPERATOR WITH POTENTIAL FUNCTIONS FROM $L_2[0,\pi]$

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Summary. This paper deals with non-self-adjoint second-order differential operators with two constant delays. We consider four boundary value problems $D_{i,k}, i=0,1, k=1,2$

$$-y''(x)+q_1(x)y(x-\tau_1)+(-1)^i q_2(x)y(x-\tau_2)=\lambda y(x), x\in[0,\pi]$$

$$y'(0)-hy(0)=0, \qquad y'(\pi)+H_k y(\pi)=0,$$

where $\frac{\pi}{3}\le\tau_2<\frac{\pi}{2}\le 2\tau_2\le\tau_1<\pi,\ h,H_1,H_2\in R\backslash\{0\}$ and λ is a spectral parameter. We assume that q_1,q_2 are real-valued potential functions from $L_2[0,\pi]$ such that $q_1(x)=0, x\in[0,\tau_1)$ and $q_2(x)=0,\ x\in[0,\tau_2)$. The inverse spectral problem of recovering operators from their spectra has been studied. We prove that delays τ_1,τ_2 and parameters h,H_1,H_2 are uniquely determined from the spectra. Then we prove that potentials are uniquely determined by Volterra linear integral equations.

1 INTRODUCTION

The theory of differential equations with delays is a very important branch of the theory of ordinary differential equations and has been studied in detail in [1] and the references therein. For a number of results relating to the inverse spectral problems for classical Sturm-Liuville operators we refer the reader to [2], while some aspects of the direct and inverse problems for operators with a delay can be found in [3] - [13]. While there are a number results about both direct and inverse problems for operators with one delay, there are just a few results related to the operators with two or more delays (see [14]-[18]). The motivation behind this paper is to initiate further research in the inverse spectral theory for differential operators with delays. In what follows, we always take $i=0,1$ and $k=1,2$. In this paper we consider the boundary value problems $D_{i,k}$

$$-y''(x)+q_1(x)y(x-\tau_1)+(-1)^i q_2(x)y(x-\tau_2)=\lambda y(x), x\in[0,\pi] \quad (1)$$

$$y'(0)-hy(0)=0, \quad (2)$$

$$y'(\pi)+H_k y(\pi)=0 \quad (3)$$

where $\frac{\pi}{3}\le\tau_2<\frac{\pi}{2}\le 2\tau_2\le\tau_1<\pi,\ h,H_1,H_2\in R\backslash\{0\}$ and λ is a spectral parameter. We assume that q_1,q_2 are real-valued potential functions from $L_2[0,\pi]$ such that

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$q_1(x) = 0,\ x \in [0, \tau_1)$ and $q_2(x) = 0,\ x \in [0, \tau_2)$. We study the inverse spectral problem of recovering operators from the spectra of $D_{i,k}$. Let $\left(\lambda_{n,i,k}\right)_{n=0}^{\infty}$ be the eigenvalues of $D_{i,k}$. The inverse problem is formulated as follows.

Inverse problem: Given $\left(\lambda_{n,i,k}\right)_{n=0}^{\infty}$, determine delays τ_1, τ_2, parameters h, H_1, H_2 and potential functions q_1, q_2.

To solve this inverse problem, we use the method of Fourier coefficients. This method based on determination of direct relations between Fourier coefficients of the potentials or some functions containing the potentials, and Fourier coefficients of some known functions.

In Section 2, we study the spectral properties of the boundary value problems $D_{i,k}$. In Section 3, we prove that delays and parameters are uniquely determined from the spectra. Then we prove that potentials are uniquely determined by the system of two Volterra linear integral equations.

2 SPECTRAL PROPERTIES

One can easily show that differential equation (1) under the initial condition (2) and conditions $q_1(x) = 0, x \in [0, \tau_1)$ and $q_2(x) = 0,\ x \in [0, \tau_2)$ is eqivalent to the integral equation

$$\begin{aligned} y_i(x,z) = cosxz + \frac{h}{z} sinzx + \frac{1}{z}\int_{\tau_1}^{x} q_1(t) sinz(x-t) y(t-\tau_1, z)dt + \\ + (-1)^i \frac{1}{z}\int_{\tau_2}^{x} q_2(t) sinz(x-t) y(t-\tau_2, z)dt \end{aligned} \tag{4}$$

Here and in the sequel $\lambda = z^2$. By the method of steps it can be easily verified that the solution of integral equation (4) on $(\tau_1, \pi]$ is

$$\begin{aligned} y_i(x,z) = coszx + \frac{h}{z} sinzx + \frac{1}{z}\left(b_{sc}^{(1)}(x,z) + (-1)^i b_{sc}^{(2)}(x,z)\right) + \\ + \frac{h}{z^2}\left(b_{s^2}^{(1)}(x,z) + (-1)^i b_{s^2}^{(2)}(x,z)\right) + \frac{1}{z^2} b_{s^2c}^{(2)}(x,z) + \frac{h}{z^3} b_{s^3}^{(2)}(x,z) \end{aligned} \tag{5}$$

where

$$b_{sc}^{(k)}(x,z) = \int_{\tau_k}^{x} q_k(t) sinz(x-t) cosz(t-\tau_k)dt,$$

$$b_{s^2}^{(i)}(x,z) = \int_{\tau_k}^{x} q_k(t) sinz(x-t) sinz(t-\tau_k)dt,$$

$$b_{s^3}^{(2)}(x,z) = \int_{2\tau_2}^{x} q_2(t) sinz(x-t) b_{s^2}^{(2)}(t-\tau_2, z)dt,$$

$$b_{s^2c}^{(2)}(x,z) = \int_{2\tau_2}^{x} q_2(t) sinz(x-t) b_{sc}^{(2)}(t-\tau_2, z)dt.$$

Denote

$$\Delta_{i,k}(\lambda) = F_{i,k}(z) = y_i{}'(\pi, z) + H_k y(\pi, z).$$ [1]

From (5) we obtain

$$F_{i,k}(z) = \left(-z + \frac{hH_k}{z}\right) sin\pi z + (h + H_k)cos\pi z + b_{c^2}^{(1)}(z) + (-1)^i b_{c^2}^{(2)}(z) +$$

$$+\frac{h}{z}\left(b_{cs}^{(1)}(z) + (-1)^i b_{cs}^{(2)}(z)\right) + \frac{H_k}{z}\left(b_{sc}^{(1)}(z) + (-1)^i b_{sc}^{(2)}(z)\right) +$$

$$+\frac{H_k h}{z^2}\left(b_{s^2}^{(1)}(z) + (-1)^i b_{s^2}^{(2)}(z)\right) + \frac{1}{z} b_{csc}^{(2)}(z) + \frac{h}{z^2} b_{cs^2}^{(2)}(z) + \frac{H_k}{z^2} b_{s^2c}^{(2)}(z) + \frac{Hh}{z^3} b_{s^3}^{(2)}(z)$$

where

$$b_{cs}^{(k)}(z) = \int_{\tau_k}^{\pi} q_k\,(t) cosz(x-t) sinz(t-\tau_k)dt,$$
$$b_{c^2}^{(k)}(z) = \int_{\tau_k}^{\pi} q_k\,(t) cosz(x-t) cosz(t-\tau_k)dt$$

$$b_{s^2c}^{(2)}(z) = \int_{2\tau_2}^{\pi} q_2\,(t) sinz(x-t) b_{sc}^{(2)}(t-\tau_2, z\,)dt,$$
$$b_{csc}^{(2)}(z) = \int_{2\tau_2}^{\pi} q_2\,(t) cosz(x-t) b_{sc}^{(2)}(t-\tau_2, z\,)dt,$$
$$b_{cs^2}^{(2)}(z) = \int_{2\tau_2}^{\pi} q_2\,(t) cosz(x-t) b_{s^2}^{(2)}(t-\tau_2, z\,)dt.$$

To simplify further consideration we define so called *the transitional functions* $\tilde{q}_i$ as follows

$$\tilde{q}_i(t) = \begin{cases} q_1\left(t + \frac{\tau_1}{2}\right) + (-1)^i q_2\left(t + \frac{\tau_2}{2}\right), t \in \left[\frac{\tau_1}{2}, \pi - \frac{\tau_1}{2}\right] \\ (-1)^i q_2\left(t + \frac{\tau_2}{2}\right), t \in \left[\frac{\tau_2}{2}, \tau_2\right) \cup \left(\pi - \frac{\tau_1}{2}, \pi - \frac{\tau_2}{2}\right]. \\ 0, t \in \left[0, \frac{\tau_2}{2}\right) \cup \left(\pi - \frac{\tau_2}{2}, \pi\right] \end{cases} \tag{6}$$

Let us also define functions $K^{(2)}$ and $U^{(2)}$ by

$$K^{(2)}(t) = q_2(t+\tau_2)\int_{\tau_2}^{t} q_2(s)ds - q_2(t)\int_{t+\tau_2}^{\pi} q_2(s)ds - \int_{t+\tau_2}^{\pi} q_2(s-t)q_2(s)ds,$$
$$t \in [\tau_2, \pi - \tau_2], \qquad K^{(2)}(t) = 0, t \in [0, \tau_2) \cup (\pi - \tau_2, \pi],$$

and

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[1] Below, instead of the argument (π, z) we write argument (z)

$$U^{(2)}(t) = q_2(t+\tau_2)\int_{\tau_2}^{t} q_2(s)ds - q_2(t)\int_{t+\tau_2}^{\pi} q_2(s)ds + \int_{t+\tau_2}^{\pi} q_2(s-t)q_2(s)ds,$$
$$t \in [\tau_2, \pi-\tau_2], \qquad U^{(2)}(t) = 0, t \in [0,\tau_2) \cup (\pi-\tau_2, \pi].$$

and introduce notations

$$J_1^{(k)} = \int_{\tau_k}^{\pi} q_i\,(t)dt, \qquad J_2^{(2)} = \int_{2\tau_2}^{\pi} q_2\,(t)\left(\int_{\tau_2}^{t-\tau_2} q_2\,(s)ds\right)dt$$

and functions

$$\tilde{a}_{i,c}(z) = \int_0^{\pi} \tilde{q}_i(t)cosz(\pi-2\,t)dt, \quad \tilde{a}_{i,s}(z) = \int_0^{\pi} \tilde{q}_i(t)sinz(\pi-2\,t)dt,$$
$$k_s(z) = \int_0^{\pi} K^{(2)}(t)\,sinz(\pi-2t)dt, \quad k_c(z) = \int_0^{\pi} K^{(2)}(t)\,cosz(\pi-2t)dt,$$
$$u_s(z) = \int_0^{\pi} U^{(2)}(t)\,sinz(\pi-2t)dt, \quad u_c(z) = \int_0^{\pi} U^{(2)}(t)\,cosz(\pi-2t)dt.$$

One can easily show that folowing relations hold

$$\int_{\tau_2}^{\pi-\tau_2} K^{(2)}(t)\,dt = -J_2^{(2)}, \qquad \int_{\tau_2}^{\pi-\tau_2} U^{(2)}(t)\,dt = J_2^{(2)}. \tag{7}$$

Using aforementioned tags and relations (7), we can rewrite characteristic functions $F_{i,k}(z)$ as follows

$$F_{i,k}(z) = \left(-z + \frac{hH_k}{z}\right)sin\pi z + (h+H_k)cos\pi z + \frac{1}{2}\left(\tilde{a}_{i,c}(z) + J_{i,c}(z)\right) +$$
$$+\frac{h}{2z}\left(-\tilde{a}_{i,s}(z) + J_{i,s}(z)\right) + \frac{H_k}{2z}\left(\tilde{a}_{i,s}(z) + J_{i,s}(z)\right) + \frac{hH_k}{2z^2}\left(\tilde{a}_{i,c}(z) - J_{i,c}(z)\right)$$
$$+\frac{1}{4z}\left(J_{2,s}(z) - u_s(z)\right) - \frac{h}{4z^2}\left(J_{2,c}(z) + k_c(z)\right) - \frac{H_k}{4z^2}\left(J_{2,c}(z) - u_c(z)\right)$$
$$-\frac{hH_k}{4z^3}\left(J_{2,s}(z) + k_s(z)\right) \tag{8}$$

where

$$J_{i,c}(z) = J_1^{(1)}cosz(\pi-\tau_1) + (-1)^i J_1^{(2)}cosz(\pi-\tau_2),$$
$$J_{i,s}(z) = J_1^{(1)}sinz(\pi-\tau_1) + (-1)^i J_1^{(2)}sinz(\pi-\tau_2)$$
$$J_{2,c}(z) = J_2^{(2)}cosz(\pi-2\tau_2), \qquad J_{2,s}(z) = J_2^{(2)}sinz(\pi-2\tau_2).$$

The functions $F_k(z)$ are entire in λ of order $1/2$. Using (8), by the well known method (see [2]), we obtain the asymptotic formulas for $(\lambda_{n,i,k})_{n=0}^{\infty}$ of $D_{i,k}$:

$$\lambda_{n,i,k} = n^2 + \frac{2}{\pi}(h + H_k) + \frac{J_1^{(1)}}{\pi} cosn\tau_1 + (-1)^i \frac{J_1^{(2)}}{\pi} cosn\tau_2 + o(1), n \to \infty. \tag{9}$$

Now, by Hadamard's factorization theorem, from the spectra of $D_{i,k}$, we can construct the characteristic functions $F_{i,k}$. The next lemma holds.

Lemma 2.1. The specification of spectrum $(\lambda_{n,i,k})_{\mathrm{n=0}}^{\infty}$ of the boundary value problems $\mathrm{D_{i,k}}$ uniquely determines the characteristic functions $F_{i,k}(z)$ by the formulas

$$F_{i,k}(z) = \pi(\lambda_{0,i,k} - z^2) \prod_{n=1}^{\infty} \frac{\lambda_{n,i,k} - z^2}{n^2}. \tag{10}$$

3 MAIN RESULTS

Lemma 3.1. The delays τ_k, integrals $J_1^{(k)}$ and sums $h + H_k$ are uniquely determined by eigenvalues $(\lambda_{n,i,k})_{n=0}^{\infty}$.

Proof. Let us consider the sequences

$$\rho_{n,k} = \tfrac{1}{2}(\lambda_{n,0,k} + \lambda_{n,1,k})$$

and

$$\sigma_n = \frac{1}{2}(\lambda_{n,0,1} - \lambda_{n,1,1}).$$

From (9) we obtain the next asymptotic formulas

$$\rho_{n,k} = n^2 + \frac{2}{\pi}(h + H_k) + \frac{J_1^{(1)}}{\pi} cosn\tau_1 + o(1)$$

and

$$\sigma_n = \frac{J_1^{(2)}}{\pi} cosn\tau_2 + o(1).$$

Obviously, the delays τ_1, τ_2 and integrals $J_1^{(1)}$, $J_1^{(2)}$ can be determined from sequences $(\rho_{n,k})_{n=0}^{\infty}$ and $(\sigma_n)_{n=0}^{\infty}$ in the same way as for the operators with one delay (see [13]). Lemma 3.1. is proved. □

Lemma 3.2. Parameters h and H_k are uniquely determined by eigenvalues $(\lambda_{n,0,k})_{n=0}^{\infty}$.

Proof. By virtue of Lemma 3.1., functions $J_{0,c}(z)$ and $J_{0,s}(z)$ are known. Since the characteristic functions are uniquely determined by the spectra, putting $\lambda = \left(\frac{4m+1}{2}\right)^2$ into functions $F_{0,k}$ from (10), we can define functions

$$F^*{}_{0,k}(m) = F_{0,k}\left(\frac{4m+1}{2}\right) + \frac{4m+1}{2} - \frac{1}{2}J_{0,c}\left(\frac{4m+1}{2}\right) - \frac{H_k + h}{4m+1}J_{0,s}\left(\frac{4m+1}{2}\right).$$

Then, using the form of the characteristic functions $F_{0,k}$ from (8), we get

$$h = \frac{1}{2}\lim_{m\to\infty}\frac{4m+1}{H_2 - H_1}\left(F^*{}_{0,2}(m) - F^*{}_{0,1}(m)\right)$$

At the end, we determine H_k from $h + H_k$, thus proving Lemma 3.2. □

In order to recover the potential functions from the spectra by the method of Fourier coefficients, we should transform the characteristic functions (8). For this purpose, we use the method of partial integration in (8), once in integrals $\tilde{a}_{i,s}(z), \tilde{a}_{i,c}(z), u_s(z)$ and $u_c(z)$, and twice in the integrals $k_c(z)$ and $k_s(z)$. This is where the next function appears

$$K^{(2)*}(t) = \begin{cases} \int_{\tau_2}^{t} K^{(2)}(u)\,du, t \in [\tau_2, \pi - \tau_2] \\ 0, t \in [0, \tau_2) \cup (\pi - \tau_2, \pi] \end{cases}.$$

One can show that following relation holds

$$\int_{\tau_2}^{\pi-\tau_2}\left(\int_{\tau_2}^{t} K^{(2)}(u)\,du\right)dt = -(\pi - 2\tau_2)J_2^{(2)}.$$

Then we obtain the characteristic functions in the form

$$\begin{aligned} F_{i,k}(z) &= \left(-z + \frac{H_k h}{z}\right)sin\pi z + (h + H_k)cos\pi z + \frac{1}{2}\left(\tilde{a}_{i,c}(z) + \frac{H_k}{z}\tilde{a}_{i,s}(z)\right) - \\ &-h\left(\tilde{q}_{i,c}{}^{(1)}(z) + \frac{H_k}{z}\tilde{q}_{i,s}{}^{(1)}(z)\right) - \frac{1}{2}\left(u_c^*(z) + \frac{H_k}{z}u_s^*(z)\right) + h\left(k_c^{**}(z) + \frac{H_k}{z}k_s^{**}(z)\right) \\ &+\frac{J_{i,c}(z)}{2} + \frac{2h + H_k}{2z}J_{i,s}(z) + \frac{1}{2z}\left(1 - \frac{H_k h}{z^2}\right)J_{2,s}(z) + \frac{h}{2z}(\pi - 2\tau_2)J_2^{(2)}sinz(\pi - 2\tau_2) \\ &+\frac{H_k h}{2z^2}(\pi - 2\tau_2)J_2^{(2)}cosz(\pi - 2\tau_2) \end{aligned} \tag{11}$$

where

$$\tilde{q}_{i,c}{}^{(1)}(z) = \int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}} \left(\int_{\frac{\tau_2}{2}}^{t} \tilde{q}_i(s)\, ds \right) cosz(\pi - 2t)dt,$$

$$\tilde{q}_{i,s}{}^{(1)}(z) = \int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}} \left(\int_{\frac{\tau_2}{2}}^{t} \tilde{q}_i(s)\, ds \right) sinz(\pi - 2t)dt\ ,$$

$$u_c^*(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} U^{(2)}(s)\, ds \right) cosz(\pi - 2t)dt,$$

$$u_s^*(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} U^{(2)}(s)\, ds \right) sinz(\pi - 2t)dt$$

and

$$k_c^{**}(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} K^{(2)*}(s)\, ds \right) cosz(\pi - 2t)dt,$$

$$k_s^{**}(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} K^{(2)*}(s)\, ds \right) sinz(\pi - 2t)dt.$$

In order to recover the potential functions from the spectra, at the beginning we define functions

$$A_i(z) = \frac{2}{H_2 - H_1}\Big(H_2 F_{i,1}(z) - H_1 F_{i,2}(z)\Big) + 2zsin\pi z - 2hcos\pi z - J_{i,c}(z) -$$

$$- \frac{2hJ_1^{(2)}}{z} sinz(\pi - \tau_2) \qquad (12)$$

and

$$B_i(z) = \frac{2z}{H_2 - H_1}\Big(F_{i,2}(z) - F_{i,1}(z)\Big) - 2hsin\pi z - 2zcos\pi z - J_{i,s}(z). \qquad (13)$$

From (11) we obtain

$$A_i(z) = \tilde{a}_{i,c}(z) - 2h\tilde{q}_{i,c}{}^{(1)}(z) - u_c^*(z) + 2hk_c^{**}(z) + \alpha(z) \qquad (14)$$

$$B_i(z) = \tilde{a}_{i,s}(z) - 2h\tilde{q}_{i,s}{}^{(1)}(z) - u_s^*(z) + 2hk_s^{**}(z) + \beta(z) \qquad (15)$$

where

$$\alpha(z) = \frac{J_2^{(2)}}{z}\big(h(\pi - 2\tau_2) + 1\big) sinz(\pi - 2\tau_2)$$

and

$$\beta(z) = \frac{hJ_2^{(2)}}{z^2}\Big(z(\pi - 2\tau_2)cosz(\pi - 2\tau_2) - sinz(\pi - 2\tau_2)\Big).$$

One can easily show that

$$\lim_{z\to 0}\beta(z)=0,$$

and

$$\lim_{z\to 0}\alpha(z)=J_2^{(2)}(h(\pi-2\tau_2)+1)(\pi-2\tau_2).$$

Put $z=m, m\in N$ into (14) and (15) and denote

$$A_{2m,i}=\frac{2}{\pi}(-1)^m A_i(m),\quad B_{2m,i}=\frac{2}{\pi}(-1)^{m+1}B_i(m).$$

Then we obtain

$$A_{2m,i}=\frac{2}{\pi}\tilde{a}_{2m,i}-\frac{4}{\pi}h\tilde{q}_{2m,i,c}^{(1)}-\frac{2}{\pi}u_{2m,c}^{*}+\frac{4}{\pi}hk_{2m,c}^{**}-\frac{2J_2^{(2)}}{\pi m}(h(\pi-2\tau_2)+1)sin2m\tau_2, \tag{16}$$

$$B_{2m}=\frac{2}{\pi}\tilde{b}_{2m,i}-\frac{4}{\pi}h\tilde{q}_{2m,i,s}^{(1)}-\frac{2}{\pi}u_{2m,s}^{*}+\frac{4}{\pi}hk_{2m,s}^{**}-\frac{2hJ_2^{(2)}}{\pi m^2}(m(\pi-2\tau_2)cos2m\tau_2+sin2m\tau_2) \tag{17}$$

where

$$\tilde{a}_{2m,i}=\int_0^{\pi}\tilde{q}_i(t)cos2mtdt,\qquad \tilde{b}_{2m,i}=\int_0^{\pi}\tilde{q}_i(t)sin2mtdt,$$

$$u_{2m,s}^{*}=\int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}}\left(\int_{\frac{\tau_2}{2}}^{t}U^{(2)}(s)\,ds\right)sin2mtdt,$$

$$u_{2m,c}^{*}=\int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}}\left(\int_{\frac{\tau_2}{2}}^{t}U^{(2)}(s)\,ds\right)cos2mtdt$$

$$k_{2m,c}^{**}=\int_{\tau_2}^{\pi-\tau_2}\left(\int_{\tau_2}^{t}K^{(2)*}(s)\,ds\right)cos2mtdt,.$$

$$k_{2m,s}^{**}=\int_{\tau_2}^{\pi-\tau_2}\left(\int_{\tau_2}^{t}K^{(2)*}(s)\,ds\right)sin2mtdt.$$

Denote $A_{0,i}=\frac{2}{\pi}\lim_{m\to 0}A_i(m)$.

Then we obtain

$$A_{0,i}=\frac{2}{\pi}\tilde{a}_{0,i}-\frac{4}{\pi}h\tilde{q}_{0,i,c}^{(1)}-\frac{2}{\pi}u_{0,c}^{*}+\frac{4}{\pi}hk_{0,c}^{**}+\frac{2J_2^{(2)}}{\pi}(h(\pi-2\tau_2)+1)(\pi-2\tau_2). \tag{18}$$

Since sequences $\{A_{2m,i}\}$ and $\{B_{2m,i}\}$ belong to the space l_2, by virtue of Riesz-Fischer theorem, there exist functions f_i from $L_2[0,\pi]$ such that

$$f_i(t) = \frac{A_{0,i}}{2} + \sum_{m=1}^{\infty} A_{2m,i} cos2mt + B_{2m,i} sin2mt\,, t \in [0, \pi]$$

Now multiplying (18) with $\frac{1}{2}$, (16) with $cos2mt$ and (17) with $sin2mt$, and then summing-up from $m = 1$ to $m = \infty$, we get the system of integral equations

$$\tilde{q}_i(t) - 2h \int_{\frac{\tau_2}{2}}^{t} \tilde{q}_i(s)\, dt_2 - \int_{\tau_2}^{t} U^{(2)}(s) ds + 2h \int_{\tau_2}^{t} K^{(2)*}(s)\, ds + \Phi(t) = f_i(t) \tag{19}$$

where

$$\Phi(t) = -\frac{2J_2^{(2)}}{\pi}(h(\pi - 2\tau_2) + 1) \sum_{m=1}^{\infty} \frac{sin2m\tau_2}{m} cos2mt -$$

$$-\frac{2hJ_2^{(2)}}{\pi}(\pi - 2\tau_2) \sum_{m=1}^{\infty} \frac{cos2m\tau_2}{m} sin2mt - \frac{2hJ_2^{(2)}}{\pi} \sum_{m=1}^{\infty} \frac{sin2m\tau_2}{m^2} sin2mt.$$

After summing and subtracting integral equations (19), and then introducing substitution of variables, we get the system of integral equations

$$q_1(x) - 2h \int_{\tau_1}^{x} q_1(u)\, du - \int_{\tau_2 + \frac{\tau_1}{2}}^{x} U^{(2)}\left(u - \frac{\tau_1}{2}\right) du + 2h \int_{\tau_2 + \frac{\tau_1}{2}}^{x} K^{(2)*}\left(u - \frac{\tau_1}{2}\right) du +$$

$$+\Phi\left(x - \frac{\tau_1}{2}\right) = \frac{1}{2}\left(f_0\left(x - \frac{\tau_1}{2}\right) + f_1\left(x - \frac{\tau_1}{2}\right)\right) \tag{20}$$

and

$$q_2(x) - 2h \int_{\tau_2}^{x} q_2(u)\, du = \frac{1}{2}\left(f_0\left(x - \frac{\tau_2}{2}\right) - f_1\left(x - \frac{\tau_2}{2}\right)\right). \tag{21}$$

Finally, we come to our main result.

Theorem 3.1. Let $q_k \in L_2[\tau_i, \pi]$, $q_k(x) = 0$ for $x \in [0, \tau_k)$.
If $\frac{\pi}{3} \le \tau_2 < \frac{\pi}{2} \le 2\tau_2 \le \tau_1 < \pi$, then integral equations (20) and (21) have unique solutions $q_1 \in L_2[\tau_1, \pi]$ and $q_2 \in L_2[\tau_2, \pi]$, respectively.

Proof. Obviously, the integral equation (21) has a unique solution q_2 on (τ_2, π). Then we obtain that integrals $\int_{\tau_2 + \frac{\tau_1}{2}}^{x} U^{(2)}\left(u - \frac{\tau_1}{2}\right) du$ and $\int_{\tau_2 + \frac{\tau_1}{2}}^{x} K^{(2)*}\left(u - \frac{\tau_1}{2}\right) du$ are known too, as well as the integral $J_2^{(2)}$. For sums appearing in the function Φ, we have

$$\sum_{m=1}^{\infty}\frac{sin2m\tau_2}{m}cos2mt=\begin{cases}-\tau_2, & t\in(\tau_2,\pi-\tau_2),\\ \frac{\pi}{2}-\tau_2, & t\in(0,\tau_2)\cup(\pi-\tau_2,\pi),\\ \frac{\pi}{4}-\tau_2, & t=\tau_2, t=\pi-\tau_2\end{cases}$$

$$\sum_{m=1}^{\infty}\frac{cos2m\tau_2}{m}sin2mt=\begin{cases}-t, & t\in(0,\tau_2)\\ \frac{\pi}{2}-t, & t\in(\tau_2,\pi-\tau_2)\\ \pi-t, & (\pi-\tau_2,\pi)\\ \frac{\pi}{4}-\tau_2, & t=\tau_2,\\ -\frac{\pi}{4}+\tau_2, & t=\pi-\tau_2\end{cases}$$

and

$$\sum_{m=1}^{\infty}\frac{sin2m\tau_2}{m^2}sin2mt=\begin{cases}(\pi-2\tau_2)t, & t\in(0,\tau_2)\\ \tau_2(\pi-2t), & t\in(\tau_2,\pi-\tau_2)\\ (\pi-2\tau_2)(t-\pi), & (\pi-\tau_2,\pi)\\ (\pi-2\tau_2)\tau_2, & t=\tau_2\\ -(\pi-2\tau_2)\tau_2, & t=\pi-\tau_2\end{cases}$$

Then for $x\in(\tau_1,\pi)$ we obtain linear integral equation

$$q_1(x)-2h\int_{\tau_1}^{x}q_1(u)\,du=\int_{\tau_2+\frac{\tau_1}{2}}^{x}U^{(2)}\left(u-\frac{\tau_1}{2}\right)du-2h\int_{\tau_2+\frac{\tau_1}{2}}^{x}K^{(2)*}\left(u-\frac{\tau_1}{2}\right)du$$
$$-\Phi\left(x-\frac{\tau_1}{2}\right)+\frac{1}{2}\left(f_0\left(x-\frac{\tau_1}{2}\right)+f_1\left(x-\frac{\tau_1}{2}\right)\right)$$

which has a unique solution q_1 on (τ_1,π). Theorem is proved. □

4 CONCLUSION

Inverse spectral problems for classical Sturm-Liouville operators have been studied completely, while the inverse problems for differential operators with delays have not been studied enough. The main results for classical Sturm-Liouville operators is presented in [2] while some of the results for differential operators with delay can be found in ([3],[4],[5],[10],[11],[12],[13]). The class of operators with two delays has been least studied, but some of the results for this class of operators are presented in ([16], [17]). The motivation behind this paper is to initiate further research in the inverse spectral theory for differential operators with delays. We studied the inverse spectral problem of recovering operators from the spectra of $D_{i,k}$. To solved this inverse problem, we used the method of Fourier coefficients. This method is based on determination of direct relations between Fourier coefficients of the potentials or some functions containing the potentials, and Fourier coefficients of some known functions. We studied the spectral properties of the boundary value problems and proved that delays and parameters are uniquely determined from the spectra. Then we proved that potentials are uniquely determined by the system of two Volterra linear integral equations.

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