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FRAGMENTS OF DYNAMIC OF MÖEBIOUS MAPPINGS AND SOME APPLICATIONS. PART II

Ž. PAVIĆEVIĆ[1,2], J. ŠUŠIĆ[1] and M. MARKOVIĆ[*1]

[1]Faculty of Natural Sciences and Mathematics, University of Montenegro, Podgorica, Montenegro;

[1]National Research Nuclear University MEPhI

(Moscow Engineering Physics Institute), Moscow, Russia

*Corresponding author. E-mail: marijanmmarkovic@gmail.com

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Summary. Using dynamic and geometry of Möebious mappings we prove Lindelöf type theorems for much larger class of functions on the unit disk than previously considered class of meromor-phic functions.

1. INTRODUCTION

In classical theory of boundary behaviour of functions of one complex variable and in the theory of boundary sets the special important place is for the Lindelöf theorem and the Fatou theorem (we refer to [3, 12]) on radial and nontangential boundary values of holomorphicc functions. The first one concerns the local property of functions, i.e., it is about the existence of nontanegntial boundary value in a single point in the domain of a holomorphic function, the second one is about global boundary behaviour, i.e., it concerns the almost everywhere existence of radial boundary values of a holomorphic function. Nowadays there exist many proofs of these theorems but all of them use classical results of analytic theory of functions (see [3, 12, 13, 23]). Generalizations of Lindelöf theorems and Fatou theorems goes in many directions. One direction is for analytic functions by proving „stronger“ results, i.e., by proving the existence of nontangentail boundary values under weaker conditions then those in the Lindelof theorem (see [17-19]). The second direction is to consider similar theorems for broader class of functions: meromorphic functions, endomorphic mappings, holomorphic mappings of several complex variables, quasiconformal mappings in n, Rn≥ 2, harmonic functions and similar [22, 24, 25].

In this paper we prove how one can efficasely use the geometry or dynamic of Möebious mappings in order to derive the results on asymptotical behavior of holomorphic functions. Namely, we prove theorems that give necessary and sufficient conditions and criteria in order that a meromorphic function on the unit disk has tangential and nontangential boundary values.

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These theorems show that the conditions in the classical Lindelöf theorem and in the theorem of Lehto and Virtanen, Bagemihl and Seidel, Gavrilov and Burkova on angular boundary values of meromorphic functions may be relaxed. In the proofs of these theorems we use the Main Lemma 1 and the Main Lemma 2 in the Section 5 (see [18]). These results give the necessary and sufficient condition on a function defined on the unit disk in the complex plane, to has a boundary set consisted of one point, along the set which is obtain applying cyclic semi-group produced by an element in the hyperbolic or parabolic Moebius group on the unit disk. More on the topic on boundary asymptotic properties of functions one may found in [13, 17-19, 22-24].

2. PRELIMINARY NOTATIONS, DEFINITIONS AND RESULTS

By *D* we denote the open unit disk $\{z \mid |z|<1\}$ in the complex plane C, and with Γ we denote the boundary of *D*, and $D^+ = D \cap \{z \mid \operatorname{Im} z > 0\}$, $D^- = D \cap \{z \mid \operatorname{Im} z < 0\}$, and $D_r = \{z \mid |z| < r\}$, 0<r<1 is the disk with radius r. By P_θ i p_θ we denote the diameter and the radius of *D* with one endpoint in $e^{i\theta}$. Further, we denote by $d(z_1, z_2) = |z_1 - z_2|$, $z_1, z_2 \in \mathbb{C}$ the Euclidean distance on $\mathbb{C}$, $d_{ph}(z,w) = \left|\frac{z-w}{1-z\overline{w}}\right|$ and $d_h(z,w) = \frac{1}{2}\log\frac{1+d_{ph}(z,w)}{1-d_{ph}(z,w)}$, $z, w \in D$, stand for the pseudohyperbolic and hyperbolic distance between z and w in the dsik D, respecitvely, and

$$d_s(z,w) = \begin{cases} \dfrac{2|z-w|}{\sqrt{1+|z|}\cdot\sqrt{1+|w|}}, \ z, w \in C; \\ \dfrac{1}{\sqrt{1+|z|^2}}, \ z \in C, \ w = \infty \end{cases}$$

is the spherical distance on the Rimanian sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

It is well known that d_h is the metric in the Poencare model of the hyperbolic geometry on the disk *D* introduced by Lobachevsy.

All convergencies in this paper are with respect to the distances introduces above.

The set $D_{ph}(w,r') = \{z \mid z \in D, d_{ph}(z,w) < r'\}$, $w \in D$, $0 < r' < 1$, is the pseudohyperbolic disk, and $D_h(w,r) = \{z \mid z \in D, d_h(z,w) < r\}$, $w \in D$, $r > 0$, is the disk with respect to the hyperbolic distance.

Lemma 1. *We have* $D_h(w,r) = D_{ph}(w,r')$, *where* $r = \frac{1}{2}\ln\frac{1+r'}{1-r'}$ $\left(r' = \frac{e^{2r}-1}{e^{2r}+1} = thr\right)$.

The pseudohyperbolic disk $D_{ph}(w,r)$ is the Euclidean disk $D(c,R)=\{z\in D \,\|z-c|<R\}$ for $c=\dfrac{1-r^2}{1-r^2|w|^2}$, and $R=\dfrac{1-|w|^2}{1-r^2|w|^2}r.$

Therefore, the boundaries of hyperbolic and pseudohyperbolic disks are the ordinary cycles. The cycle which lies in *D* and with Γ has one common point is the *oricycle D*. The radius of *D* and arcs in *D* and in intersection with Γ have two points are *hypercycles* in *D*.

An arbitrary hypercycle will be denoted by H, an arbitrary oricycle will be denoted by O. We denote by $H^{\theta},\theta\in[0,\pi)$, the hypercycle which connects the points $-e^{i\theta}$ and $e^{i\theta}$; we denote by $O^{\theta},\ \theta\in[0,2\pi)$, the oricycle which is tangent to Γ in $e^{i\theta}$, and $O_0{}^{\theta}=\left\{\dfrac{u}{u-i}e^{i\theta}\,\middle|\,u\in(-\infty,\infty)\right\}$ is the oricycle $\left\{z\,\middle|\,\left|z-\dfrac{1}{2}e^{i\theta}\right|=\dfrac{1}{2}\right\}$.

We will also consider the family of all hypercycles with two common points in Γ.

The hyperbolic distance between a point z, $z\in D$, to the curve γ, $\gamma\subset D$, is $d_h(z,\gamma)=\inf\limits_{w\in\gamma}d_h(z,w)$.

For $\gamma=H^{\theta}$, one can prove that $d_h(z,H^{\theta})=\min\limits_{w\in H^{\theta}}d_h(z,w)$ and that $d_h(z,H^{\theta})$ does not depend on z if $z\in H$, where H is a hyper-cycle from the family of all hypercycles which is defined by the hypercycle H^{θ} (see [10]). Also one can prove (see [10]) that there exists unique point w_0 in H^{θ} such that

$$d_h(z,H^{\theta})=\min_{w\in H^{\theta}}d_h(z,w)=d_h(z,w_0). \tag{1}$$

From above, by “symmetric thinking”, it follows that for $w\in H^{\theta}$ there exists unique point z_0 in H such that

$$d_h(w,H)=\min_{z\in H}d_h(w,z)=d_h(w,z_0) \tag{2}$$

And this distance does not depend on $w\in H^{\theta}$.
From (1) and (2) it follows that for any $w\in H^{\theta}$ and $z\in H$ there exists unique points $w_0\in H^{\theta}$ and $z_0\in H$ such that

$$d_h(w,H)=d_h(z,H^{\theta})=d_h(w_0,z_0). \tag{3}$$

Having in mind all the preceding, the equality (3) define the hyperbolic distance between hypercycles H^θ *and H. Notation:* $d_h\left(H^\theta, H\right)$.

From the all given above we have:

Lemma 2 *(*see [10]*).* *The set of points in D such the hyperbolic distance between the hypercycle H is the hypercycle which belongs to the family of all hypercycles defined by the hypercycle H.*

From Lemma 2 we obtain:

Lemma 3 (see [8, 14]). *The set* $\Delta_H\left(\theta, r\right) = \bigcup_{a\in(-1,1)} D_h\left(ae^{i\theta}, r\right)$, $r \in (0, +\infty)$, *is a domain in the disk D bounded by two hyper-cycles* $H^\theta(r)$ *and* $H^\theta(-r)$ *such that their hyperbolic distance to the radius* P_θ *is equal to r, and which contains the points –th* $re^{i\left(\frac{\pi}{2}+\theta\right)}$ *and th* $re^{i\left(\frac{\pi}{2}+\theta\right)}$ *and contains the points* $-e^{i\theta}$ *and* $e^{i\theta}$ (see the Figure 1).

Lemma 4 (see [10])**.** *Let r be the hyperbolic distance of hyper-cycle* H^θ *from the diameter* P_θ *of the disk D. The angle* α *between* H^θ *and* P_θ *is equal to* $\alpha = \frac{\pi}{2} - arctg\ e^{\pm 2r\sqrt{\pi}}$.

If $h(\theta,\alpha_1)$ and $h(\theta,\alpha_2)$, $-\frac{\pi}{2} < \alpha_1 < \alpha_2 < \frac{\pi}{2}$, are arcs in *D* that with the radius p_θ of the disk *D* with endpoint in point $e^{i\theta}$ make angles α_1 and α_2, then the domain in *D* which is bounded by these arcs and by the circle $D_r = \left\{z \mid \left|z - e^{i\theta}\right| = r\right\}$ is the Stolz angle with vertex at $e^{i\theta}$. This domain is denoted by $\Delta\left(\theta, \alpha_1, \alpha_2\right)$. By $\Delta\left(\theta, \alpha\right)$ we denote the Stolz angle with boundary $h(\theta,\alpha)$ and $h(\theta,-\alpha)$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. We denote it by $\Delta\left(\theta, \alpha_1, \alpha_2\right)$. With $\Delta\left(\theta, \alpha\right)$ we denote the Stolz angle with boundary $h(\theta,\alpha)$ and $h(\theta,-\alpha)$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, i.e.,

$$\Delta\left(\theta,\alpha\right) = \left\{ z \,\middle|\, \mathrm{z} \in \mathrm{D},\ \left|\arg\left(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{z}\right)\right| < \alpha,\ 0 < \alpha < \frac{\pi}{2} \right\}.$$

Threfore, the Stolz angle is the domain which is an usualty geometic object (see Figure 1).

From Lemma 4 we obtain:

Lemma 5. *For every* α**,** $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, *there exist* r, $r \in (0, +\infty)$, r_1, $r_1 \in (0,1)$, *such that* $\left\{z \,\middle|\, \left|y - e^{i\theta}\right| < r_1\right\} \cap \Delta\left(\theta,\alpha\right) \subset \left\{z \,\middle|\, \left|y - e^{i\theta}\right| < r_1\right\} \cap \Delta_H\left(\theta, r\right)$. *For every* r, $r \in (0, +\infty)$, *there exists* α**,** $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, *such that* $\Delta_H\left(\theta, r\right) \subset \Delta\left(\theta, \alpha\right)$.

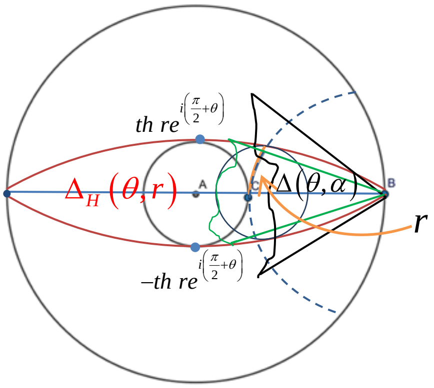

Figure 1

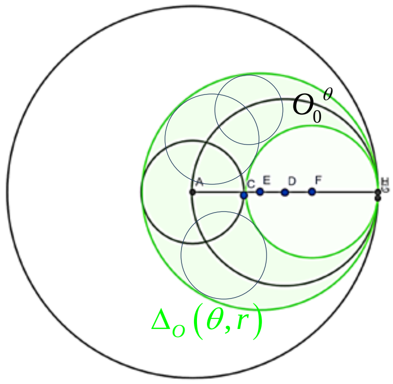

Figure 2

If $d_h(z,\gamma)=\inf\limits_{w\in\gamma} d_h(z,w)$ and if $\gamma=O^\theta$, then $d_h(z,O^\theta)=\min\limits_{w\in O^\theta} d_h(z,w)$ and $d_h(z,H^\theta)$ do not depend on $z\in O$, where O is the oricycle from the family of all oricycles generated by O^θ. One can also prove that (see[10]) there exists only one point w_0 in O^θ such that $d_h(z,O^\theta)=\min\limits_{w\in O^\theta} d_h(z,w)=d_h(z,w_0)$. Analogy one may define the distance between two ori-cycles from the same family of ori-cycles in the following way $d_h(H^\theta,H)=d_h(w,H)=d_h(z,H^\theta)$. It may be shown that there exist unique points $w_0\in O^\theta$ and $z_0\in O$ such that $d_h(H^\theta,H)=d_h(w,H)=d_h(z,H^\theta)=d_h(w_0,z_0)$.

Now, we have the following statements:

Lemma 6 (see [10])**.** *The set of points in D for which the hyperbolic distance is constant from the oricycle O is the orycycle in the family of all orycycles defined by the oricycle O.*

From lemma 6, we have:

Lemma 7 (see [14]). *The set* $\Delta_O(\theta,r)=\bigcup\limits_{u\in(-\infty,\infty)} D_h\left(\frac{u}{u+i}e^{i\theta},r\right)$, $r\in(0,+\infty)$, *is a domain in the disk D which is bounded by two ori-cycles* $O^\theta(-r)$ *and* $O^\theta(r)$ *such that the hyperbolic distance between them and the oricycle* $O_0^{\ \theta}=\left\{\frac{u}{u+i}e^{i\theta}\,\middle|\, u\in(-\infty,\infty)\right\}$ *is equal to r, and that pass throughout* $e^{i\theta}$, $-th\ re^{i\theta}$ *and* $th\ re^{i\theta}$ (see Figure 2).

3. FRAGMENTS OF THE GEOMETRY OF MÖBIOUS MAPPINGS

The Möbious group on the unit disk D is the group of all conformal automorphisms of the unit disc D, i.e., $G = G(D) = \left\{ e^{i\theta} \dfrac{z-a}{1-\overline{a}z} \mid a \in D,\ z \in \mathbb{C},\ \theta \in [0,2\pi) \right\}$.

The set

$$H_D^\theta = \left\{ g_a^\theta = g_a^\theta(z) = \frac{z + ae^{i\theta}}{1 + ae^{-i\theta}z} \mid a \in (-1,1) \right\},\ \theta \in [0,\pi) \text{ is fixed,}$$

stand for the hyperbolic subgroup of G with fixed points $e^{i\theta}$ *and* $-e^{i\theta}$,

$$P_D^\theta = \left\{ g_u^\theta = g_u^\theta(z) = \frac{(u+i)z - ue^{i\theta}}{-(u-i) + ue^{-i\theta}z} \mid u \in (-\infty,+\infty) \right\},\ \theta \in [0,2\pi) \text{ is fixed,}$$

is the parabolic subgroup of G with fixed point $e^{i\theta}$,

and finally

$$E_D^\theta = \left\{ g_\theta^{z_0} = g_\theta^{z_0}(z) = \frac{\left(1-|z_0|e^{i\theta}\right)z - z_0\left(1-e^{i\theta}\right)}{\overline{z_0}\left(1-e^{i\theta}\right)z + e^{-i\theta} - |z_0|^2} \mid \theta \in [0,2\pi) \right\},\ z_0 \in D \text{ is fixed,}$$

is the elliptic subgroup of G with fixed point z_0.

Since the hyperbolic distance is invariant with respect to $g \in G$ and from the definition of the groups H_D^θ and P_D^θ and sets P_D^θ, $\Delta_H(\theta,r)$, O_0^θ and $\Delta_O(\theta,r)$ we have the following statements:

Lemma 9. (i) $\Delta_H(\theta,r) = \bigcup_{g \in H_D^\theta} g\left(D_h(0,r)\right) = \bigcup_{g \in H_D^\theta} g\left(D_{ph}(0,thr)\right),\ r \in (0,+\infty)$

(ii) $\Delta_P(\theta,r) = \bigcup_{g \in P_D^\theta} g\left(D_h(0,r)\right) = \bigcup_{g \in P_D^\theta} g\left(D_{ph}(0,thr)\right),\ r \in (0,+\infty)$.

The set $A,\ A \subset D,$ is the stabilisator of the group H_D^θ if $g(A) = A$, for every $g \in H_D^\theta$.

Lemma 10. *For every* $g \in H_D^\theta$ *we have* $g(P_\theta) = P_\theta$, *i.e., the diameter* P_θ *is stabilisator of the group* H_D^θ.

Lemma 11. *For every* $g \in H_D^\theta$ *and* $r \in (0,+\infty)$ *we have* $g\left(\Delta_H(\theta,r)\right) = \Delta_H(\theta,r)$, *i.e., the set* $\Delta_H(\theta,r)$ *is also the stabilisator of the group* H_D^θ.

Lemma 12. *For every* $g \in P_D^\theta$ *we have* $g\left(O_0^\theta\right) = O_0^\theta$, *i.e., the ori-cycle* O_0^θ *is the stabilisator of the group* P_D^θ.

Lemma 13. *For every* $g \in P_D^\theta$ *and* $r \in (0,+\infty)$ *we have* $g(\Delta_O(\theta,r)) = \Delta_O(\theta,r)$, *i.e.,* $\Delta_O(\theta,r)$ *is the stabilisator of the group* P_D^θ.

For $g \in G(D)$ denote $g^n(z) = \underbrace{g\ (g(...(g}_{n \text{ puta}}(z)...)\ )$, $g^0(z) = i$, i is the identity and $g^{-n}(z) = (g^{-1})^n(z) = \underbrace{g^{-1}(g^{-1}(...(g^{-1}}_{n \text{ puta}}(z)...)\ )$, $n \in \mathbb{N}$.

Lemma 14 (see [2] on p. 73)**.**
(i) *Let* $g \in H_D^\theta$. *For fixed points* $e^{i\theta}$ *and* $-e^{i\theta}$ *there holds* $g^n(z) \underset{n\to\infty}{\to} e^{i\theta}$ *and* $g^{-n}(z) \underset{n\to\infty}{\to} -e^{i\theta}$, *where we mean uniform convergence on compacts sets of the disk D.*
(ii) *Let* $g \in P_D^\theta$. *Then for fixed point* $e^{i\theta}$ *we have* $g^n(z) \underset{n\to\infty}{\to} e^{i\theta}$, *where we also mean the uniform convergence of compact subsets of the disk D.*

Therefore, the point $e^{i\theta}$ is an attraction point for $g \in H_D^\theta$, and $-e^{i\theta}$ is repulsive point for g, i.e. it is an attraction point for g^{-1}. If $g \in P_D^\theta$ then the attraction point for $g \in P_D^\theta$.

For $g \in H_D^\theta$, and $\theta \in [0,\pi)$ fixed, $g \neq i$, denote $H_g^\theta = \{g^n | n \in \mathbb{Z}\}$. The set H_g^θ sa with composition operation is the cyclic subgroup of the group H_D^θ. If $g \in P_D^\theta$, then the set $P_g^\theta = \{g^n | n \in \mathbb{Z}\}$ with composition of functions is the cyclic subgroup of P_D^θ.

Let $\Delta_g(\theta,r) = \bigcup_{n\in\mathbb{Z}} g^n(D_h(0,r))$, $r \in (0,+\infty)$.

Further, from the property of invariance of the hyperbolic distance with respect to $g \in G$ we have:

Lemma15. $\Delta_g(\theta,r) = \bigcup_{n\in\mathbb{Z}} (D_h(g^n(0),r))$, $r \in (0,+\infty)$.

Lemma16. *Let* $g \in H_D^\theta$, $g \neq i$. *For every* $r \in (0,+\infty)$ *there exists* $r_1 \in (0,+\infty)$ *such that* $\Delta_H(\theta,r) \subset \Delta_g(\theta,r_1)$, *and* $\Delta_g(\theta,r) \subset \Delta_H(\theta,r)$ *for every* $r \in (0,+\infty)$.

Proof of lemma 16. Let $g \in H_D^\theta$ be arbitrary and let it be fixed and $g \neq i$. Let $z \in \Delta_H(\theta,r)$. There exists $a \in (0,+\infty)$ such that $z \in D_h(ae^{i\theta},r)$. Since $ae^{i\theta} \in P_\theta$ and $g^n(0) \in P_\theta$ for every $n \in \mathbb{Z}$, there exists $N \in \mathbb{Z}$ such that $ae^{i\theta}$ is between $g^N(0)$ and $g^{N+1}(0)$ or is equalt to one of that points. Let $0 < M = d_h(0,g(0)) = d_h(g^n(0),g^{n+1}(0))$, $n \in \mathbb{Z}$. Then we have

$$d_h\left(g^N(0),z\right)\le d_h\left(g^N(0),ae^{i\theta}\right)+d_h\left(ae^{i\theta},z\right)\le d_h\left(g^N(0),g^{N+1}(0)\right)+d_h\left(ae^{i\theta},z\right)<M+r$$

Therefore, for every $z\in\Delta_H(\theta,r)$ there exists $N\in\mathbb{Z}$ such that $d_h\left(g^N(0),z\right)<M+r$, where M and r are independent on z and N.

Since $\Delta_g(\theta,M+r)=\bigcup_{n\in\mathbb{Z}}\left(D_h\left(g^n(0),M+r\right)\right)$ (by Lemma 14) and $D_h\left(g^N(0),M+r\right)\subset\Delta_g(\theta,M+r)$, we obtain $z\in\Delta_g(\theta,M+r)$. If we take $r_1=M+r$, it follows $\Delta_H(\theta,r)\subset\Delta_g(\theta,r_1)$.

Now $\Delta_g(\theta,r)\subset\Delta_H(\theta,r)$, $r\in(0,+\infty)$ follows from Lemma 9 and Lemma 15. □

Lemma 17. *Let* $g\in P_D^\theta$, $g\ne i$. *For every* $r\in(0,+\infty)$ *there exists* $r_1\in(0,+\infty)$ *such that* $\Delta_P(\theta,r)\subset\Delta_g(\theta,r_1)$, *and* $\Delta_g(\theta,r)\subset\Delta_P(\theta,r)$ *for every* $r\in(0,+\infty)$.

Lemma 17 may be proved in a similar way as Lemma 16, instead of diameter P_θ one has to take the oricycle O_0^θ .

We will further consider the domains: $\tilde{\Delta}_H(\theta,r)=\bigcup_{a\in[0,1]}D_h\left(ae^{i\theta},r\right)$ and $\tilde{\tilde{\Delta}}_H(\theta,r)=\bigcup_{a\in[-1,0]}D_h\left(ae^{i\theta},r\right)$, $r\in(0,+\infty)$, we call them the hypercyclic domains in D and $\tilde{\Delta}_O(\theta,r)=\bigcup_{u\in(0,+\infty)}D_h\left(\frac{u}{u+i}e^{i\theta},r\right)$ and $\tilde{\tilde{\Delta}}_O(\theta,r)=\bigcup_{u\in(-\infty0,)}D_h\left(\frac{u}{u+i}e^{i\theta},r\right)$, $r\in(0,+\infty)$, which will be called the oricyclic domains in D.

Lemma 18. *Let* $g_a\in H_D^\theta$, $g_a\ne i$, *for which* $e^{i\theta}$ *is an attraction fixed point. Then for every* $r\in(0,+\infty)$ *there exists* $r_1\in(0,+\infty)$ *such that* $\bigcup_{n=0}^{\infty}g_a^n\left(D_h(0,r)\right)\subset\tilde{\Delta}_H(\theta,r)\subset\bigcup_{n=0}^{\infty}g_a^n\left(D_h(0,r_1)\right)$.

Lemma 19. *Let* $g_u\in P_D^\theta$, $g_u\ne i$, *for which* $e^{i\theta}$ *is fixed attraction point. Then for every* $r\in(0,+\infty)$ *there exists* $r_1\in(0,+\infty)$ *such that* $\bigcup_{n=0}^{\infty}g_u^n\left(D_h(0,r)\right)\subset\tilde{\Delta}_O(\theta,r)\subset\bigcup_{n=0}^{\infty}g_u^n\left(D_h(0,r_1)\right)$, *u>0, and* $\bigcup_{n=0}^{\infty}g_u^n\left(D_h(0,r)\right)\subset\tilde{\tilde{\Delta}}_O(\theta,r)\subset\bigcup_{n=0}^{\infty}g_u^n\left(D_h(0,r_1)\right)$, *u<0.*

Lemma 18 and lemma 19 may be proved in a similar way as Lemma 16.

4. CLASSICAL RESULTS FOR ASYMPTOTIC AND ANGULAR LIMIT VALUES OF ANALYTIC FUNCTIONS AT A POINT

For $A \subset D,\ \overline{\mathrm{A}} \cap \Gamma = \{e^{i\theta}\}$, we denote by $\overline{A}$ the closure of the set A, and $C(f, A, e^{i\theta}) = \left\{ \omega \;\middle|\; \omega \in \Omega,\ (\mathrm{z_n}) \subset A,\ \lim_{\mathrm{n}\to\infty} z_n = e^{i\theta},\ \lim_{n\to\infty} f(z_n) = \omega \right\}$ is the boundaty set of a function $f : D \to \Omega$ corresponding to the point $e^{i\theta}$ allong the set A. It is known that $C(f, A, e^{i\theta}) = \overline{C(f, A, e^{i\theta})}$.

The symbol $\varphi_n \rightrightarrows_K \varphi$ denotes the uniform convergence on the set $K \subset D$, of the sequence (φ_n) of functions $\varphi_n : D \to \overline{\mathbb{C}},\ n \in \mathbb{N}$, to $\varphi : D \to \overline{\mathbb{C}}$.

If $A = \Delta(e^{i\theta}, \alpha)$ is a Stolz angle in the disk D with the vertex at the point $e^{i\theta}$, then $C(f, \Delta(e^{i\theta}, \alpha), e^{i\theta})$ is the boundary set of the function f along the angle $\Delta(e^{i\theta}, \alpha)$. If for every $\alpha, 0 < \alpha < \frac{\pi}{2}$, $C(f, \Delta(e^{i\theta}, \alpha), e^{i\theta}) = \{\omega\}$, the $e^{i\theta}$ is the Fatou point of f, and $\omega \in \Omega$ is the unique nontangential boundary value.

We always denote by γ the simple Jordan curve in the disk D with endpoint in $e^{i\theta}$. If $C(f, \gamma, e^{i\theta}) = \{\omega\}$, $\omega \in \Omega$, then ω is an asympthotic boundary value of the function f in the point $e^{i\theta}$ along the curve γ.

We give now the classical assymptotic results and nontangential of analytic functions.

Theorem of Lindelöf (see [12, 23])**.** *If $f : D \to \mathbb{C}$ is a bounded analytic function. If $C(f, \gamma, e^{i\theta}) = \{\omega\}$, $\omega \in \mathbb{C}$, then $C(f, \Delta(e^{i\theta}, \alpha), e^{i\theta}) = \{\omega\}$, i.e., $e^{i\theta}$ is the Fatou point of function f.*

There are many proofs of the Lindelöf theorem. A proof based on maximum principle of analytic functions may be found in [23].

One generalization of the Lindelöf theorem is given by Lehto and Virtanen in [11]. The used results from normal function theory and results in harmonic function theory and harmonic measure.

For a family of functions $\mathfrak{I} = \left\{ f \;\middle|\; f : O \to \overline{\mathbb{C}} \right\}$ we say that it is *normal family* on a domain *O*, $O \subset \mathbb{C}$, if for every sequence (f_n) in that familty $\mathfrak{I}$ there exists a subsequence (f_{n_k}) which convege uniformly on compact subsets of *O* to a function $f: O \to \overline{\mathbb{C}}$. This is normality in the sense of $\mathfrak{I}$ of Montel. The family of functions $\mathfrak{I} = \left\{ f \middle| f: O \to \overline{\mathbb{C}} \right\}$ is normal in the point $z \in O$ if it is normal familiy in a neiborhoud of *z*.

It is well known that a family of functions $\mathfrak{I} = \left\{ f \middle| f: O \to \overline{\mathbb{C}} \right\}$ is normal family in the domain *O* if and only if it is normal in every point in the domain O (see [16, 20]).

If $O \subset \overline{\mathbb{C}}$, i.e. if $\infty \in O$, then the family of functions $\mathfrak{I} = \left\{ f \middle| f: O \to \overline{\mathbb{C}} \right\}$ is normal in the point ∞ if we have normality of the family $\mathfrak{I}' = \left\{ f\left(\frac{1}{z}\right) \mid f \in \mathfrak{I} \right\}$ in 0. The family of functions $\mathfrak{I} = \left\{ f \middle| f: O \to \overline{\mathbb{C}} \right\}$ is normal on *O* if it is normal in every point of the domain *O*. The theory of normal functions is well exposed in [16, 20].

If $f: D \to \mathbb{C}$ *is a bounded analytic maping, then the family* $\left\{ f \circ g \middle| g \in G \right\}$ *is normal family of functions on the disk D.*

Theorem of Lehto and Virtanen (see [11])**.** *Let* $f: D \to \overline{\mathbb{C}}$ *be a meromorphic function. If* $\left\{ f \circ g \middle| g \in G \right\}$ *is normal family of functions on the disk D and* $C\left(f, \gamma, e^{i\theta}\right) = \{\omega\}$, $\omega \in \mathbb{C}$, *then we have* $C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{\omega\}$ *, i.e.,* $e^{i\theta}$ *is the Fatou point of the function f.*

For the proof of the theroem Lehto and Virtanen used the results from harmonic function theory and harmonic measures (the theorem on two constants) and the propery of the normal meromorphic functions (see [3, 11]).

A meromorphic function $f: D \to \overline{\mathbb{C}}$ for which the family $\left\{ f \circ g \middle| g \in G \right\}$ is normal family of functions on D is the class of very well understood normal meromorphic functions N which contains the Bloch class of holomorphic functions denoted by B.

In the following theorems proved by Bagemihl and Seidel [1], it is proved the existence of angular boundary values under weaker asymptotical conditions then these in the preceding theorems. But these theorems are based on the theorems of Lehto and Virtanen.

Theorem of Bagemihl and Seidel 1 (see [1])**.** *Let* $f: D \to \overline{\mathbb{C}}$ *be a meromorphic function. If* $\left\{ f \circ g \middle| g \in G \right\}$ *is a normal family of functions on the disk D and if for every* $z \in D$ *we have* $f(z) \neq w,\ w \in \overline{\mathbb{C}}$, *and if there exists a sequence* (z_n), $z_n \in D$, $n \in \mathbb{N}$, *such that:*

$\lim_{n\to\infty} z_n = e^{i\theta}$, $d_h(z_n, z_{n+1}) < M$, $n \in \mathbb{N}$, i $\lim_{n\to\infty} f(z_n) = \omega$, $\omega \in \overline{\mathbb{C}}$, *then* $C(f, \Delta(e^{i\theta}, \alpha), e^{i\theta}) = \{\omega\}$ *for every* α, $0 < \alpha < \frac{\pi}{2}$, *i.e.,* $e^{i\theta}$ *is the Fatou point of the function f.*

Theorem of Bagemihl and Seidel 2 (see [1])**.** *Let* $f : D \to \overline{\mathbb{C}}$ *be a meromorphic function. If* $\{f \circ g \mid g \in G\}$ *is a normal family of functions on the disk D and if there exists a sequence* (z_n), $z_n \in D$, $n \in \mathbb{N}$, *such that* $\lim_{n\to\infty} z_n = e^{i\theta}$, $\lim_{n\to\infty} d_h(z_n, z_{n+1}) = 0$, i $\lim_{n\to\infty} f(z_n) = \omega$, $\omega \in \overline{\mathbb{C}}$, *then* $C(f, \Delta(e^{i\theta}, \alpha), e^{i\theta}) = \{\omega\}$ *for every* α, $0 < \alpha < \frac{\pi}{2}$, *i.e.,* $e^{i\theta}$ *is the Fatou point of function f.*

Bagemihl and **Seidel** [1] constructed an analytic functions in order to show that the condition concerning the hypervbolic distance $d_h(z_n, z_{n+1})$ in Theorem 5 and Theorem 6 is not possible to remove.

In the following theorem proved by Gavrilov and Burkova in [8], it is proved the existence of angular boundary values for the broader class of meromorphic functions then the class in the theorem of Lehto and Virtanen. In [Gavrilov and Burkova 11] it is given an example of meromorphic function for which $\{f \circ g_a^\theta \mid g_a^\theta \in H_D^\theta\}$ is normal on *D* but the family $\{f \circ g \mid g \in G\}$ is not normal on *D*.

A construction is based on the theorem which says that for a meromorphic function $f : D \to \overline{C}$ the family $\{f \circ g \mid g \in G\}$ is normal on the disku *D* if and only if the disk *D* does not contain the so called P-sequences for the function f, dok je $\{f \circ g_a^\theta \mid g_a^\theta \in H_D^\theta\}$ is normal family on the disk *D* if and only if in the domain $\Delta_g(\theta, r) \subset \Delta_H(\theta, r)$, $r \in (0, +\infty)$, does not exist the P-sequences for the function f.

A sequence (z_n), $z_n \in D$, $\lim_{n\to\infty} |z_n| = 1$, is a *P-sequence* for a function $f : D \to \overline{C}$ if for every subsequence $(z_{n_k})_{k\in N}$ and for every ε, $0 < \varepsilon < 1$, the function f on $\bigcup_{k\in N} D_h(z_{n_k}, \varepsilon)$, takes inifinity many times all velues in $\overline{C}$, except possibly at most two (see definition, Gavrilov[6]).

In the sequel we will need the following theorems concerning the *P*-sequences:

Theorem on P-sequences 1 (see [6, Lemma 1])**.** *Let* (z_n) *be a P-sequence for a meromorphic function* $f: D \to \overline{C}$. *If for a sequence* $(z'_n) \subset D$ *we have* $\lim_{n\to\infty} d_h\left(z_n, z_n{}'\right) = 0$, *then the sequence* (z'_n) *is P-sequence for* f.

Theorem on P-sequences 2 (see [26])**.** *Let* $f: D \to \overline{C}$ *be a meromorphic fuction on D and let* $(z_n) \subset D$ *a sequence such that* $\lim_{n\to\infty}|z_n| = 1$ *and* $\lim_{n\to\infty} f(z_n) = c$ *for some a* $c \in \overline{\mathbb{C}}$. *Further, let* $\left(z_n{}'\right) \subset D$ *be a sequence such that* $\lim_{n\to\infty}\left|z_n{}'\right| = 1, \lim_{n\to\infty} d_h\left(z_n, z_n{}'\right) = 0$, *and* $(f(z_n))$ *does not sonverge to a c as* $n \to \infty$. *Then* (z_n) *and* $\left(z_n{}'\right)$ *are both P-sequences of the fuction f.*

Theorem of Gavrilov and Burkova (see [8])**.** *Let* $f: D \to \overline{\mathbb{C}}$ *be a meromorphic function. If* $\left\{ f \circ g_a^\theta \mid g_a^\theta \in H_D^\theta \right\}$ *is a normal family on the disk D and* $C\left(f, \gamma, e^{i\theta}\right) = \{\omega\}$, $\omega \in \mathbb{C}$, *then* $C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{\omega\}$ *for every* $\alpha, 0 < \alpha < \frac{\pi}{2}$, *i.e.,* $e^{i\theta}$ *is the Fatou point of the function f.*

A proof of theorem of Gavrilov and Burkova goes in the same way as the, by using the result from harmonic function theory as well as using properties of harmonic measure, as in the proof of theorem Lehto – Virtanen.

In [1] are given theorems that are analigies to the theorems of Bagemil and Seidel for meromorphic functions on *D* i.e., functions for which $\left\{ f \circ g_a^\theta \mid g_a^\theta \in H_D^\theta \right\}$ is normal on the disk *D*.

5. MAIN RESULT

The main lemma 1. *For any function* $f: D \to \overline{\mathbb{C}}$, *any compact set* K, $K \subset D$, *and any mapping* $g_a \in H_D^\theta$, $g_a \neq i$, *the following conditions are equivalent:*

i) $f \circ (g_a)^n \rightrightarrows_K c$;

ii) $C\left(f, \bigcup_{n=0}^{\infty} g_a^n(K), e^{i\theta}\right) = \{c\}$.

Proof of main lemma 1. Let $c \in \mathbb{C}$.

i)⇒ ii). From i) we have

$$(\forall \varepsilon > 0)(\exists N_1 = N_1(\varepsilon))(\forall n \geq N_1)(\forall z \in K)\left(\left| f \circ g_a^n(z) - c \right| < \varepsilon\right) \tag{4}$$

i.e., $f\left(\bigcup_{n=N_1}^{\infty} g_a^n(K)\right) \subset \{w \in \mathbb{C} \mid |w-c| < \varepsilon\}.$

From lemma 8 we have

$$(\forall \delta > 0)(\exists N_2 = N_2(\delta))(\forall n \geq N_2)(\forall z \in K)\left(\left|g_a^n(z) - e^{i\theta}\right| < \delta\right), \qquad (5)$$

i.e.,
$$\left(\forall z \in \bigcup_{n=N_1}^{\infty} g_a^n(K)\right)\left|z - e^{i\theta}\right| < \delta. \qquad (5')$$

Let (z_n) be any sequence in $\bigcup_{n=1}^{\infty} g_a^n(K)$ for which $\lim_{n\to\infty} z_n = e^{i\theta}$. From (5), i.e., from (5′) we obtain

$$(\exists N_3 = N_3((z_n), N_2))(\forall n \geq N_3)\left(z_n \in \bigcup_{k=N_2}^{\infty} g_a^k(K)\right). \qquad (6)$$

If $N_2 \leq N_1$, then we have $\bigcup_{n=N_2}^{\infty} g_a^n(K) \subset \bigcup_{n=N_1}^{\infty} g_a^n(K)$, from this and from (6) it follows that $z_n \in \bigcup_{n=N_1}^{\infty} g_a^n(K)$ for every $n \geq N_3$. Having in mind now (4) it follows that $(\forall n \geq N_3)(|f(z_n) - c| < \varepsilon)$, which means that $\lim_{n\to\infty} f(z_n) = c.$

If $N_1 \leq N_2$, then from the sequence (z_n), except $z_1, \ldots, z_{N_3}$, remove those that are in the set $\bigcup_{k=1}^{N_1} g_a^k(K)$, there are only finite many of them. Therefore, there exists N_4 such that $z_n \in \bigcup_{n=N_1}^{\infty} g_a^n(K)$ for every $n \geq N_4$. Now, according to (4) we obtain $(\forall n \geq N_3)(|f(z_n) - c| < \varepsilon)$, i.e., in this case we also have $\lim_{n\to\infty} f(z_n) = c.$

Therefore, for every sequence (z_n) in $\bigcup_{n=0}^{\infty} g_a^n(K)$ we have $\lim_{n\to\infty} f(z_n) = c$, do we may conclude that $C\left(f, \bigcup_{n=0}^{\infty} g_a^n(K), e^{i\theta}\right) = \{c\}.$

ii)⇒ i). From ii) we have $\forall (z_n) \subset \bigcup_{n=0}^{\infty} g_a^n(K) \wedge \lim_{n\to\infty} z_n = e^{i\theta} \Rightarrow \lim_{n\to\infty} f(z_n) = c$ i.e.,

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon))\left(\forall z \in \bigcup_{n=0}^{\infty} g_a^n(K)\right)\left(\left|z - e^{i\theta}\right| < \delta \Rightarrow |f(z) - c| < \varepsilon\right) \qquad (7)$$

Since $g_a^n \rightrightarrows_K e^{i\theta}$, for $\delta > 0$ the exists $N = N(\delta)$ such that for any $n \geq N$ and every $z \in K$ holds $\left|g_a^n(z) - e^{i\theta}\right| < \delta$, i.e.,

$$\bigcup_{n=N}^{\infty} g_a^n(K) \subset \left\{ z \in D \middle| \; \left| z - e^{i\theta} \right| < \delta \right\}. \qquad (8)$$

From (7) and (8) it follows that $f\left(\bigcup_{n=N_1}^{\infty} g_a^n(K)\right) \subset \left\{ w \in \mathbb{C} \middle| \; |w-c| < \varepsilon \right\}$, and therefore $(\forall n \geq N)(\forall z \in K)\left(\left| f\left(g_a^n(z)\right) - c\right| < \varepsilon\right)$, i.e., $f\circ(g_a)^n \rightrightarrows_K c$.

If $c = \infty \in \overline{\mathbb{C}}$, the proof goes in the same way as in the case $c \in \mathbb{C}$ instead of the Eucilean metric we have to take the spherical distance. $\square$

Main lemma 2. *For any function $f : D \to \overline{\mathbb{C}}$, and a compact set K, $K \subset D$, and any mapping $g_u \in P_D^\theta$, $g_u \neq i$, the following conditions are equivalent:*

i) $f\circ(g_u)^n \rightrightarrows_K c$, $c \in \mathbb{C}$.

ii) $C\left(f, \bigcup_{n=0}^{\infty} g_u^n(K), e^{i\theta}\right) = \{c\}$.

Main Lemma 2 may be proved in the same way as Main Lemma 1.

6. APPLICATIONS

For $g_a \in H_D^\theta$, $g_a \neq i$, for which $e^{i\theta}$ is an attraction fixed point $H_{g_a}^\theta = \left\{ g_a^n \middle| \; n \in \mathbb{N} \cup \{0\} \right\}$, $a \in (-1,1)$ is fixed, is the hyperbolic semigroup of G with fixed attraction point $e^{i\theta}$, and $P_{g_u}^\theta = \left\{ g_u^n \middle| \; n \in \mathbb{N} \cup \{0\} \right\}$, $u \in (-\infty, +\infty)$ is fixed, is the parabolic semigroup of G with attraction fixed point $e^{i\theta}$.

6.1. Angular boundary values of meromorphic functions

Theorem 1. *Let $f : D \to \overline{\mathbb{C}}$ be a meromorphic function. If*

$$\left\{ f \circ g \middle| \; g \in H_{g_a}^\theta \right\} = \left\{ f \circ g_a^n \middle| \; n \in \mathbb{N} \cup \{0\} \right\}, \; a \in (-1,1)$$

(a is fixed), is normal family of functions on the disk D, γ is simple Jordan curve with one endpoint in $e^{i\theta}$ and $\gamma \subset \tilde{\Delta}_H(\theta, r)$ and $C\left(f, \gamma, e^{i\theta}\right) = \{c\}$, $c \in \overline{\mathbb{C}}$, then

$$C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{c\} \text{ for every } \alpha, 0 < \alpha < \frac{\pi}{2},$$

i.e., $e^{i\theta}$ is the Fatou point of the function f.

Proof of Theorem 1. Since $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$, $a \in (-1,1)$, a is fixed, is a normal family of functions on the disk D, there exists a sequence $\left(f \circ g_a^{n_k} \right)$ which uniformly on compact sete convegre to a meromorphic function φ on $\overline{D_{r_1}}$. .i.e., $f \circ (g_a)^{n_k} \rightrightarrows_{\overline{D_{r_1}}} \varphi$.

Since $\gamma \subset \tilde{\Delta}_H(\theta, r)$, the sets $\gamma \cap g_a^{n_k}\left(D_{r_1} \setminus D_r \right)$, $0 < r < r_1 < 1$, $n \in \mathbb{N}$, are made of two simple curves. By γ_k we denote one of them Then we have $\gamma_k \cap \gamma_{k+1} = \varnothing$, and $\left[\Gamma_k = g_a^{-n_k}(\gamma_k) \right] \cap \left[\Gamma_{k+1} = g_a^{-n_{k+1}}(\gamma_{k+1}) \right] = \varnothing$, $n \in \mathbb{N}$, since the Moebius transforms g_a^n are bijections

For every $m \in \mathrm{N}$ let us select a sequence $\left(z_k^m \right)$, $z_k^m \in \Gamma_k$, such that $\lim_{k \to \infty} z_k^m = z_0^m \in \overline{D_{ph}}(0, r_1)$ and $z_0^i \neq z_0^j$ for $i \neq j$. We will show that $\varphi\left(z_0^m \right) = c$, $c \in \overline{C}$, for every $m \in \mathrm{N}$.

For every $m \in \mathrm{N}$ there holds

$$d_S\left(\varphi\left(z_0^m \right), c \right) \leq d_S\left(\varphi\left(z_0^m \right), \varphi\left(z_k^m \right) \right) + d_S\left(\varphi\left(z_k^m \right), f_{n_k}\left(z_k^m \right) \right) + d_S\left(f_{n_k}\left(z_k^m \right), c \right). \quad (9)$$

Let ε be any positive real number. Since of coninuily of φ we have $d_S\left(\varphi\left(z_0^m \right), \varphi\left(z_k^m \right) \right) < \frac{\varepsilon}{3}$, if k is enough big.

Since the sequence $\left(f_{n_k} \right)$ converge uniformly on compact sets of the disk D to φ, we have $d_S\left(\varphi(z), f_{n_k}(z) \right) < \frac{\varepsilon}{3}$ for every $z \in \overline{D_{r_1}}$ and enough big k.

Since $z_k^m \in \overline{D_r}$ we have $d_S\left(\varphi\left(z_k^m \right), f_{n_k}\left(z_k^m \right) \right) < \frac{\varepsilon}{3}$. Since $z_k^m \in \Gamma_k$ it follows that $\varphi_{n_k}\left(z_k^m \right) = w_k^m \in \gamma_k \subset \gamma$ and $\lim_{k \to \infty} w_k^m = e^{i\theta}$.

Since c is the asymptotic value of f snd since $\lim_{k \to \infty} f_{n_k}\left(w_k^m \right) = \lim_{k \to \infty} f \circ \varphi_{n_k}\left(z_k^m \right) = \lim_{k \to \infty} f_{n_k}\left(z_k^m \right) = c$, for enough big k we have $d_S\left(f_{n_k}\left(z_k^m \right), c \right) < \frac{\varepsilon}{3}$ for every $m \in \mathrm{N}$.

From (9) and obtained inequality it follows that $d_S\left(\varphi\left(z_0^m \right), c \right) < \varepsilon$ for every m. Since ε is any number, we have $\varphi\left(z_0^m \right) = c$ for every m.

Since the sequence $\left(z_0^m \right)$ is in $\overline{D_{r_1}}$ and i $\overline{D_{r_1}}$ has an accumulation point, from the uniqness theorem we have $\varphi \equiv c$.

Therefore, we have proved that any sequence in the family $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$ which is uniformly convergent on compact sets in *D*, is convergent to the constant c.

Now we will show that any sequence in the family $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$ converge uniformly on compact sets of D the the constant c. Assume conctrary, that thereexist a sequence postoji (f_n), $f_n \in \left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$, which uniformly on compacts does not converge to the constant c. Then there exists a number $\varepsilon > 0$ such that for every $k \in \mathrm{N}$ we have $n_k \in \mathrm{N}$ and $z_{n_k} \in \overline{D_r}$ such that $d_S\left(f_{n_k}\left(z_{n_k} \right), c \right) \geq \varepsilon$. Since tha family $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$ is normal, f_{n_k} has a subsequence $f_{n_{kl}}$ which uniformly on compact sets of D converge, according to the preceding consideration it follows that it converge to the constant c, which is contrary with the assumption $d_S\left(f_{n_k}\left(z_k^m \right), c \right) \geq \varepsilon$. This contradiction shows that every sequence in $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$ uniformly on compact sets of D converge to the constant c. Having in mind the Lemma 1 it follows that $C\left(f, \bigcup_{n=0}^{\infty} g_a^n(K), e^{i\theta} \right) = \{c\}$, for every compact set K, $K \subset D$, and form Lemma 5 and Lemma 18 we have that $C\left(f, \Delta(\theta, \alpha), e^{i\theta} \right) = \{c\}$ for every α, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, i.e., the function f in the point $e^{i\theta}$ has angular boundary value c. $\square$

In [27] it is proved that $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$, $a \in (-1,1)$ is normal family on D if and only if in the domain $\tilde{\Delta}_H(\theta, r)$, $r \in (0, +\infty)$, does not exist P-sequences for *f*.

Theorem 2. *Let* $f : D \to \overline{\mathbb{C}}$ *be a meromorphic function. If* $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$, $a \in (-1,1)$, *where a is fixed, normal family of functions on the disk D and if for a sequence* $(z_n) \subset \tilde{\Delta}_H(\theta, r)$ *holds* $\lim_{n\to\infty} z_n = e^{i\theta}$ *and* $\lim_{n\to\infty} f(z_n) = c$, $c \in \overline{\mathbb{C}}$, *then for any sequence* (r_n) *for which* $\lim_{n\to\infty} r_n = 0$ *and* $\bigcup_{n=1}^{\infty} D_h(z_n, r_n) \subset \tilde{\Delta}_H(\theta, r_1)$ *for* $r_1 > 0$, $C\left(f, \bigcup_{n=1}^{\infty} D_h(z_n, r_n), e^{i\theta} \right) = \{c\}$.

Theorem 2 follows directly from theorem on P-sequences which is formulated in the Section 4 and the criteria for normality formulated above of the familiy of functions $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$, $a \in (-1,1)$ on D.

Theorem 3. *Let* $f : D \to \overline{\mathbb{C}}$ *be meromorphic function. If* $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$, $a \in (-1,1)$ *where a is fixed, is normal family of functions on the disk D and if for a sequence* $(z_n) \subset \tilde{\Delta}_H(\theta, r)$ *holds:* $\lim_{n\to\infty} z_n = e^{i\theta}$, $\lim_{n\to\infty} d_h(z_n, z_{n+1}) = 0$ *and* $\lim_{n\to\infty} f(z_n) = c$, $c \in \overline{\mathbb{C}}$, *then* $C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta} \right) = \{c\}$ *for every* $\alpha, 0 < \alpha < \frac{\pi}{2}$, *i.e.,* $e^{i\theta}$ *is the Fatou point of the function f.*

Proof of Theorem 3. Let $x_n = 2d_h(z_n, z_{n+1})$. Then we have $\lim_{n\to\infty} x_n = 0$. If $D_h(z_n, x_n) = \{z \mid d_h(z, z_n) < x_n\}$, then form Theorem 2 follows that $C\left(f, \bigcup_{n=1}^{\infty} D_h(z_n, x_n), e^{i\theta}\right) = \{c\}\ c \in \overline{\mathbb{C}}$. Since the curve (poligonal line) $\gamma = \overline{z_1 z_2 ... z_n ... e^{i\theta}} \subset \bigcup_{n=1}^{\infty} D_h(z_n, x_n)$, we have $C(f, \gamma, e^{i\theta}) = \{c\}$. Since it is possible to chose r>0, such that $\bigcup_{n=1}^{\infty} D_h(z_n, x_n) \subset \tilde{\Delta}_H(\theta, r)$, from Theorem 1 we conclude $C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{c\}$ for every $\alpha, 0 < \alpha < \frac{\pi}{2}$, i.e., $e^{i\theta}$ is the Fatou point of *f*. $\square$

Theorem 4. *Let $f : D \to \overline{\mathbb{C}}$ be meromorphic function such that $f(z) \neq c,\ c \in \overline{\mathbb{C}},\ z \in D$. If $\left\{ f \circ g_a^n \mid n \in \mathbb{N} \cup \{0\} \right\}$, $a \in (-1,1)$, a is fixed, normal family f functions on th disk D and if holds: $\lim_{n\to\infty} z_n = e^{i\theta}$ and $\lim_{n\to\infty} f\left(g_a^n(0)\right) = c,\ c \in \overline{\mathbb{C}}$, then we have $C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{c\}$ for every $\alpha, 0 < \alpha < \frac{\pi}{2}$, i.e., $e^{i\theta}$ is the Fatou point of f.*

Proof of Theorem 4. Form normality of meromorphic functions $\left\{ f \circ g_a^n \mid n \in \mathbb{N} \cup \{0\} \right\}$ and the condition $\lim_{n\to\infty} f\left(g_a^n(0)\right) = c,\ c \in \overline{\mathbb{C}}$, from Hurwitz theorem (see [20]) $f \circ g_u^n \rightrightarrows_K c$ for every compact set $K \subset D$. If we take $K = \overline{D_h(0, r)}$, 0<r<1, then from the Main Lemma 1, Lemma 5 and Lemma 18 we have $C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{c\}$ for every $\alpha, 0 < \alpha < \frac{\pi}{2}$, i.e., $e^{i\theta}$ is the Fatou point of f. $\square$

6.2. Tangentialy oricyclic boundary values of meromorphic functions

If for every $r \in (0, +\infty)$ holds $C\left(f, \tilde{\Delta}_o(\theta, r), e^{i\theta}\right) = \{\omega\}$, $\omega \in \overline{\mathbb{C}}$, then we will call ω the upper oricyclic boundary value of f in the point $e^{i\theta}$. On the other hand, if for every $r \in (0, +\infty)$ holds $C\left(f, \approx{\Delta}_o(\theta, r), e^{i\theta}\right) = \{\omega\}$, $\omega \in \overline{\mathbb{C}}$, $\omega \in \overline{\mathbb{C}}$, then we will call ω the lower o oricyclic boundary value for f in the point $e^{i\theta}$. If $C\left(f, \tilde{\Delta}_o(\theta, r) \cup \approx{\Delta}_o(\theta, r), e^{i\theta}\right) = \{\omega\}$ then we call se ω the oricyclic boundary value of f in the point $e^{i\theta}$.

Theorem 5. *Let $f : D \to \overline{\mathbb{C}}$ be meromorphic function. If*

$$\left\{ f\circ g \,\middle|\, g\in P^{\theta}_{g_u} \right\} = \left\{ f\circ g^n_u \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(0,\infty),$$

u is fixed, normal family of functions on the disk D and if for a sequence $(z_n)\subset\tilde{\Delta}_O(\theta,r)$ holds $\lim\limits_{n\to\infty} z_n = e^{i\theta}$ and $\lim\limits_{n\to\infty} f(z_n)=c,\ c\in\overline{\mathbb{C}}$, then for every sequence (r_n) for which $\lim\limits_{n\to\infty} r_n = 0$ and $\bigcup\limits_{n=1}^{\infty} D_h(z_n,r_n)\subset\tilde{\Delta}_O(\theta,r_1)$ for a $r_1>0$, $C\left(f,\bigcup\limits_{n=1}^{\infty} D_h(z_n,r_n),e^{i\theta}\right)=\{c\}$.

Theorem 5′ *Let $f:D\to\overline{\mathbb{C}}$ be a meromorphic function. If*

$$\left\{ f\circ g \,\middle|\, g\in P^{\theta}_{g_u} \right\} = \left\{ f\circ g^n_u \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(-\infty,0)\ ,$$

where u is fixed, normal family of functions on the disk D and if for a sequence $(z_n)\subset\tilde{\tilde{\Delta}}_o(\theta,r)$ holds: $\lim\limits_{n\to\infty} z_n = e^{i\theta}$ and $\lim\limits_{n\to\infty} f(z_n)=c,\ c\in\overline{\mathbb{C}}$, then for every (r_n) for which $\lim\limits_{n\to\infty} r_n = 0$ and $\bigcup\limits_{n=1}^{\infty} D_h(z_n,r_n)\subset\tilde{\tilde{\Delta}}_o(\theta,r)$ for $r_1>0$, $C\left(f,\bigcup\limits_{n=1}^{\infty} D_h(z_n,r_n),e^{i\theta}\right)=\{c\}$.

U [27] it is proved that $\left\{ f\circ g^n_u \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(-\infty,0)$ is normal family on the disk D if and only if in the domain $\tilde{\tilde{\Delta}}_o(\theta,r)\ \tilde{\Delta}_O(\theta,r),\ r\in(0,+\infty)$, does not exist P-sequences for the function *f*. On the other hand, the family $\left\{ f\circ g^n_u \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(0,+\infty)$ is normal on the disk D if and only if in the domain $\tilde{\Delta}_O(\theta,r),\ r\in(0,+\infty)$, does not exist *P*-sequences for the function f.

Theorem 5 and Thorem 5′ follows directly from theorem on *P*-sequences 1 which is formulated in Section 4 and the above formulated criterion for normality of the family of functions $\left\{ f\circ g^n_u \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(-\infty,0)$ $\left\{ f\circ g \,\middle|\, g\in P^{\theta}_{g_u} \right\} = \left\{ f\circ g^n_u \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(-\infty,0)$.

Theorem 6. *Let $f:D\to\overline{\mathbb{C}}$ be a meromorphic function. If*

$$\left\{ f\circ g \,\middle|\, g\in P^{\theta}_{g_u} \right\} = \left\{ f\circ g^n_u \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(0,\infty),\ u \text{ is fixed,}$$

is normal family on the disk D, γ a simple Jordan curve with one endpoint in $e^{i\theta}$ and $\gamma\subset\tilde{\Delta}_O(\theta,r)$ and $C\left(f,\gamma,e^{i\theta}\right)=\{\omega\}$, $\omega\in\mathbb{C}$, then $C\left(f,\tilde{\Delta}_O(\theta,r),e^{i\theta}\right)=\{\omega\}$ for every $r\in(0,+\infty)$, i.e., ω is the upper oricyclic boundary value for the function f in the point $e^{i\theta}$.

Theorem 6′. *Let* $f : D \to \overline{\mathbb{C}}$ *be a meromorphic function. If*

$$\left\{ f \circ g \,\middle|\, g \in P_{g_u}^{\theta} \right\} = \left\{ f \circ g_u^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\},\ u \in (-\infty, 0)\ ,\ u \text{ is fixed,}$$

is normal family of functions on the disku D, γ *is simple Jordan curve with one endpoint in* $e^{i\theta}$ *and* $\gamma \subset \tilde{\tilde{\Delta}}_o(\theta, r)$ *and* $C\left(f, \gamma, e^{i\theta}\right) = \{\omega\}$, $\omega \in \mathbb{C}$, *then* $C\left(f, \tilde{\tilde{\Delta}}_o(\theta, r), e^{i\theta}\right) = \{\omega\}$ *for every* $r \in (0, +\infty)$, *i.e.,* ω *is the lower oricyclic boundary value of the function f in the point* $e^{i\theta}$.

Theorem 6 and Theorem 6′ may be proved using the Main Lemma 2 and Lemma 19 in the same way as Theorema 1 is derived from the Main Lemma 1 and Lemma 18.

Theorem 7 and Theorem 7′ may be proved using Theorem 6 and Theorem 6′ in the same way as Theorem 3 using Theorem 1.

Theorem 7. *Let* $f : D \to \overline{\mathbb{C}}$ *be a meromorphic function. If*

$$\left\{ f \circ g \,\middle|\, g \in P_{g_u}^{\theta} \right\} = \left\{ f \circ g_u^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\},\ u \in (0, +\infty)\ ,\ u \text{ is fixed,}$$

is a normal family of functions on the disk D and if for a sequence $(z_n) \subset \tilde{\Delta}_O(\theta, r)$ *holds:* $\lim\limits_{n\to\infty} z_n = e^{i\theta}$, $\lim\limits_{n\to\infty} d_h(z_n, z_{n+1}) = 0$ *and* $\lim\limits_{n\to\infty} f(z_n) = c$, $c \in \overline{\mathbb{C}}$, *then* $C\left(f, \tilde{\Delta}_O(\theta, r), e^{i\theta}\right) = \{\omega\}$ *for every* $r \in (0, +\infty)$, *i.e.,* ω *is the upper oricyclic boundary value of function f in the point* $e^{i\theta}$.

Theorem 7′. *Let* $f : D \to \overline{\mathbb{C}}$ *be a meromorphic function. If*

$$\left\{ f \circ g \,\middle|\, g \in P_{g_u}^{\theta} \right\} = \left\{ f \circ g_u^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\},\ u \in (-\infty, 0)\ ,\ u \text{ is fixed,}$$

is normal family of functions on the disk D and if for a sequence $(z_n) \subset \tilde{\tilde{\Delta}}_o(\theta, r)$ *holds:* $\lim\limits_{n\to\infty} z_n = e^{i\theta}$, $\lim\limits_{n\to\infty} d_h(z_n, z_{n+1}) = 0$ *and* $\lim\limits_{n\to\infty} f(z_n) = c$, $c \in \overline{\mathbb{C}}$, *then* $C\left(f, \tilde{\tilde{\Delta}}_o(\theta, r), e^{i\theta}\right) = \{\omega\}$ *for every* $r \in (0, +\infty)$, *i.e.,* ω *is the lower oricyclic boundary value of the function f in the point* $e^{i\theta}$.

Theorem 8. *Let* $f : D \to \overline{\mathbb{C}}$ *be a meromorphic function such that* $f(z) \neq c$, $c \in \overline{\mathbb{C}}$, $z \in D$. $\left\{ f \circ g \,\middle|\, g \in P_{g_u}^{\theta} \right\} = \left\{ f \circ g_u^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$, $u \in (0, +\infty)$ *, u is fixed, normal family of functions on D and if* $\lim\limits_{n\to\infty} z_n = e^{i\theta}$ *and* $\lim\limits_{n\to\infty} f\left(g_u^n(0)\right) = c$, $c \in \overline{\mathbb{C}}$, *then we have* $C\left(f, \tilde{\Delta}_O(\theta, r), e^{i\theta}\right) = \{\omega\}$ *for every* $r \in (0, +\infty)$, *i.e.,* ω *is the upper oricyclic boundary value of the function f in the point* $e^{i\theta}$.

Theorem 8′. *Let* $f: D \to \overline{\mathbb{C}}$ *be a meromorphic function such that* $f(z) \neq c,\ c \in \overline{\mathbb{C}},\ z \in D$. $\left\{ f \circ g \,\middle|\, g \in P^{\theta}_{g_u} \right\} = \left\{ f \circ g^n_u \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\},\ u \in (-\infty, 0)$ *, u is fixed, normal family of functions on the disk D and if* $\lim\limits_{n\to\infty} z_n = e^{i\theta}$ *and* $\lim\limits_{n\to\infty} f\left(g^n_u(0)\right) = c,\ c \in \overline{\mathbb{C}}$, *then* $C\left(f, \tilde{\tilde{\Delta}}_O(\theta, r), e^{i\theta} \right) = \{\omega\}$ *for every* $r \in (-\infty, 0)$, *i.e..* ω *is the lower oricyclic boundary value of the function f in* $e^{i\theta}$.

Theorem 8 and Theorem 8′ may be proved using the Main Lemma 2 and Lemma 19 in the same way as Theorem 4 is derived from the Main Lemma 1, Lemma 5 and Lemma 18.

7. CONSTRUCTION OF ONE EXAMPLE

We will construct an example of meromorphic function $f: D \to \overline{\mathbb{C}}$ for which $\left\{ f \circ g^n_a \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\},\ a \in (-1, 1)$, a is fixed, is normal family of functions on the disk *D*, and $\left\{ f \circ g^{\theta}_a \,\middle|\, g^{\theta}_a \in H^{\theta}_D \right\}$ is not normal family of functions on *D*. This construction is similar as one in the work [8].

Let $z_k = -\rho_k e^{i\theta},\ \rho_k > 0,\ k \in \mathbb{N},$ be such that $\lim\limits_{k\to\infty} \rho_k = 1$ and $\lim\limits_{k\to\infty} d(z_k, z_{k+1}) = 0.$ The elements of the sequence (z_k) are in the set $P_\theta \cap \Delta_{g^{-1}}(-\theta, r)$. Let a sequence (ε_k) be a such one that we have:

$$0 < \varepsilon_{k+1} < \varepsilon_k; \quad \lim_{k\to\infty} \varepsilon_k = 0; \quad D(z_k, \varepsilon_k) \cap D(z_{k+1}, \varepsilon_{k+1}) = \varnothing,\ k \in \mathbb{N}; \quad \lim_{k\to\infty} \left(\sup_{z \in D(z_k, \varepsilon_k)} d(z_k, z_{k+1}) \right) = 0;$$

$$\sum_{k=1}^{\infty} \varepsilon_k < +\infty .$$

Let $a_k = \varepsilon_k^3,\ k \in \mathbb{N}$, and $f(z) = \sum\limits_{k=1}^{\infty} a_k (z - z_k)^{-1}$. The function f is meromorphic on the disk D, with the poles in $z_k,\ k \in \mathbb{N}$. Since $f(z_k) = \infty,\ \left| f(z_k + \varepsilon) \right| < M,\ k \in \mathbb{N},$ and $\lim\limits_{n\to\infty} d_h(z_k, z_k + \varepsilon) = 0$ from Theorem 2 on *P*-sequences it follows that $(z_k) \subset \Delta_{g^{-1}}(-\theta, r) \subset \Delta(\theta, r)$ is *P*-sequence of *f*. Therefore, we may conclude that $\left\{ f \circ g^{\theta}_a \,\middle|\, g^{\theta}_a \in H^{\theta}_D \right\}$ is not normal family of functions on the disk *D*.

Since for every $z', z'' \in D \setminus \bigcup\limits_{k=1}^{\infty} D(z_k, \varepsilon_k)$ we have $\left| f(z') - f(z'') \right| \le \left| z_k{}' - z_k{}'' \right| \sum\limits_{k=1}^{\infty} \varepsilon_k = C < +\infty$ and since $\Delta_g(\theta, r),\ r > 0,$ contains finite number of points z_k, and since $\Delta_g(\theta, r),\ r > 0,$ is invariant set with respect to g_a^n, $n\in\mathbb{N}$, it follows that $\limsup\limits_{\Delta_g(\theta,r)\ni z\to e^{i\theta}}\left|f(z)\right|=c_f(r)<\infty$, $0<r<1$. Therefore, for every $r, 0<r<1$, the function f is bounded on $O_r\cap\Delta_g(\theta,r)$, where $O_r=\left\{z\,\middle\|\,\left|z-e^{i\theta}\right|<1-r\right\}$ so we have that $\left\{f\circ g_a^n\,\middle|\, n\in\mathbb{N}\cup\{0\}\right\}$, $a\in(-1,1)$ is normal family on the disku D (see [20], p. 35, Montel's theorem).

CONCLUSION

In this paper it is given a new approach in deriving theorems from the theory of asymptotical behavior of analytic functions. Namely, our theorems are proved using some results from the dynamic and the geometry of Möebious mappings and classical uniqueness theorem for analytic mappings, but in the preceding time these theorems were proved by using the approach and the results from the theory of harmonic mappings and harmonic measure theory.

The Main Lemma 1 and the Main Lemma 2 prove that the necessary condition for a function $f:D\to\overline{\mathbb{C}}$ to has the angular or oricyclic boundary value in $e^{i\theta}$ is that the following two families of functions

$$\left\{f\circ g_a^n\,\middle|\, n\in\mathbb{N}\cup\{0\}\right\},\ a\in(-1,1),\ \left\{f\circ g\,\middle|\, g\in P_{g_u}^{\theta}\right\}=\left\{f\circ g_u^n\,\middle|\, n\in\mathbb{N}\cup\{0\}\right\},\ u\in(-\infty,0),$$

are normal on the disk D.

The constructed example in Section 7 shows that the angular boundary values exist for broader class of meromorphic functions then the class considered in the theorems of Lehto-Virtanen and Gavrilov-Burkova. We have proved theorems of type of Bagemihl-Seidel for broader class of functions.

From Theorem 6 and Theorem 6′ it follows that the upper and the lower oricyclic boundary values of meromorphic function $f:D\to\overline{\mathbb{C}}$ in $e^{i\theta}$ are equal ω, $\omega\in\overline{\mathbb{C}}$, then f has tangential – oricyclic boundary value ω in $e^{i\theta}$. In general case it is possible to occur that one of these boundary values exists but the other not. This may be proved by an example which may be constructed in a similar way as the example in the Section 7.

For further consideration it remains to consider is it possible to use the approach of this paper in order to derive results concerning the asymptotic behavior of harmonic functions on the unit disk D in the complex plane $\mathbb{C}$.

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INVERSE PROBLEMS FOR STURM – LIOUVILLE OPERATOR WITH POTENTIAL FUNCTIONS FROM $L_2[0,\pi]$

BILJANA VOJVODIĆ[1], NATAŠA PAVLOVIĆ KOMAZEC[2*]

[1]University of Banja Luka, Faculty of Mechanical Engineering
Vojvode Stepe Stepanovića 71, Banja Luka 78000, Bosnia and Herzegovina
[2] University of East Sarajevo, Faculty of Electrical Engineering
Vuka Karadžića 30, East Sarajevo 71126, Bosnia and Herzegovina
*Corresponding author. E-mail: natasa.pavlovic@etf.ues.rs.ba

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Summary. This paper deals with non-self-adjoint second-order differential operators with two constant delays. We consider four boundary value problems $D_{i,k}, i = 0,1, k = 1,2$

$$-y''(x) + q_1(x)y(x-\tau_1) + (-1)^i q_2(x)y(x-\tau_2) = \lambda y(x), x \in [0,\pi]$$

$$y'(0) - hy(0) = 0, \qquad y'(\pi) + H_k y(\pi) = 0,$$

where $\frac{\pi}{3} \leq \tau_2 < \frac{\pi}{2} \leq 2\tau_2 \leq \tau_1 < \pi,\ h, H_1, H_2 \in R\backslash\{0\}$ and λ is a spectral parameter. We assume that q_1, q_2 are real-valued potential functions from $L_2[0,\pi]$ such that $q_1(x) = 0, x \in [0,\tau_1)$ and $q_2(x) = 0,\ x \in [0,\tau_2)$. The inverse spectral problem of recovering operators from their spectra has been studied. We prove that delays τ_1, τ_2 and parameters h, H_1, H_2 are uniquely determined from the spectra. Then we prove that potentials are uniquely determined by Volterra linear integral equations.

1 INTRODUCTION

The theory of differential equations with delays is a very important branch of the theory of ordinary differential equations and has been studied in detail in [1] and the references therein. For a number of results relating to the inverse spectral problems for classical Sturm-Liuville operators we refer the reader to [2], while some aspects of the direct and inverse problems for operators with a delay can be found in [3] - [13]. While there are a number results about both direct and inverse problems for operators with one delay, there are just a few results related to the operators with two or more delays (see [14]-[18]). The motivation behind this paper is to initiate further research in the inverse spectral theory for differential operators with delays. In what follows, we always take $i = 0,1$ and $k = 1,2$. In this paper we consider the boundary value problems $D_{i,k}$

$$-y''(x) + q_1(x)y(x-\tau_1) + (-1)^i q_2(x)y(x-\tau_2) = \lambda y(x), x \in [0,\pi] \tag{1}$$

$$y'(0) - hy(0) = 0, \tag{2}$$

$$y'(\pi) + H_k y(\pi) = 0 \tag{3}$$

where $\frac{\pi}{3} \leq \tau_2 < \frac{\pi}{2} \leq 2\tau_2 \leq \tau_1 < \pi,\ h, H_1, H_2 \in R\backslash\{0\}$ and λ is a spectral parameter. We assume that q_1, q_2 are real-valued potential functions from $L_2[0,\pi]$ such that

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$q_1(x) = 0,\ x \in [0, \tau_1)$ and $q_2(x) = 0,\ x \in [0, \tau_2)$. We study the inverse spectral problem of recovering operators from the spectra of $D_{i,k}$. Let $\left(\lambda_{n,i,k}\right)_{n=0}^{\infty}$ be the eigenvalues of $D_{i,k}$. The inverse problem is formulated as follows.

Inverse problem: Given $\left(\lambda_{n,i,k}\right)_{n=0}^{\infty}$, determine delays τ_1, τ_2, parameters h, H_1, H_2 and potential functions q_1, q_2.

To solve this inverse problem, we use the method of Fourier coefficients. This method based on determination of direct relations between Fourier coefficients of the potentials or some functions containing the potentials, and Fourier coefficients of some known functions.

In Section 2, we study the spectral properties of the boundary value problems $D_{i,k}$. In Section 3, we prove that delays and parameters are uniquely determined from the spectra. Then we prove that potentials are uniquely determined by the system of two Volterra linear integral equations.

2 SPECTRAL PROPERTIES

One can easily show that differential equation (1) under the initial condition (2) and conditions $q_1(x) = 0, x \in [0, \tau_1)$ and $q_2(x) = 0,\ x \in [0, \tau_2)$ is eqivalent to the integral equation

$$\begin{aligned} y_i(x,z) = cosxz + \frac{h}{z} sinzx + \frac{1}{z}\int_{\tau_1}^{x} q_1\,(t) sinz(x-t) y(t-\tau_1, z)dt + \\ + (-1)^i \tfrac{1}{z}\textstyle\int_{\tau_2}^{x} q_2\,(t) sinz(x-t) y(t-\tau_2, z)dt \end{aligned} \tag{4}$$

Here and in the sequel $\lambda = z^2$. By the method of steps it can be easily verified that the solution of integral equation (4) on $(\tau_1, \pi]$ is

$$\begin{aligned} y_i(x,z) = coszx + \frac{h}{z} sinzx + \frac{1}{z}\left(b_{sc}^{(1)}(x,z) + (-1)^i b_{sc}^{(2)}(x,z)\right) + \\ + \frac{h}{z^2}\left(b_{s^2}^{(1)}(x,z) + (-1)^i b_{s^2}^{(2)}(x,z)\right) + \frac{1}{z^2} b_{s^2c}^{(2)}(x,z) + \frac{h}{z^3} b_{s^3}^{(2)}(x,z) \end{aligned} \tag{5}$$

where

$$b_{sc}^{(k)}(x,z) = \int_{\tau_k}^{x} q_k\,(t) sinz(x-t) cosz(t-\tau_k)dt,$$

$$b_{s^2}^{(i)}(x,z) = \int_{\tau_k}^{x} q_k\,(t) sinz(x-t) sinz(t-\tau_k)dt,$$

$$b_{s^3}^{(2)}(x,z) = \int_{2\tau_2}^{x} q_2\,(t) sinz(x-t) b_{s^2}^{(2)}(t-\tau_2, z\,)dt,$$

$$b_{s^2c}^{(2)}(x,z) = \int_{2\tau_2}^{x} q_2\,(t) sinz(x-t) b_{sc}^{(2)}(t-\tau_2, z\,)dt.$$

Denote

$$\Delta_{i,k}(\lambda) = F_{i,k}(z) = {y_i}'(\pi, z) + H_k y(\pi, z).$$ [1]

From (5) we obtain

$$F_{i,k}(z) = \left(-z + \frac{hH_k}{z}\right) sin\pi z + (h + H_k) cos\pi z + b_{c^2}^{(1)}(z) + (-1)^i b_{c^2}^{(2)}(z) +$$

$$+\frac{h}{z}\left(b_{cs}^{(1)}(z) + (-1)^i b_{cs}^{(2)}(z)\right) + \frac{H_k}{z}\left(b_{sc}^{(1)}(z) + (-1)^i b_{sc}^{(2)}(z)\right) +$$

$$+\frac{H_k h}{z^2}\left(b_{s^2}^{(1)}(z) + (-1)^i b_{s^2}^{(2)}(z)\right) + \frac{1}{z} b_{csc}^{(2)}(z) + \frac{h}{z^2} b_{cs^2}^{(2)}(z) + \frac{H_k}{z^2} b_{s^2c}^{(2)}(z) + \frac{Hh}{z^3} b_{s^3}^{(2)}(z)$$

where

$$b_{cs}^{(k)}(z) = \int_{\tau_k}^{\pi} q_k\,(t) cosz(x - t) sinz(t - \tau_k) dt,$$
$$b_{c^2}^{(k)}(z) = \int_{\tau_k}^{\pi} q_k\,(t) cosz(x - t) cosz(t - \tau_k) dt$$

$$b_{s^2c}^{(2)}(z) = \int_{2\tau_2}^{\pi} q_2\,(t) sinz(x - t) b_{sc}^{(2)}(t - \tau_2, z\,) dt,$$
$$b_{csc}^{(2)}(z) = \int_{2\tau_2}^{\pi} q_2\,(t) cosz(x - t) b_{sc}^{(2)}(t - \tau_2, z\,) dt,$$
$$b_{cs^2}^{(2)}(z) = \int_{2\tau_2}^{\pi} q_2\,(t) cosz(x - t) b_{s^2}^{(2)}(t - \tau_2, z\,) dt.$$

To simplify further consideration we define so called *the transitional functions* $\tilde{q}_i$ as follows

$$\tilde{q}_i(t) = \begin{cases} q_1\left(t + \frac{\tau_1}{2}\right) + (-1)^i q_2\left(t + \frac{\tau_2}{2}\right), t \in \left[\frac{\tau_1}{2}, \pi - \frac{\tau_1}{2}\right] \\ (-1)^i q_2\left(t + \frac{\tau_2}{2}\right), t \in \left[\frac{\tau_2}{2}, \tau_2\right) \cup \left(\pi - \frac{\tau_1}{2}, \pi - \frac{\tau_2}{2}\right]. \\ 0, t \in \left[0, \frac{\tau_2}{2}\right) \cup \left(\pi - \frac{\tau_2}{2}, \pi\right] \end{cases} \tag{6}$$

Let us also define functions $K^{(2)}$ and $U^{(2)}$ by

$$K^{(2)}(t) = q_2(t + \tau_2) \int_{\tau_2}^{t} q_2(s) ds - q_2(t) \int_{t+\tau_2}^{\pi} q_2(s) ds - \int_{t+\tau_2}^{\pi} q_2(s - t) q_2(s) ds,$$
$$t \in [\tau_2, \pi - \tau_2], \qquad K^{(2)}(t) = 0, t \in [0, \tau_2) \cup (\pi - \tau_2, \pi],$$

and

[1] Below, instead of the argument (π, z) we write argument (z)

$$U^{(2)}(t) = q_2(t+\tau_2)\int_{\tau_2}^{t} q_2(s)ds - q_2(t)\int_{t+\tau_2}^{\pi} q_2(s)ds + \int_{t+\tau_2}^{\pi} q_2(s-t)q_2(s)ds,$$
$$t \in [\tau_2, \pi-\tau_2], \qquad U^{(2)}(t) = 0, t \in [0,\tau_2) \cup (\pi-\tau_2, \pi].$$

and introduce notations

$$J_1^{(k)} = \int_{\tau_k}^{\pi} q_i\,(t)dt, \qquad J_2^{(2)} = \int_{2\tau_2}^{\pi} q_2\,(t)\left(\int_{\tau_2}^{t-\tau_2} q_2\,(s)ds\right)dt$$

and functions

$$\tilde{a}_{i,c}(z) = \int_0^{\pi} \tilde{q}_i(t) cosz(\pi - 2\,t)dt, \quad \tilde{a}_{i,s}(z) = \int_0^{\pi} \tilde{q}_i(t) sinz(\pi - 2\,t)dt,$$
$$k_s(z) = \int_0^{\pi} K^{(2)}(t)\, sinz(\pi-2t)dt, \quad k_c(z) = \int_0^{\pi} K^{(2)}(t)\, cosz(\pi-2t)dt,$$
$$u_s(z) = \int_0^{\pi} U^{(2)}(t)\, sinz(\pi-2t)dt, \quad u_c(z) = \int_0^{\pi} U^{(2)}(t)\, cosz(\pi-2t)dt.$$

One can easily show that folowing relations hold

$$\int_{\tau_2}^{\pi-\tau_2} K^{(2)}(t)\,dt = -J_2^{(2)}, \qquad \int_{\tau_2}^{\pi-\tau_2} U^{(2)}(t)\,dt = J_2^{(2)}. \tag{7}$$

Using aforementioned tags and relations (7), we can rewrite characteristic functions $F_{i,k}(z)$ as follows

$$F_{i,k}(z) = \left(-z + \frac{hH_k}{z}\right) sin\pi z + (h+H_k)cos\pi z + \frac{1}{2}\left(\tilde{a}_{i,c}(z) + J_{i,c}(z)\right) +$$
$$+\frac{h}{2z}\left(-\tilde{a}_{i,s}(z) + J_{i,s}(z)\right) + \frac{H_k}{2z}\left(\tilde{a}_{i,s}(z) + J_{i,s}(z)\right) + \frac{hH_k}{2z^2}\left(\tilde{a}_{i,c}(z) - J_{i,c}(z)\right)$$
$$+\frac{1}{4z}\left(J_{2,s}(z) - u_s(z)\right) - \frac{h}{4z^2}\left(J_{2,c}(z) + k_c(z)\right) - \frac{H_k}{4z^2}\left(J_{2,c}(z) - u_c(z)\right)$$
$$-\frac{hH_k}{4z^3}\left(J_{2,s}(z) + k_s(z)\right) \tag{8}$$

where

$$J_{i,c}(z) = J_1^{(1)} cosz(\pi-\tau_1) + (-1)^i J_1^{(2)} cosz(\pi-\tau_2),$$
$$J_{i,s}(z) = J_1^{(1)} sinz(\pi-\tau_1) + (-1)^i J_1^{(2)} sinz(\pi-\tau_2)$$
$$J_{2,c}(z) = J_2^{(2)} cosz(\pi-2\tau_2), \qquad J_{2,s}(z) = J_2^{(2)} sinz(\pi-2\tau_2).$$

The functions $F_k(z)$ are entire in λ of order $1/2$. Using (8), by the well known method (see [2]), we obtain the asymptotic formulas for $\left(\lambda_{n,i,k}\right)_{n=0}^{\infty}$ of $D_{i,k}$:

$$\lambda_{n,i,k} = n^2 + \frac{2}{\pi}(h + H_k) + \frac{J_1^{(1)}}{\pi} cosn\tau_1 + (-1)^i \frac{J_1^{(2)}}{\pi} cosn\tau_2 + o(1), n \to \infty. \tag{9}$$

Now, by Hadamard's factorization theorem, from the spectra of $D_{i,k}$, we can construct the characteristic functions $F_{i,k}$. The next lemma holds.

Lemma 2.1. The specification of spectrum $\left(\lambda_{n,i,k}\right)_{n=0}^{\infty}$ of the boundary value problems $D_{i,k}$ uniquely determines the characteristic functions $F_{i,k}(z)$ by the formulas

$$F_{i,k}(z) = \pi\left(\lambda_{0,i,k} - z^2\right) \prod_{n=1}^{\infty} \frac{\lambda_{n,i,k} - z^2}{n^2}. \tag{10}$$

3 MAIN RESULTS

Lemma 3.1. The delays τ_k, integrals $J_1^{(k)}$ and sums $h + H_k$ are uniquely determined by eigenvalues $\left(\lambda_{n,i,k}\right)_{n=0}^{\infty}$.

Proof. Let us consider the sequences

$$\rho_{n,k} = \frac{1}{2}\left(\lambda_{n,0,k} + \lambda_{n,1,k}\right)$$

and

$$\sigma_n = \frac{1}{2}\left(\lambda_{n,0,1} - \lambda_{n,1,1}\right).$$

From (9) we obtain the next asymptotic formulas

$$\rho_{n,k} = n^2 + \frac{2}{\pi}(h + H_k) + \frac{J_1^{(1)}}{\pi} cosn\tau_1 + o(1)$$

and

$$\sigma_n = \frac{J_1^{(2)}}{\pi} cosn\tau_2 + o(1).$$

Obviously, the delays τ_1, τ_2 and integrals $J_1^{(1)}$, $J_1^{(2)}$ can be determined from sequences $\left(\rho_{n,k}\right)_{n=0}^{\infty}$ and $(\sigma_n)_{n=0}^{\infty}$ in the same way as for the operators with one delay (see [13]). Lemma 3.1. is proved. □

Lemma 3.2. Parameters h and H_k are uniquely determined by eigenvalues $\left(\lambda_{n,0,k}\right)_{n=0}^{\infty}$.

Proof. By virtue of Lemma 3.1., functions $J_{0,c}(z)$ and $J_{0,s}(z)$ are known. Since the characteristic functions are uniquely determined by the spectra, putting $\lambda = \left(\frac{4m+1}{2}\right)^2$ into functions $F_{0,k}$ from (10), we can define functions

$$F^*{}_{0,k}(m) = F_{0,k}\left(\frac{4m+1}{2}\right) + \frac{4m+1}{2} - \frac{1}{2}J_{0,c}\left(\frac{4m+1}{2}\right) - \frac{H_k + h}{4m+1}J_{0,s}\left(\frac{4m+1}{2}\right).$$

Then, using the form of the characteristic functions $F_{0,k}$ from (8), we get

$$h = \frac{1}{2}\lim_{m\to\infty}\frac{4m+1}{H_2 - H_1}\left(F^*{}_{0,2}(m) - F^*{}_{0,1}(m)\right)$$

At the end, we determine H_k from $h + H_k$, thus proving Lemma 3.2. □

In order to recover the potential functions from the spectra by the method of Fourier coefficients, we should transform the characteristic functions (8). For this purpose, we use the method of partial integration in (8), once in integrals $\tilde{a}_{i,s}(z), \tilde{a}_{i,c}(z), u_s(z)$ and $u_c(z)$, and twice in the integrals $k_c(z)$ and $k_s(z)$. This is where the next function appears

$$K^{(2)*}(t) = \begin{cases} \int_{\tau_2}^{t} K^{(2)}(u)\,du, t \in [\tau_2, \pi - \tau_2] \\ 0, t \in [0, \tau_2) \cup (\pi - \tau_2, \pi] \end{cases}.$$

One can show that following relation holds

$$\int_{\tau_2}^{\pi-\tau_2}\left(\int_{\tau_2}^{t} K^{(2)}(u)\,du\right)dt = -(\pi - 2\tau_2)J_2^{(2)}.$$

Then we obtain the characteristic functions in the form

$$\begin{aligned} F_{i,k}(z) = {} & \left(-z + \frac{H_k h}{z}\right) sin\pi z + (h + H_k)cos\pi z + \frac{1}{2}\left(\tilde{a}_{i,c}(z) + \frac{H_k}{z}\tilde{a}_{i,s}(z)\right) - \\ & -h\left(\tilde{q}_{i,c}{}^{(1)}(z) + \frac{H_k}{z}\tilde{q}_{i,s}{}^{(1)}(z)\right) - \frac{1}{2}\left(u_c^*(z) + \frac{H_k}{z}u_s^*(z)\right) + h\left(k_c^{**}(z) + \frac{H_k}{z}k_s^{**}(z)\right) \\ & + \frac{J_{i,c}(z)}{2} + \frac{2h + H_k}{2z}J_{i,s}(z) + \frac{1}{2z}\left(1 - \frac{H_k h}{z^2}\right)J_{2,s}(z) + \frac{h}{2z}(\pi - 2\tau_2)J_2^{(2)} sinz(\pi - 2\tau_2) \\ & + \frac{H_k h}{2z^2}(\pi - 2\tau_2)J_2^{(2)} cosz(\pi - 2\tau_2) \end{aligned} \tag{11}$$

where

$$\tilde{q}_{i,c}{}^{(1)}(z) = \int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}} \left(\int_{\frac{\tau_2}{2}}^{t} \tilde{q}_i(s)\, ds \right) cosz(\pi - 2t)dt,$$
$$\tilde{q}_{i,s}{}^{(1)}(z) = \int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}} \left(\int_{\frac{\tau_2}{2}}^{t} \tilde{q}_i(s)\, ds \right) sinz(\pi - 2t)dt\ ,$$
$$u_c^*(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} U^{(2)}(s)\, ds \right) cosz(\pi - 2t)dt,$$
$$u_s^*(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} U^{(2)}(s)\, ds \right) sinz(\pi - 2t)dt$$

and

$$k_c^{**}(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} K^{(2)*}(s)\, ds \right) cosz(\pi - 2t)dt,$$
$$k_s^{**}(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} K^{(2)*}(s)\, ds \right) sinz(\pi - 2t)dt.$$

In order to recover the potential functions from the spectra, at the beginning we define functions

$$A_i(z) = \frac{2}{H_2 - H_1}\Big(H_2 F_{i,1}(z) - H_1 F_{i,2}(z)\Big) + 2zsin\pi z - 2hcos\pi z - J_{i,c}(z) -$$
$$- \frac{2hJ_1^{(2)}}{z} sinz(\pi - \tau_2) \tag{12}$$

and

$$B_i(z) = \frac{2z}{H_2 - H_1}\Big(F_{i,2}(z) - F_{i,1}(z)\Big) - 2hsin\pi z - 2zcos\pi z - J_{i,s}(z). \tag{13}$$

From (11) we obtain

$$A_i(z) = \tilde{a}_{i,c}(z) - 2h\tilde{q}_{i,c}{}^{(1)}(z) - u_c^*(z) + 2hk_c^{**}(z) + \alpha(z) \tag{14}$$

$$B_i(z) = \tilde{a}_{i,s}(z) - 2h\tilde{q}_{i,s}{}^{(1)}(z) - u_s^*(z) + 2hk_s^{**}(z) + \beta(z) \tag{15}$$

where

$$\alpha(z) = \frac{J_2^{(2)}}{z}(h(\pi - 2\tau_2) + 1)sinz(\pi - 2\tau_2)$$

and

$$\beta(z) = \frac{hJ_2^{(2)}}{z^2}\big(z(\pi - 2\tau_2)cosz(\pi - 2\tau_2) - sinz(\pi - 2\tau_2)\big).$$

One can easily show that

$$\lim_{z\to 0}\beta(z)=0,$$

and

$$\lim_{z\to 0}\alpha(z)=J_2^{(2)}(h(\pi-2\tau_2)+1)(\pi-2\tau_2).$$

Put $z=m, m\in N$ into (14) and (15) and denote

$$A_{2m,i}=\frac{2}{\pi}(-1)^m A_i(m),\quad B_{2m,i}=\frac{2}{\pi}(-1)^{m+1}B_i(m).$$

Then we obtain

$$A_{2m,i}=\frac{2}{\pi}\tilde{a}_{2m,i}-\frac{4}{\pi}h\tilde{q}^{(1)}_{2m,i,c}-\frac{2}{\pi}u^*_{2m,c}+\frac{4}{\pi}hk^{**}_{2m,c}-\frac{2J_2^{(2)}}{\pi m}(h(\pi-2\tau_2)+1)sin2m\tau_2, \tag{16}$$

$$B_{2m}=\frac{2}{\pi}\tilde{b}_{2m,i}-\frac{4}{\pi}h\tilde{q}^{(1)}_{2m,i,s}-\frac{2}{\pi}u^*_{2m,s}+\frac{4}{\pi}hk^{**}_{2m,s}-\frac{2hJ_2^{(2)}}{\pi m^2}(m(\pi-2\tau_2)cos2m\tau_2+sin2m\tau_2) \tag{17}$$

where

$$\tilde{a}_{2m,i}=\int_0^\pi \tilde{q}_i(t)cos2mtdt\,,\qquad \tilde{b}_{2m,i}=\int_0^\pi \tilde{q}_i(t)sin2mtdt\,,$$

$$u^*_{2m,s}=\int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}}\left(\int_{\frac{\tau_2}{2}}^{t}U^{(2)}(s)\,ds\right)sin2mtdt,$$

$$u^*_{2m,c}=\int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}}\left(\int_{\frac{\tau_2}{2}}^{t}U^{(2)}(s)\,ds\right)cos2mtdt$$

$$k^{**}_{2m,c}=\int_{\tau_2}^{\pi-\tau_2}\left(\int_{\tau_2}^{t}K^{(2)*}(s)\,ds\right)cos2mtdt,.$$

$$k^{**}_{2m,s}=\int_{\tau_2}^{\pi-\tau_2}\left(\int_{\tau_2}^{t}K^{(2)*}(s)\,ds\right)sin2mtdt.$$

Denote $A_{0,i}=\frac{2}{\pi}\lim_{m\to 0}A_i(m)$.

Then we obtain

$$A_{0,i}=\frac{2}{\pi}\tilde{a}_{0,i}-\frac{4}{\pi}h\tilde{q}^{(1)}_{0,i,c}-\frac{2}{\pi}u^*_{0,c}+\frac{4}{\pi}hk^{**}_{0,c}+\frac{2J_2^{(2)}}{\pi}(h(\pi-2\tau_2)+1)(\pi-2\tau_2). \tag{18}$$

Since sequences $\{A_{2m,i}\}$ and $\{B_{2m,i}\}$ belong to the space l_2, by virtue of Riesz-Fischer theorem, there exist functions f_i from $L_2[0,\pi]$ such that

$$f_i(t) = \frac{A_{0,i}}{2} + \sum_{m=1}^{\infty} A_{2m,i} cos2mt + B_{2m,i} sin2mt \, , t \in [0, \pi]$$

Now multiplying (18) with $\frac{1}{2}$, (16) with $cos2mt$ and (17) with $sin2mt$, and then summing-up from $m = 1$ to $m = \infty$, we get the system of integral equations

$$\tilde{q}_i(t) - 2h \int_{\frac{\tau_2}{2}}^{t} \tilde{q}_i(s) \, dt_2 - \int_{\tau_2}^{t} U^{(2)}(s) ds + 2h \int_{\tau_2}^{t} K^{(2)*}(s) \, ds + \Phi(t) = f_i(t) \tag{19}$$

where

$$\Phi(t) = -\frac{2J_2^{(2)}}{\pi} (h(\pi - 2\tau_2) + 1) \sum_{m=1}^{\infty} \frac{sin2m\tau_2}{m} cos2mt -$$

$$-\frac{2hJ_2^{(2)}}{\pi} (\pi - 2\tau_2) \sum_{m=1}^{\infty} \frac{cos2m\tau_2}{m} sin2mt - \frac{2hJ_2^{(2)}}{\pi} \sum_{m=1}^{\infty} \frac{sin2m\tau_2}{m^2} sin2mt.$$

After summing and subtracting integral equations (19), and then introducing substitution of variables, we get the system of integral equations

$$q_1(x) - 2h \int_{\tau_1}^{x} q_1(u) \, du - \int_{\tau_2 + \frac{\tau_1}{2}}^{x} U^{(2)} \left(u - \frac{\tau_1}{2}\right) du + 2h \int_{\tau_2 + \frac{\tau_1}{2}}^{x} K^{(2)*} \left(u - \frac{\tau_1}{2}\right) du +$$

$$+\Phi \left(x - \frac{\tau_1}{2}\right) = \frac{1}{2} \left(f_0 \left(x - \frac{\tau_1}{2}\right) + f_1 \left(x - \frac{\tau_1}{2}\right) \right) \tag{20}$$

and

$$q_2(x) - 2h \int_{\tau_2}^{x} q_2(u) \, du = \frac{1}{2} \left(f_0 \left(x - \frac{\tau_2}{2}\right) - f_1 \left(x - \frac{\tau_2}{2}\right) \right). \tag{21}$$

Finally, we come to our main result.

Theorem 3.1. Let $q_k \in L_2[\tau_i, \pi]$, $q_k(x) = 0$ for $x \in [0, \tau_k)$.
If $\frac{\pi}{3} \leq \tau_2 < \frac{\pi}{2} \leq 2\tau_2 \leq \tau_1 < \pi$, then integral equations (20) and (21) have unique solutions $q_1 \in L_2[\tau_1, \pi]$ and $q_2 \in L_2[\tau_2, \pi]$, respectively.

Proof. Obviously, the integral equation (21) has a unique solution q_2 on (τ_2, π). Then we obtain that integrals $\int_{\tau_2 + \frac{\tau_1}{2}}^{x} U^{(2)} \left(u - \frac{\tau_1}{2}\right) du$ and $\int_{\tau_2 + \frac{\tau_1}{2}}^{x} K^{(2)*} \left(u - \frac{\tau_1}{2}\right) du$ are known too, as well as the integral $J_2^{(2)}$. For sums appearing in the function Φ, we have

$$\sum_{m=1}^{\infty} \frac{sin2m\tau_2}{m} cos2mt = \begin{cases} -\tau_2, & t \in (\tau_2, \pi - \tau_2), \\ \frac{\pi}{2} - \tau_2, & t \in (0, \tau_2) \cup (\pi - \tau_2, \pi), \\ \frac{\pi}{4} - \tau_2, & t = \tau_2, t = \pi - \tau_2 \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{cos2m\tau_2}{m} sin2mt = \begin{cases} -t, & t \in (0, \tau_2) \\ \frac{\pi}{2} - t, & t \in (\tau_2, \pi - \tau_2) \\ \pi - t, & (\pi - \tau_2, \pi) \\ \frac{\pi}{4} - \tau_2, & t = \tau_2, \\ -\frac{\pi}{4} + \tau_2, & t = \pi - \tau_2 \end{cases}$$

and

$$\sum_{m=1}^{\infty} \frac{sin2m\tau_2}{m^2} sin2mt = \begin{cases} (\pi - 2\tau_2)t, & t \in (0, \tau_2) \\ \tau_2(\pi - 2t), & t \in (\tau_2, \pi - \tau_2) \\ (\pi - 2\tau_2)(t - \pi), & (\pi - \tau_2, \pi) \\ (\pi - 2\tau_2)\tau_2, & t = \tau_2 \\ -(\pi - 2\tau_2)\tau_2, & t = \pi - \tau_2 \end{cases}$$

Then for $x \in (\tau_1, \pi)$ we obtain linear integral equation

$$q_1(x) - 2h\int_{\tau_1}^{x} q_1(u)\,du = \int_{\tau_2+\frac{\tau_1}{2}}^{x} U^{(2)}\left(u - \frac{\tau_1}{2}\right)du - 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x} K^{(2)*}\left(u - \frac{\tau_1}{2}\right)du$$
$$-\Phi\left(x - \frac{\tau_1}{2}\right) + \frac{1}{2}\left(f_0\left(x - \frac{\tau_1}{2}\right) + f_1\left(x - \frac{\tau_1}{2}\right)\right)$$

which has a unique solution q_1 on (τ_1, π). Theorem is proved. □

4 CONCLUSION

Inverse spectral problems for classical Sturm-Liouville operators have been studied completely, while the inverse problems for differential operators with delays have not been studied enough. The main results for classical Sturm-Liouville operators is presented in [2] while some of the results for differential operators with delay can be found in ([3],[4],[5],[10],[11],[12],[13]). The class of operators with two delays has been least studied, but some of the results for this class of operators are presented in ([16], [17]). The motivation behind this paper is to initiate further research in the inverse spectral theory for differential operators with delays. We studied the inverse spectral problem of recovering operators from the spectra of $D_{i,k}$. To solved this inverse problem, we used the method of Fourier coefficients. This method is based on determination of direct relations between Fourier coefficients of the potentials or some functions containing the potentials, and Fourier coefficients of some known functions. We studied the spectral properties of the boundary value problems and proved that delays and parameters are uniquely determined from the spectra. Then we proved that potentials are uniquely determined by the system of two Volterra linear integral equations.

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AMENABILITY OF $A \oplus_T X$ AS AN EXTENSION OF BANACH ALGEBRA

M. GHORBAI, D. E. BAGHA*

Department of Mathematics, Central Tehran Branch, Islamic Azad university, Tehran, Iran.
*Corresponding author. E-mail: e_bagha@yahoo.com

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Summary. Let $A, X, \mathfrak{U}$ be Banach algebras and A be a Banach $\mathfrak{U}$-bimodule also X be a Banach $A-\mathfrak{U}$-module. In this paper we study the relation between module amenability, weak module amenability and module approximate amenability of Banach algebra $A \oplus_T X$ and that of Banach algebras A, X. Where T: $A \times A \to X$ is a bounded bi-linear mapping with specific conditions.

1 INTRODUCTION

The notation of amenability of Banach algebras was introduced by B.Johnson in [9]. A Banach algebra A is amenable if every bounded derivation from A into any dual Banach A-bimodule is inner, equivalently if $H(A,\ X^*) = \{0\}$ for any Banach A-bimodule X, where $H(A,\ X^*)$ is the first Hochschild co- homology group of A with coefficient in X^*. Also, a Banach algebra A is weakly amenable if $H(A,\ A^*) = \{0\}$. Bade, Curtis and Dales introduced the notion of weak amenability on Banach algebras in [5]. They considered this concept only for commutative Banach algebras. After a while, Johnson defined the weak amenability for arbitrary Banach algebras [8].

For a morphism $T: B \to A$ from a Banach algebra B to a commutative Banach algebra A. The notion of module amenability of Banach algebras was introduced by Amini in [1]. Amini and Ebrahimi Bagha in [3] studied the concept of weak module amenability. In [10] the notation of module approximate amenability and contractibility as modules over of another Ba- nach algebra was introduced for the notion of Banach algebras.

M. Sangani-Monfared in [11] defined a product on $A \times B$ and obtained the Banach algebra $A \times_\theta B$ using a character $\theta \in \sigma(B)$, for Banach algebras in a fairly general setting.

Later, S.J. Bhatt and P.A. Dabhi in [6] defined a product on $A \times B$ and obtained a Banach algebra $A \times_T B$ for a morphism $T : B \to A$ from a Banach algebra B to a commutative Banach algebra A.

The first and the second authors generalized all these constructions, and de- fined the module Lau product $A \times_\alpha B$ for Banach algebras A and B such that A is a Banach B-bimodule. They studied the ideal amenability of $A \times_\alpha B$ in [4].

T.Yazdan panah in [12] studied the concept of expanded modular of Banach algebra denoted by $A \oplus_T X$. He showed that $A \oplus_T X$ is amenable if and only if A is amenable and $X = \{0\}$. In this paper, we define a new Banach algebra different from of all above Banach algebras, named $A \oplus_T X$ in section 2. Then, some required basic properties of the following part are studied. In section 3, as the main section of paper, we study the relationship between module amenability of $A \oplus_T X$ and module amenability of A and X. We show that If $T(A,\ 0) = X$ and $A^2 = A$, then the module amenability of A implies module amenability of $A \oplus_T X$. Furthermore, it's

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conversly obtained that the module amenability of $A \oplus_T X$ implies module amenability of A and moreover if $T(A,\ 0) = X$, then X also is module amenable. In sectiones 4 and 5 respectively we study the relationship between weak mod- ule amenability (based as definition in [1] and [2]) and module approximte amenability of $A \oplus_T X$ and weak module amenability and module approximte amenability of A, X.

2 DEFINITIONS AND BASIC PROPERTIES

Throughout this paper it's assumed that $\mathfrak{U}$ be a Banach algebra, A be a Banach $\mathfrak{U}$-bimodule and X be a Banach A-U-bimodule. Module actions are assumed as follow too:

$$A \times \mathfrak{U} \to A; (a,\quad \alpha) \longmapsto a \circ \alpha, \mathfrak{U} \times A \to A; (\alpha,\quad a) \longmapsto \alpha \cdot a.$$
$$X \times \mathfrak{U} \to X; (x,\quad \alpha) \longmapsto x \vartriangle \alpha, \mathfrak{U} \times X \to X; (\alpha,\quad x) \longmapsto \alpha \nabla x.$$
$$X \times A \to X; (x,\quad a) \longmapsto x \circ a, A \times X \to X; (a,\quad x) \longmapsto a \cdot x.$$

Consider the bounded bilinear map $T : A \times A \to X$, which has the following properties:

$$a \cdot T(a_1 a_2, 0) = T(aa_1, 0) \circ a_2, T(a_1 a_2, 0) = T(a_1, 0) T(a_2, 0)\ ,$$
$$T(\alpha \cdot a,\quad \alpha \nabla x) = \alpha \cdot T(a,\quad x), T(a \circ \alpha,\quad x \vartriangle \alpha) = T(a,\quad x) \cdot \alpha,$$

$\| T(a,\ 0) \| = \| a \|$, for all $a, a_1, a_2 \in A, x \in X, \alpha \in \mathfrak{U}$.

Module extension $A \oplus X$, with the product

$$(a,\ x)(a_1,\ x_1) = (aa_1,\ a \cdot x_1 + x \circ a_1 + T(aa_1, 0))$$

and the norm $\| (a,\ x) \| = \| a \| + \| x \|$ is a Banach algebra denoted by $A \oplus_T X$.

Definition 2.1 *The bounded map $D: A \to X^*$ with $D(a + b) = D(a) + D(b), D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in A$, and $D(\alpha \cdot a) = \alpha \cdot D(a), D(a \cdot \alpha) = D(a) \cdot \alpha (\alpha \in \mathfrak{U},\ a \in A)$, is called module derivation.*

Note that X^* is also Banach module over A and $\mathfrak{U}$ with compatible actions under the canonical actions of A and $\mathfrak{U}$, $\alpha \cdot (a \cdot f) = (\alpha \cdot a) \cdot f, (a \in A,\ \alpha \in \mathfrak{U},\ f \in X^*)$, and the same for right action. Here the canonical actions of A and $\mathfrak{U}$ on X^* are defined by $(\alpha \cdot f)(x) = f(x \vartriangle \alpha), (a \cdot f)(x) = f(x \circ a), (\alpha \in \mathfrak{U},\ a \in A,\ f \in X^*,\ x \in X)$ and it's the same for right actions. As in [1] we call A- module X which have a compatible $\mathfrak{U}$-action as above, a $A - \mathfrak{U}$ modules, above assertion is to say that if X is an $A - \mathfrak{U}$- module, then so is X^*. Also we use the notation $Z_{\mathfrak{U}}(A,\ X^*)$ for the set of all module derivations $D: A \to X^*$, and $N_{\mathfrak{U}}(A,\ X^*)$ for those which are inner and $H_{\mathfrak{U}}(A,\ X^*)$ for the quotient group.

Proposition 2.2 *$A \oplus_T X$ is a Banach $\mathfrak{U}$- bimodule.*

Proof. Consider the module actions as follow:

$\mathfrak{U} \times (A \oplus_T X) \to A \oplus_T X; \alpha \cdot (a,\ x) = (\alpha \cdot a,\ \alpha \nabla x)$, and $(A \oplus_T X) \times \mathfrak{U} \to A \oplus_T X; (a,\ x) \bullet \alpha = (\alpha \circ a,\ \alpha \vartriangle x)$. It is easy to check the satification of the properties. ■

Proposition 2.3 *If Y is an A-U-module, then $Y \oplus \{O\}$ is a Banach $A \oplus_T X - \mathfrak{U}$-bimodule.*

Proof. Assume that the module actions on Y, are as follows:

$\mathfrak{U} \times Y \to Y; (\alpha,\ y) \longmapsto \alpha \vartriangle y, Y \times \mathfrak{U} \to Y; (y,\ \alpha) \longmapsto y \bullet \alpha.$ And $A \times Y \to Y; (a,\ y) \longmapsto a \cdot y, Y \times A \to Y; (y,\ a) \longmapsto y.a$. Define the module actions as: $(Y \oplus \{0\}) \times \mathfrak{U} \to Y \oplus$

$\{0\}; (y,\ 0) \bullet \alpha = (y \bullet \alpha,\ 0), \mathfrak{U} \times (Y \oplus \{0\}) \to Y \oplus \{0\};\quad \alpha \bullet (y,\ 0) = (\alpha \vartriangle y,\ 0)$. And $(A \oplus_T X) \times (Y \oplus \{0\}) \to Y \oplus \{0\}; (a,\ x).(y,\ 0) = (a.y,\ 0), (Y \oplus \{0\}) \times (A \oplus_T X) \to Y \oplus \{0\}; (y,\ 0) \circ (a,\ x) = (y.\ a,\ 0)$. We only need to show that the actions are compatible.

$$\begin{aligned} 1)\alpha.\ ((a,\ x).(y,\ 0)) &= \alpha.\ (a.y,\ 0) \\ &= (\alpha \vartriangle (a.y),\ 0) = ((\alpha \cdot a) \cdot y,\ 0) \\ &= ((\alpha \cdot a,\ \alpha \nabla x) \cdot (y,\ 0) = (\alpha \cdot (a,\ x)).(y,\ 0)\ . \end{aligned}$$

$$\begin{aligned} 2)((a,\ x) \cdot (y,\ 0)) \bullet \alpha &= (a \cdot y,\ 0) \bullet \alpha \\ &= ((a \cdot y) \bullet \alpha,\ 0)) = (a \cdot (y \bullet \alpha),\ 0) \\ &= (a,\ x) \cdot (y \ \bullet\ \alpha,\ 0) = (a,\ x) \cdot ((y,\ 0)\ \bullet\ \alpha)\ . \end{aligned}$$

$$\begin{aligned} 3)(\alpha.\ (y,\ 0)) \cdot (a,\ x) &= (\alpha \vartriangle y,\ 0) \cdot (a,\ x) \\ &= ((\alpha \vartriangle y).\ a,\ 0) = (\alpha \vartriangle (y.\ a),\ 0)) \\ &= \alpha.\ (y.\ a,\ 0) = \alpha.\ ((y,\ 0) \circ (a,\ x)) \end{aligned}$$

■

Proposition 2.4 *Let $M \oplus N$ be a Banach $A \oplus_T X - \mathfrak{U}$-bimodule, then M is a Banach $A - \mathfrak{U}$-bimodule.*

Proof. Consider the map $Q_M : M \oplus N \to M; (m,\ n) \longmapsto m$ and define the module actions as: $M \times \mathfrak{U} \to M; (m,\ n) \longmapsto m \cdot \alpha = Q_M((m,\ 0) \bullet \alpha), \mathfrak{U} \times M \to M; (\alpha,\ m) \longmapsto \alpha \circ m = Q_M(\alpha \cdot (m,\ 0\ \ M \times A \to M; (m,\ a) \longmapsto m.a = Q_M((m,\ 0) \circ (a,\ 0))$ and $A \times M \to A; (a, m) \longmapsto a \bullet m = Q_M((a,\ 0) \cdot (m, 0))$ ■

Proposition 2.5 *Let M be a Banach $A - \mathfrak{U}$-module and N be a Banach $X - \mathfrak{U}$-bimodule, $M \oplus N$ is a Banach $A \oplus_T X - \mathfrak{U}$-bimodule.*

Proof. Given module actions on $M \oplus N$ as follows:

$(M \oplus N) \times \mathfrak{U} \to M \oplus N; (m,\ n) \cdot \alpha = (m.\alpha,\ n\nabla\alpha), \mathfrak{U} \times (M \oplus N) \to M \oplus N;\quad \alpha \bullet (m,\ n) = (\alpha \bullet m,\ \alpha \vartriangle m), (M \oplus N) \times (A \oplus_T X) \to (M \oplus N); (m,\ n) \cdot (a,\ x) = (m\ \ \alpha, n.\ T(a,\ 0\ \ (A \oplus_T X) \times (M \oplus N) \to M \oplus N; (a,\ x) \bullet (m,\ n) = (a\ \ \bullet\ \ m,\ T(a,\ 0) \odot n)$.

■

Proposition 2.6 *For each $(f,\ g) \in M^* \oplus N^*, (a,\ x) \in A \oplus_T X, (m,\ n) \in M \oplus N$ we have $(f,\ g) \cdot (a,\ x) = (f \cdot a,\ g \cdot T(a,\ 0))$ and $(a,\ x).(f,\ g) = (a.f,\ T(a,\ 0).g)$.*

Proof.

$$\begin{aligned} \langle (f,\ g).(a,\ x),\ (m,\ n) \rangle &= \langle (f,\ g),\ (a,\ x) \bullet (m,\ n) \rangle \\ &= \langle (f,\ g),\ (a \circ m,\ T(a,\ 0) \odot n) \rangle \\ &= \langle f,\ a \circ m \rangle + \langle g,\ T(a,\ 0) \odot n \rangle \\ &= \langle f.a,\ m \rangle + \langle g.T(a,\ 0),\ n \rangle \\ &= \langle (f.a,\ g.T(a,\ 0\ \ (m,\ n))) \end{aligned}$$

■

Proposition 2.7 *If N is a Banach $X - \mathfrak{U}$- bimodule, then is a Banach $A - \mathfrak{U}$-bimodule.*

Proof. The module actions are defined as follow:

$A \times N \to N; a.n = T(a,\ 0) \odot n$ and $N \times A \to N; n \bullet a = n.T(a,\ \odot)$. ■

3 MODULE AMENABILITY

Lemma 3.1 *$D \in Z_{\mathfrak{U}}(A \oplus_T X,\ M^* \oplus N^*)$ if and only if there are $D_1 \in Z_{\mathfrak{U}}(A,\ M^*), D_3 \in Z_{\mathfrak{U}}(X,\ N^*), R \in Z_{\mathfrak{U}}(A,\ N^*)$ and linear map $D_2 : X \to M^*$ such that*

1) $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D_3(x))$,

2) $D_2(a \bullet x) = a \cdot D_2(x)$,

3) $D_2(x \circ a) = D_2(x) \cdot a$,

4) $R(bd) = R(b) \cdot T(d,\ 0) + T(b,\ 0) \cdot R(d) = R(b).d + b.R(d)$,

5) $D_2(T(ab,\ 0)) = 0$,

6) $D_3(a.x) = T(a,\ 0).D_3(x)$,

7) $D_3(x \circ a) = D_3(x).T(a,\ 0)$,

8) $D_3(T(ab,\ 0)) = 0$.

Proof. Suppose that $D \in Z_{\mathfrak{U}}(A \oplus_T X,\ M^* \oplus N^*)$ then there are $d_1 : A \oplus \tau X \to M^*$, $d_2 : A \oplus_T X \to N^*$ such that $\mathrm{D} = (d_1,\ d_2)$, Set

$$D_1: A \to M^*; D_1(a) = d_1(a,\ 0),$$

$$D_2: X \to N^*; D_2(x) = d_1(0,\ x),$$

$$D_3: X \to N^*; D_3(x) = d_2(0,\ x), R: A \to N^*; R(a) = d_2(a,\ 0).$$

Now

$$\begin{aligned} D(a,\ x) &= (d_1,\ d_2)((a,\ 0) + (0,\ x)) = (d_1,\ d_2)(a,\ 0) + (d_1,\ d_2)(0,\ x) \\ &= (d_1(a,\ 0),\ d_2(a,\ 0)) + ((d_1(0,\ x),\ d_2(0,\ x)) \\ &= (d_1(a,\ 0) + d_1(0,\ x)) + (d_2(a,\ 0) + d_2(0,\ x)) \\ &= (D_1(a) + D_2(x),\ R(a) + D_3(x)), \end{aligned} \tag{1}$$

Now

$$\begin{aligned} D((a,\ x)(m,\ x')) &= D(am,\ a \cdot x' + x \circ m + T(am,\ 0)) \\ &= (D_1(am) + D_2(a \cdot x') + D_2(x \circ m) + D_2(T(am,\ 0)), R(am) + D_3(a.x') \\ &\quad + D_3(x \circ m) + D_3(T(am, 0))), \end{aligned} \tag{2}$$

since D is module derivation so

$$\begin{aligned} D((a,\ x)(m,\ x')) &= D(a,\ x) \cdot (m,\ x') + (a,\ x) \cdot D(m,\ x') \\ &= (D_1(a) + D_2(x),\ R(a) + D_3(x)).(m,\ x') \\ &\quad + (a,\ x).(D_1(m) + D_2(x),\ R(m) + D_3(x')) \\ &= ((D_1(a) \cdot m + D_2(x)) \cdot m + a \cdot D_2(x') + D_2(x).m,\ R(a) \cdot T(m,\ 0) \\ &\quad + T(a,\ 0).R(m) + D_3(x).T(m,\ 0) + T(a,\ 0).D_3(x')). \end{aligned} \tag{3}$$

In 3, 2 Take $x = x' = 0$ to get $D_1 \in Z_{\mathfrak{U}}(A,\ M^*)$, (5), (4) and (8). Take $\mathrm{a} = 0$ to get (3) and (6). Take $\mathrm{m} = 0$ to get (2), (7). And in a similar way we can get other parameters. Conversely is in a same way. ■

Corollary 3.2 *Let $X = \{0\}$ and D, D_1 and R be as in perivious lemma, then $D = \delta_{(f,g)}$ if and only if $D_1 = \delta_f$ and $g = \overline{\delta}_g$. Where $\overline{\delta}_g(a) = gT(a,\ 0) - T(a,\ 0) \cdot g$.*

Proof. Since $X = \{0\}$ and $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D(x))$ so $D(a,\ 0) = (D_1(a),\ R(a))$. If $D = \delta_{(f,g)}$ then

$$\begin{aligned} D(a,\ 0) &= \delta_{(f,g)}(a,\ 0) \\ &= (f,\ g)\cdot(a,\ 0) - (a,\ 0)\cdot(f,\ g) \\ &= (f\cdot a,\ g\cdot T(a,\ 0)) - (a\cdot f,\ T(a,\ 0)\cdot g) \\ &= (f\cdot a - a\cdot f,\ g\cdot T(a,\ 0) - T(a,\ 0)\cdot g) = (\delta_f(a),\ \overline{\delta}_g(a)) \ . \end{aligned}$$

So $D_1 = \delta_f$ and $R = \overline{\delta}_g$. Conversely

$$\begin{aligned} D(a,\ 0) &= (D_1(a),\ R(a)) \\ &= (\delta_f(a),\ \overline{\delta}_g(a)) \\ &= (f\cdot a - a\cdot f,\ g\cdot T(a,\ 0) - T(a,\ 0)\cdot g) \\ &= (f,\ g)\cdot(a,\ 0) - (a,\ 0)\cdot(f,\ g) \\ &= \delta_{(f,g)}(a,\ 0) \ . \end{aligned}$$

■

Theorem 3.3 *The module amenability of $A \oplus_T X$ implies module amenability of A. Moreover if $T(A,\ 0) = X$, then X is also module amenable.*

Proof. Assume that M, N are Banach $A - \mathfrak{U}$-bimodule and Banach $X - \mathfrak{U}$-bimodule respectively. Let $D_1 : A \to M^*$ and $D_2 : X \to N^*$ be module derivations. By Proposition 2.5, $M \oplus N$ is a Banach $A \oplus_T X - \mathfrak{U}$- bimodule. Define $D' : A \oplus_T X \to M^* \oplus N^*; D'(a,\ x) = (D_1(a), D_2(T(a, 0))$. Now

$$\begin{aligned} D'((a,\ x)(m,\ x')) &= D'\ (am,\ a.x' + x \circ m + T(am,\ 0)) \\ &= (D_1(am), D_2(T(am, 0))) \\ &= (D_1(a).m + a.D_1(m),\ D_2(T(a,\ 0)).T(m,\ 0) + T(a,\ 0).D_2(T(m,\ 0))) \\ &= (D_1(a),\ D_2(T(a,\ 0))).(m,\ x') + (a,\ x).(D_1(m),\ D_2(T(m,\ 0))) \\ &= D'(a,\ x).(m,\ x') + (a,\ x).D'(m,\ x)). \end{aligned}$$

Also

$$\begin{aligned} D'(\alpha\cdot(a,\ x)) &= D'(\alpha\cdot a,\ \alpha\nabla x) \\ &= ((D_1(\alpha\cdot a),\ D_2(T(\alpha\cdot a,\ 0))) \\ &= ((\alpha\cdot D_1(a),\ \alpha\cdot D_2(T(a,\ 0))) \\ &= (\alpha\cdot D'(a,\ x)\ . \end{aligned}$$

And

$$\begin{aligned} D'((a,\ x) + (m,\ x') &= D'((a+m,\ x+x')) \\ &= (D_1(a+m),\ D_2(T(a+m),\ 0)) \\ &= (D_1(a) + D_1(m),\ D_2(T(a,\ 0)) + D_2(T(m,\ 0)) \\ &= (D_1(a),\ D_2(T(a,\ 0)) + (D_1(m),\ D_2(T(m,\ 0)) \\ &= D'(a,\ x) + D'(m,\ x'). \end{aligned}$$

So D' is a module derivation. Sice $A \oplus_T X$ is module amenable, there exists $(f,\ g) \in M^* \oplus N^*$ such that $D' = \delta_{(f,g)}$. Thus

$$\begin{aligned} D'(a,\ x) &= \delta_{(f,g)}(a,\ x) \\ &= (f,\ g)\cdot(a,\ x) - (a,\ x)\cdot(f,\ g) \\ &= (f\cdot a + g\cdot T(a,\ 0)) - (a\cdot f,\ T(a,\ 0)\cdot g) \\ &= (f\cdot a - a\cdot f,\ g\cdot T(a,\ 0) - T(a,\ 0)\cdot g)\ . \end{aligned}$$

Consequently $D_1(a) = f \cdot a - a \cdot f$ i.e. $D_1 = \delta_f$ and $D_2(T(a,\ 0)) = \delta_g(T(a, 0))$. Since $T(A,\ 0) = X$, $D_2(x) = \delta_g(x)$ for all $x \in X$. ∎

Theorem 3.4 *The module amenability of A implies module amenability of $A \oplus_T \{0\}$.*

Proof. Let $M \oplus N$ be a Banch $A \oplus_T X - \mathfrak{U}$-bimodule and $D: A \oplus_T X \to M^* \oplus N^*$ be a module derivation. By lemma 3.1, there are D_1, D_2, D_3 and R such that $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D_3(x))$. Since here $X = \{0\}$ so $D(a,\ 0) = (D_1(a),\ R(a))$. Module amenability of A implies there exist $f \in M^*$ such that $D_1 = \delta_f$ and since N^* is an A-U- bimodule and R is module derivation, there exist $g \in N^*$ such that $R = \delta_g$. Thus $D = \delta_{(f,g)}$. ∎

Theorem 3.5 *If $T(A,\ 0) = X$ and $A^2 = A$, then the module amenability of A implies module amenability of $A \oplus_T X$.*

Proof. Let $M \oplus N$ be a Banch $A \oplus_T X - \mathfrak{U}$-bimodule and $D: A \oplus_T X \to M^* \oplus N^*$ be a module derivation. By lemma 3.1, there are D_1, D_2, D_3 and R such that $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D_3(x))$. Since here $T(A,\ 0) = X$ and $A^2 = A$ so $D(a,\ x) = (D_1(a),\ R(a))$. Module amenability of A implies there exist $f \in M^*$ such that $D_1 = \delta_f$ and since N^* is an $A - \mathfrak{U}$-bimodule and R is module derivation, there exist $g \in N^*$ such that $R = \delta_g$. Thus $D = \delta_{(f,g)}$. ∎

Example 3.6 *Let $\mathbb{N}$ be the set of positive integers. Consider $S = (\mathbb{N},\ \vee)$ with the maximum operation $m \vee n = max\{m,\ n\}$, then S is a amenable countable, abelian inverse semigroup with the identity 1. Clearly $E_S = S$. This semigroup is denoted by $\mathbb{N}_\vee$. $l^1(\mathbb{N}_\vee)$ is unital with unit δ_1. Since $\mathbb{N}_\vee$ is amenable and $l^1(\mathbb{N}_\vee)$ is unital so $l^1(\mathbb{N}_\vee)$ is module amenable (as an $l^1(\mathbb{N}_\vee) - l^1(\mathbb{N}_\vee)$)-bimodule. Define $T: l^1(S) \times l^1(S) \to l^1(S)$; $T(\delta_x,\ \delta_y) = \begin{cases} \delta_x, & \delta_y = 0 \\ \delta_{x \vee y}, & \delta_y \neq 0. \end{cases}$ Then $l^1(\mathbb{N}_\vee) \oplus_T l^1(\mathbb{N}_\vee)$ is module amenable.*

4 WEAK MODULE AMENABILITY

The Banach algebra A is called weak module amenable (as an $\mathfrak{U}$-bimodule), if $H_{\mathfrak{U}}(A,\ X) = \{0\}$, where X is a commutative $\mathfrak{U}$-submodule of A^*([2]).

Theorem 4.1 *The weak module amenability of $A \oplus_T X$ implies weak module amenability of A. In addition if $T(A,\ 0) = X$ then X is also weak module amenable.*

Proof. Assume that M, N are commutative $\mathfrak{U}$-submodule of A^* and X^*, respectively. we can show that $M \oplus N$ is a commutative $\mathfrak{U}$-submodule of $(A \oplus_T X)^*$. Let $D_1 \in Z_{\mathfrak{U}}(A,\ M)$ and $D_2 \in Z_{\mathfrak{U}}(X,\ N)$. Define $D : A \oplus_T X \to M \oplus N; D(a,\ x) = (D_1(a), D_2(T(a, 0)))$, it is easy to see that $D \in Z_{\mathfrak{U}}(A \oplus \tau X, M \oplus N)$. Since $A \oplus_T X$ is weak module amenable there is $(f,\ g) \in M \oplus N$ such that $D = \delta_{(f,g)}$ and

$$\begin{aligned} (D_1(a), D_2(T(a,\ 0))) &= D(a,\ x) \\ &= \delta_{(f,g)}(a,\ x) \end{aligned}$$

$$= (f,\ g) \cdot (a,\ x) - (a,\ x) \cdot (f,\ g)$$
$$= (f \cdot a - a \cdot f,\ g \cdot T(a,\ 0) - T(a,\ 0) \cdot g)$$
$$= (\delta_f(a),\ \delta_g(T(a,\ 0)))$$

Hence A, X are weak module amenable. ■

Theorem 4.2 *The weak module amenability of A implies the weak module amenability of* $A \oplus_T \{0\}$.

Proof. Suppose that $M \oplus N$ is a commutative Banach $\mathfrak{U}$-submodule of $(A \oplus \tau\{0\})^*$, and $D \in Z_{\mathfrak{U}}(A \oplus \tau\{0\},\ M \oplus N)$. Then M and N are commutative $\mathfrak{U}$-submodule of A^*. Since $D \in Z_{\mathfrak{U}}(A \oplus \tau\{0\},\ M \oplus N)$, by lemma 3.1 there are $D_1 \in Z_{\mathfrak{U}}(A,\ M)$, and $R \in Z_{\mathfrak{U}}(A,\ N)$, such that $D(a,\ 0) = (D_1(a),\ R(a))$. Since A is weak module amenable so there are $m \in M$ and $n \in N$ such that $D_1 = \delta_m, R = \delta_n$, where $\delta_n(a) = a \cdot n - n \bullet a = T(a,\ 0) \odot n - n.T(a,\ 0)$.

Now

$$\begin{aligned} D(a,\quad x) &= (D_1(a),\quad R(a)) \\ &= (\delta_m(a),\ \delta_n(a)) \\ &= (a \cdot m - a \bullet m,\ T(a,\ 0) \odot n - n.\ T(a,\ 0)) \\ &= (a,\ 0) \cdot (m,\ n) - (m,\ n) \circ (a,\ 0) \\ &= \delta_{(m,n)}(a,\ 0)\ . \end{aligned}$$

■

Theorem 4.3 *If* $T(A,\ 0) = X$ *and* $A^2 = A$, *then the weak module amenability of A implies the weak module amenability of* $A \oplus_T X$.

Proof. The proof is as above theorem. ■

Example 4.4 *Let* $S = \mathbb{N}_\vee$ *be as in Example 3.6, since* $l^1(S)$ *is* $l^1(S) - l^1(S) -$ *module and* $l^1(S)$ *is weak module amenable. Let* $T: l^1(S) \times l^1(S) \to l^1(S)$ *have the properties as above theorems, then* $l^1(S) \oplus_T l^1(S)$ *is weak module amenable.*

5 MODULE APPROXIMATE AMENABILITY

Let A be as above, then A is module approximately amenable (as an $\mathfrak{U}$- bimodule), if for any commutative Banach $A - \mathfrak{U}$-bimodule X, each module derivation $D: A \to X^*$ is approximately inner.

A derivation $D: A \to X$ is said to be approximately inner if there exists a net $(x_i)_i \subseteq X$ such that $D(a) = \lim_i (a \cdot x_i - x_i \cdot a), a \in A$.([10]) .

Lemma 5.1 *Let* D_1, R, D_3 *and* D_2 *are such as in the Lemma 3.1, and* $D(a,\ b) = (D_1(a) + D_2(b),\ R(a) + D_3(b))$. *If* $T(A,\ 0) = X$ *and* $A^2 = A$ *then: D is approximately inner if and only if* D_1 *and R are approximately inner.*

Proof. Assume that M is a commutative $A - \mathfrak{U}$-bimodule and also N is com- mutative $X - \mathfrak{U}$-bimodule, then $M \oplus N$ is a commutative $A \oplus_T X - \mathfrak{U}$-bimodule. Let D be approximately inner so there is $(f_i,\ g_i)_i \subseteq M^* \oplus N^*$ such that

$$
\begin{aligned}
D(a,\quad x) &= T(a',\quad 0)\\
&= \lim_i((a,\ x)\cdot(f_i,\ g_i)-(f_i,\ g_i)\cdot(a,\ x))\\
&= \lim_i((a\cdot f_i,\ T(a,\ 0)\cdot g_i)-(f_i\cdot a,\ g_i.T(a,\ 0)))\\
&= \lim_i(a\cdot f_i-f_i\cdot a,\ T(a,\ 0)\cdot g_i-g_i\cdot T(a,\ 0)),
\end{aligned}
$$

i.e. $D(a)=\lim_i(a\cdot f_i-f_i\cdot a)$ and $R(a)=\lim_i(T(a,\ 0)\cdot g_i-g_i\cdot T(a,0))$.

Conversely, let $D_1(a)=\lim_{i\in I}(a\cdot f_i-f_i\cdot a)$ and $R(a)=\lim_{j\in J}(T(a,\ 0)\cdot g_j-g_j\cdot T(a,0))$

Since the index sets $(I,\ \leq),(J,\ \leq)$ are ordered sets, so the set $\Lambda=I\times J=\{(i,\ j)\ :\ i\in I,\ j\in J\}$ is ordered as follows

$$(i,\ j)\leq(i',\ j')\Leftrightarrow(i\leq i',\ j\leq j).$$

For $\lambda=(i,\ j)\in\Lambda$ set $t_\lambda=(f_i,\ g_j)$. Let $\epsilon>0$ be given. Since $D_1(a)=\lim_i(a\cdot f_i-f_i\cdot a)$ and $R(a)=\lim_j(T(a,\ 0)\cdot g_j-g_j\cdot T(a,0))$ there are $i_0\in I,j_0\in J$ such that

1) For all $i\geq i_0$, $\| D_1(a)-(a\cdot f_i-f_i\cdot a)\|\leq\frac{\epsilon}{3}$.

2) For all $j\geq j_0$, $\| R(a)-(T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\leq\frac{\epsilon}{3}$.

Now set $\lambda_0=(i_0,\ j_0)$, then for all $\lambda\geq\lambda_0$, since $D(a,\ T(a',\ 0))=(D_1(a),\ R(a))$, we have

$$
\begin{aligned}
\| D(a,\quad x)-((a,\quad x)\cdot t_\lambda-t_\lambda\cdot(a,\quad x))\| &=\| D(a,\quad x)-((a,\quad x)\cdot(f_i,\ g_j)-(f_i,\quad g_j)\cdot(a,\quad x))\|\\
&=\| D(a,\ x)-(a\cdot f_i-f_i\cdot a,\ T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\\
&=\|(D_1(a),\ R(a))-(a\cdot f_i-f_i\cdot a,\ T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\\
&=\|((D_1(a)-(a\cdot f_i-f_i\cdot a)),R(a)-(T(a,\ 0)\cdot g_j-g_j\cdot T(a,0)))\|\\
&\leq\| D_1(a)-(a\cdot f_i-f_i\cdot a)\|+\| R(a)-(T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\\
&<\epsilon.
\end{aligned}
$$

Hence $D(a,\ x)=\lim_\lambda((a,\ x)\cdot t_\lambda-t_\lambda(a,\ x$ where $x=T(a',\ 0)$ i.e. D is approximately inner. ■

Theorem 5.2 *If $A\oplus\tau X$ is module approximately amenable then A is module approximately amenable. Furthermore, if $T(A,\ 0)=X$ also X is module approximately amenable.*

Proof. In an argument as in the proof of Theorem 3.3 and the application, the usage of above lemma. ■

Theorem 5.3 *If $T(A.O)=X$ and $A^2=A$ then the module approximate amenability of A implies the module approximate amenability of $A\oplus_T X$.*

Proof. Let $M\oplus N$ be a commutative $A\oplus_T X-\mathfrak{U}$-bimodule and $D\in Z_{\mathfrak{U}}(A\oplus\tau X,M^*\oplus N^*)$. There are $D_1\in Z_{\mathfrak{U}}(A,\ M^*),D_3\in Z_{\mathfrak{U}}(X,\ N^*),R\in Z_{\mathfrak{U}}(A,\ N^*)$ and $D_2:X\to N^*$ such that $D(a,\ x)=(D_1(a)+D_2(x),\ R(a)+D(x))$ and since $T(A,\ 0)=X$ and $A^2=A$ we have $D(a,\ x)=(D_1(a),\ R(a))$. Since A,X are module approximate amenable, so D_1 and R are approximatly inner. Thus by the above lemma, D is approximately inner. ■

Example 5.4 *Let S be an amenable inverse semigroup such that the set of idempotents E_S be equal to S and $l^1(S)$ has approximately unit. Since S is amenable, $l^1(S)$ is module approximately amenable, [10]. Also $l^1(S)$ is $l^1(S) - l^1(S)$-bimodule, thus $l^1(S) \oplus_T l^1(S)$ is module approximately amenable. Where $T: l^1(S) \times l^1(S) \to l^1(S)$ is defined by*

$$T(\delta_x,\ \delta_y) = \begin{cases} \delta_x, & \delta_y = 0 \\ \delta_{xy}, & \delta_y \neq 0. \end{cases}$$

6 CONCLUSIONS

The module amenability of $A \oplus_T X$ implies module amenability of A and The module amenability of A implies module amenability of $A \oplus_T \{0\}$. Also If $T(A,\ 0) = X$ and $A^2 = A$, then the module amenability of A implies module amenability of $A \oplus_T X$. meanwhile, The weak module amenability of $A \oplus_T X$ implies weak module amenability of A. On the contrary, if $T(A,\ 0) = X$ and $A^2 = A$, then the weak module amenability of A implies the weak module amenability of $A \oplus_T X$.

Considering approximately, if $A \oplus \tau X$ is module approximately amenable then A is module approximately amenable. On the contrary, if $T(A,\ 0) = X$ and $A^2 = A$ then the module approximate amenability of A implies the module approximate amenability of $\oplus_T X$. For example, we have S be an amenable inverse semigroup such that the set of idempotents E_S be equal to S and $l^1(S)$ has approximately unit. Since S is amenable, $l^1(S)$ is module approximately amenable.

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SOME NEW INTEGRAL INEQUALITIES THROUGH THE STEKLOV OPERATOR

B. BENAISSA[1], A. SENOUCI [2*]

[1]Faculty of Material Sciences, University of Tiaret-Algeria.
[2]Department of Mathematics, University of Tiaret-Algeria.
Laboratory of Informatics and Mathematics, University of Tiaret-Algeria.

*Corresponding author. E-mail: kamer295@yahoo.fr

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Summary. Hardy and Copson type inequalities have been studied by a large number of authors during the twentieth century and has motivated some important lines of study which are currently active. A large number of papers have been appeared involving Copson and Hardy inequalities (see [2-16] for more details).

In this paper some Hardy-Steklov and Copson-Steklov type integral inequalities were established. Namely the integral inequalities were proved there.

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(\mathcal{F}_s f)^p(x)dx \le \left(\frac{p}{|p-\alpha-1|}\right)^p \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)}|K(x)|^q dx\right)^{\frac{p}{q}}, \quad (*)$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx \le \left(\frac{p}{|p-\alpha-1|}\right)^p \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)}|J(x)|^q dx\right)^{\frac{p}{q}}. \quad (**)$$

Where $(\mathcal{F}_s f)$ is the Hardy-Steklov type operator and $(\mathcal{C}_s f)$ is the Copson-Steklov type operator (see the main results for more details).

Several Hardy-Steklov type, Hardy-type and Hardy integral inequalities were derived from $(*)$. Similarly, some Copson-Steklov type and Copson type integral inequalities are deduced from $(**)$.

1. INTRODUCTION

In 1928, G.H. Hardy proved the following integral inequalities [6]. Let f non-negative measurable function on $(0, \infty)$

$$(\mathcal{F}f)(x) = \begin{cases} \int_0^x f(t)dt & \text{for } \alpha < p-1, \\ \int_x^\infty f(t)dt & \text{for } \alpha > p-1, \end{cases}$$

then

$$\int_0^\infty x^{\alpha-p}(\mathcal{F}f)^p(x)dx \le \left(\frac{p}{|p-\alpha-1|}\right)^p \int_0^\infty x^\alpha f^p(x)dx, \quad \text{for } p > 1. \quad (1)$$

In 1976, E.T. Copson proved the following integral inequalities (see [4], Theorem 1, Theorem 3). Let f, ϕ non-negative measurable functions on $(0, \infty)$

$$\Phi(x) = \int_0^x \phi(t)dt, \qquad (\mathcal{C}f)(x) = \begin{cases} \int_0^x f(t)\phi(t)dt, & \text{for } c > 1, \\ \int_x^\infty f(t)\phi(t)dt, & \text{for } c < 1, \end{cases}$$

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then

$$\int_0^b (\mathcal{C}f)^p(x)\Phi^{-c}(x)\phi(x)dx \le \left(\frac{p}{|c-1|}\right)^p \int_0^b f^p(x)\Phi^{p-c}(x)\phi(x)dx, \quad \text{for } p \ge 1. \tag{2}$$

Inequality (2) can be easily rewritten in the following form

$$(\mathcal{C}f)(x) = \begin{cases} \int_0^x f(t)\phi(t)dt, & \text{for } \alpha < p-1, \\ \int_x^\infty f(t)\phi(t)dt, & \text{for } \alpha > p-1, \end{cases}$$

then

$$\int_0^b (\mathcal{C}f)^p(x)\Phi^{\alpha-p}(x)\phi(x)dx \le \left(\frac{p}{|p-1-\alpha|}\right)^p \int_0^b f^p(x)\Phi^{\alpha}(x)\phi(x)dx, \quad \text{for } p \ge 1. \tag{3}$$

The Hardy-Steklov operator is defined by

$$(Tf)(x) = g(x)\int_{r(x)}^{h(x)} f(t)dt, \qquad f \ge 0,$$

where g is a positive measurable function and r, h are functions defined on an interval (a,b) such that $r(x) < h(x)$ for all $x \in (a,b)$.

Particular cases of this operator are Hardy operator $(\mathcal{F}f)(x) = \int_0^x f(t)dt$, the Hardy averaging operator $(F_\mu f)(x) = x^\mu \int_0^x f(t)dt$ and the Steklov operator $(Sf)(x) = \int_{x-1}^{x+1} f(t)dt$, which has been studied intensively (see [9] for example).

Let f, v, ϕ be non-negative measurable functions on $(0,\infty)$. Suppose that r and h are increasing differentiable functions on $[0,\infty)$, such that

$$\begin{cases} 0 < r(x) < h(x) < \infty \quad \text{for all} \quad x \in (0,\infty), \\ r(0) = h(0) = 0 \quad \text{and} \quad r(\infty) = h(\infty) = \infty. \end{cases} \tag{4}$$

The Hardy-Steklov and Copson-Steklov type operators are defined as follows,

$$(\mathcal{F}_s f)(x) = \int_{r(x)}^{h(x)} f(y)v(y)dy, \qquad x > 0, \tag{5}$$

$$(\mathcal{C}_s f) = \int_{r(x)}^{h(x)} \frac{f(y)\phi(y)}{\Phi(y)}dy, \qquad x > 0, \tag{6}$$

where

$$\Phi(x)=\int_0^x \phi(t)dt, \quad \text{for} \quad x\in (0,\infty).$$

We adopt the usual convention: $\frac{0}{0}=\frac{\infty}{\infty}=0$.

2. MAIN RESULTS

Let $0<b\leq\infty$. Throughout the paper, we will assume that the integrals exist and are finite. The following lemma is needed in the proof of the main results (was proved in [1]).

Lemma 2.1. Let $1<p\leq q<\infty$ and f,g,w be non-negative measurable functions on (a,b) such that $W(x)=\int_0^x w(t)dt$. If $m\in\mathbb{R}, m\neq 1$, then

$$\int_a^b \frac{w(x)}{W^m(x)} g^p(f(x))dx \leq \left(\int_a^b w(x)dx\right)^{1-\frac{p}{q}} \left(\int_a^b \frac{w(x)}{W^{\frac{mq}{p}}(x)} g^q(f(x))dx\right)^{\frac{p}{q}}. \tag{7}$$

Remark 2.1. Let $V(x)=\int_0^x v(t)dt$. By putting $m=p-\alpha$ in inequality (7), $w(x)=v(x)$, $W(x)=V(x)$ (respectively $w(x)=\phi(x)$, $W(x)=\Phi(x)$ and $f(x)=g(f(x))$), we obtain

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} f^p(x)dx \leq \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} f^q(x)dx\right)^{\frac{p}{q}}. \tag{8}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} f^p(x)dx \leq \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} f^q(x)dx\right)^{\frac{p}{q}}. \tag{9}$$

The main results are presented in the following Theorem and Corollaries.

Theorem 2.1. Let f,v,ϕ be non-negative measurable functions on $(0,\infty)$, $1<p\leq q<\infty$ and $r(x)$, $h(x)$ satisfied the conditions (4). If $\alpha<p-1$, then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_s f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} |K(x)|^q dx\right)^{\frac{p}{q}}, \tag{10}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} |J(x)|^q dx\right)^{\frac{p}{q}}, \tag{11}$$

where

$$K(x)=\frac{V(x)}{v(x)}\left\{[V(h(x))]'f(h(x))-[V(r(x))]'f(r(x))\right\},$$

$$J(x)=\frac{\Phi(x)}{\phi(x)}\left\{\frac{[\Phi(h(x))]'}{\Phi(h(x))}f(h(x))-\frac{[\Phi(r(x))]'}{\Phi(r(x))}f(r(x))\right\}.$$

Proof. We consider the inequality (11), then

$$(\mathcal{C}_s f)'(x) = h'(x)\frac{\phi\left(h(x)\right)}{\Phi\left(h(x)\right)} f\big(h(x)\big) - r'(x)\frac{\phi\left(r(x)\right)}{\Phi\left(r(x)\right)} f\big(r(x)\big)$$

$$= \frac{\phi(x)J(x)}{\Phi(x)},$$

integrating by part in the left-hand side of (11), we get

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx = \left[\frac{-(\mathcal{C}_s f)^p(x)}{(p-\alpha-1)\Phi^{p-\alpha-1}(x)}\right]_0^b + \frac{p}{p-\alpha-1}$$

$$\times \int_0^b \frac{\phi(x)J(x)(\mathcal{C}_s f)^{p-1}(x)}{\Phi^{p-\alpha}(x)}dx.$$

Since $\alpha < p-1$, we have

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx \leq \frac{p}{p-\alpha-1}\int_0^b \frac{\phi(x)J(x)(\mathcal{C}_s f)^{p-1}(x)}{\Phi^{p-\alpha}(x)}dx.$$

The Hölder integral inequality for $\frac{1}{p}+\frac{1}{p'} = 1$, gives

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx \leq \frac{p}{p-\alpha-1}\left(\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx\right)^{\frac{1}{p'}}$$

$$\times\left(\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}|J(x)|^p dx\right)^{\frac{1}{p}},$$

therefore

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(\mathcal{C}_s f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}|J(x)|^p dx.$$

Finally, by using inequality (9), we get (11).

The proof of inequality (10) is similar. So, the proof of Theorem is complete.

Now let $r(x) = 0$ in (5) and (6), thus

$$\left(\mathcal{F}_{s,1}f\right)(x) = \int_0^{h(x)} f(y)v(y)dy, \qquad x > 0,$$

$$(\mathcal{C}_{s,1}f)(x) = \int_0^{h(x)} \frac{f(y)\phi(y)}{\Phi(y)} dy, \qquad x > 0 .$$

If we set q = p in (10) and (11), we obtain the following corollary.

Corollary 2.1. Let f, v, ϕ be non-negative measurable functions on $(0,\infty)$, $p > 1$, $\alpha < p-1$ and

$$\begin{cases} 0 < h(x) < \infty \quad \text{for all} \quad x \in (0,\infty), \\ h(0) = 0 \quad \text{and} \quad h(\infty) = \infty. \end{cases} \tag{12}$$

Then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_{s,1}f)^p(x)dx \le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{v(x)}{V^{p-\alpha}(x)} |K_1(x)|^p dx, \tag{13}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_{s,1}f)^p(x)dx \le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} |J_1(x)|^p dx, \tag{14}$$

where

$$K_1(x) = \frac{V(x)[V(h(x))]' f(h(x))}{v(x)},$$

$$J_1(x) = \frac{\Phi(x)}{\phi(x)} \frac{[\Phi(h(x))]'}{\Phi(h(x))} f(h(x)) .$$

Remark 2.2. If $h(x) = x$ in Corollary 2.1, we obtain the following weighted Hardy inequality and Copson-type inequality

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_{s,2}f)^p(x)dx \le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{v(x)}{V^{-\alpha}(x)} f^p(x)dx, \tag{15}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_{s,2}f)^p(x)dx \le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} f^p(x)dx, \tag{16}$$

where

$$(\mathcal{F}_{s,2}f)(x) = \int_0^x f(y)v(y)dy, \qquad x > 0 ,$$

$$(\mathcal{C}_{s,2}f)(x) = \int_0^x \frac{f(y)\phi(y)}{\Phi(y)} dy, \qquad x > 0 .$$

If we put $h(x) = \lambda x$ and $r(x) = \beta x$ and $q = p$ in Theorem 2.1, we get following corollary.

Corollary 2.2. Let f, v, ϕ be non-negative measurable functions on $(0, \infty)$, $0 < \beta < \lambda < \infty$, $p > 1$ and

$$\left(\mathcal{F}_{s,3} f\right)(x) = \int_{\beta x}^{\lambda x} f(y) v(y) dy, \qquad x > 0,$$

$$\left(\mathcal{C}_{s,,3} f\right)(x) = \int_{\beta x}^{\lambda x} \frac{f(y)\phi(y)}{\Phi(y)} dy, \qquad x > 0.$$

If $\alpha < p - 1$, then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} \left(\mathcal{F}_{s,3} f\right)^p (x) dx \leq \left(\frac{p}{p - \alpha - 1}\right)^p \int_0^b \frac{v(x)}{V^{p-\alpha}(x)} |K_2(x)|^p dx, \tag{17}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} \left(\mathcal{C}_{s,3} f\right)^p (x) dx \leq \left(\frac{p}{p - \alpha - 1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} |J_2(x)|^p dx. \tag{18}$$

Where

$$K_3(x) = \frac{V(x)[\lambda\, v(\lambda x) f(\lambda x) - \beta\, v(\beta x) f(\beta x)]}{v(x)},$$

$$J_3(x) = \frac{\Phi(x)}{\phi(x)} \left\{ \frac{\lambda\, \phi(\lambda x)}{\Phi(\lambda x)} f(\alpha x) - \frac{\beta\, \phi(\beta x)}{\Phi(\beta x)} f(\beta x) \right\}.$$

Remark 2.3. One can prove the boundedness of the operator $\mathcal{F}_{s,3}$ from $L_p(0, \infty)$ to $L_p(0, \infty)$ by using the Minkowski integral inequality for $p > 1$, it means that $\left\| \left(\mathcal{F}_{s,3} f\right)(x) \right\|_{L_p(0,\infty)} \leq C_{(\lambda,\beta,p)} \|f(x)\|_{L_{p,v}(0,\infty)}$, where $L_p(0, \infty)$ is the classical Lebesgue space and $L_{p,v}(0, \infty)$ is the weighted Lebesgue space, with the following norm $\|f(x)\|_{L_{p,v}(0,\infty)} = \left(\int_0^\infty |f(x) v(x)|^p dx\right)^{\frac{1}{p}}$ and $C_{(\lambda,\beta,p)}$ is a positive constant depending only on λ, β and p.

Remark 2.4. For $\lambda = 1$ and $\beta = \frac{1}{2}$, we get a Pachpatte-type inequality.

Let

$$(\mathcal{F}_s^* f)(x) = \int_{r(x)}^{\infty} f(y) v(y) dy, \qquad (\mathcal{C}_s^* f)(x) = \int_{r(x)}^{\infty} \frac{f(y)\phi(y)}{\Phi(y)} dy, \qquad x > 0,$$

with

$$\begin{cases} 0 < r(x) < \infty \quad \text{for all} \quad x \in (0,\infty), \\ \quad r(0) = 0 \quad \text{and} \quad r(\infty) = \infty. \end{cases} \tag{19}$$

By setting $h(x) = \infty$ and reasoning a manner analogous to the proof of Theorem 2.1, we get the following corollary.

Corollary 2.3 Let f, v, ϕ be non-negative measurable functions on $(0,\infty)$, $1 < p \leq q < \infty$. If $\alpha > p - 1$, then

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (\mathcal{F}_s^* f)^p(x)dx \leq \left(\frac{p}{\alpha - p + 1}\right)^p \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} |K^*(x)|^q dx\right)^{\frac{p}{q}}, \tag{20}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (\mathcal{C}_s^* f)^p(x)dx \leq \left(\frac{p}{\alpha - p + 1}\right)^p \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} |J^*(x)|^q dx\right)^{\frac{p}{q}}, \tag{21}$$

where

$$K^*(x) = -\frac{V(x)\,[V(r(x))]' f(r(x))}{v(x)},$$

$$J^*(x) = -\frac{\Phi(x)}{\phi(x)} \frac{[\Phi(r(x))]'}{\Phi(r(x))} f(r(x)).$$

Remark 2.5. The following particular case of Corollary 2.3 can be derived by taking $\mathrm{r}(x) = x$ *and* $q = p$.

$$\int_0^b \frac{v(x)}{\mathrm{V}^{p-\alpha}(x)} \left(\widetilde{\mathcal{F}_s^*} f\right)^p(x)dx \leq \left(\frac{p}{\alpha - p + 1}\right)^p \int_0^b \frac{v(x)}{\mathrm{V}^{-\alpha}(x)} f^p(x)dx, \tag{22}$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} \left(\widetilde{\mathcal{C}_s^*} f\right)^p(x)dx \leq \left(\frac{p}{\alpha - p + 1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} f^p(x)dx, \tag{23}$$

where

$$\left(\widetilde{\mathcal{F}_s^*} f\right)(x) = \int_x^\infty f(y)v(y)dy, \qquad x > 0, \quad \left(\widetilde{\mathcal{C}_s^*} f\right)(x) = \int_x^\infty \frac{f(y)\phi(y)}{\Phi(y)} dy, \qquad x > 0.$$

Remark 2.6 We note that if $v(x) = 1$ in the inequalities (15) and (22), we get the Hardy inequalities (1).

3. CONCLUSION

By using Hardy-Steklov and Copson-Steklov type operators and by introducing a second parameter of integrability q, some new integral inequalities were established and proved. These integral inequalities generalize certain classical inequalities like those of Hardy Copson and Pachpatte. As a perspective, we propose to extended these results to $\mathbb{R}^n$ or subsets of $\mathbb{R}^n$ for $n \geq 2$. Also it would of interest to try apply some of this integral inequalities in the study of deferent fields of mathematics (partial deferential equations, functional spaces, mathematical modeling, …).

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TWO-LAYER FINITE-DIFFERENCE SCHEMES FOR THE KORTEWEG-DE VRIES EQUATION IN EULER VARIABLES

V.I. MAZHUKIN*, A.V. SHAPRANOV, E.N. BYKOVSKAYA

Keldysh Institute of Applied Mathematics of the RAS, Moscow, Russia.

* Corresponding author. E-mail: vim@modhef.ru

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Summary. A family of weighted two-layer finite-difference schemes is presented. Using the example of the numerical solution of model problems on the propagation of a single soliton and the interaction of two solitons, the high quality of explicit-implicit schemes of the Crank-Nichols type with the parameter σ = 0.5 and the order of approximation $O(\Delta t^2 + \Delta x^2)$ is shown. Completely implicit two-layer difference schemes with the parameter σ = 1 and O ($\Delta t + \Delta x^2$) are characterized by absolute stability with a low solution accuracy due to a high approximation error. The family of explicitly implicit difference schemes is absolutely unstable if the explicitness parameter σ <0.5 prevails. Analysis of the structure of the approximation error, performed using the modified equation method, confirmed the results of numerical simulation.

1 INTRODUCTION

The theory of nonlinear waves was originally associated with the study of problems in gas and hydrodynamics, which include a number of varied and striking problems of applied and fundamental nature [1], which lead to the need to analyze the huge and growing data associated with multidimensional nonlinear dynamics.

Initially, the Korteweg de Vries equation (KdV) arose from the needs of hydrodynamics [2], [3] associated with the propagation of nonlinear solitary waves in shallow water [4, 5], which ended with the discovery of solitons [6]. The KdV equation was the first nonlinear wave equation to have soliton solutions. Note that the discovery of solitons [6] was carried out on the basis of a computational experiment. As it turned out, solitons, which are stable formations, have a number of amazing properties. Thus, the propagation of a soliton in the form of a nonlinear solitary wave allows it to maintain its shape and speed during its motion. In addition, solitons are characterized by elastic interaction with each other. In the course of a collision, they first deform and then restore their original parameters and their original shape. Taking into account that the propagation of a soliton is described by a nonlinear equation, then the principle of superposition, as it is understood in linear systems, according to which the sum of particular solutions is also a solution, does not hold for it. Solitons exactly interact with each other, first deforming, and then, restoring their original parameters, in contrast to linear solution systems, which pass through each other. The only result of the interaction of solitons may be some phase shift. This confirms that solitons are precisely nonlinear solutions.

Due to the rapid development of high-performance computing technology, computational algorithms and methods of modern mathematical modeling, it became possible to study more

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and more complex problems of hydrodynamics [7], nonlinear optics [8], plasma physics [9, 10] and solids [11]. However, since the complexity of the problems under study is ahead of the development of the used computer technology, the problem of increasing the efficiency of mathematical methods and approaches remains relevant.

The active use of solitons in the study and solution of nonlinear wave equations [12] describing physical phenomena in many areas [13] stimulated interest in methods for solving the KdV equation. The KdV equation was solved numerically by various methods, such as the Galerkin method [14-16], the collocation method [17, 18], the finite element method [19-21], the finite-difference method [22-30], etc. The choice of one or another numerical solution method largely determines the quality of numerical modeling. It should also be kept in mind that it is far from being indifferent, at the expense of what costs the final result of modeling is achieved. Therefore, it is quite natural to impose on computational algorithms the requirement not only of stability and efficiency, but also of simplicity of implementation. Finite-difference methods possess the greatest combination of these properties.

Earlier, in [24, 25], the results of the analysis of two-layer difference schemes for the KdV equation from the point of view of integral conservation laws were reported. The concept of L_2-conservatism of a difference scheme was used as the ability of its solution to satisfy the grid analogue of conservation laws [31, 32]. The L_2-conservatism principle makes it possible, when constructing efficient algorithms, to ensure that they satisfy the grid analogs of the basic properties of the differential problem.

Based on the L_2-conservatism principle for the Korteweg-de Vries equation, it was shown that explicit two-layer difference schemes do not satisfy the L_2-conservatism condition and, moreover, are absolutely unstable even in the weakest L_2 norm. In the same papers, this principle was applied to construct a family of three-layer completely conservative (conservative and L_2-conservative) weighted difference schemes.

In this paper, we numerically and analytically study a family of two-layer difference schemes for the KdV equation, which includes both explicit and implicit schemes.

2 STATEMENT OF THE PROBLEM

KdV equation in the divergence form

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) + \beta \frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

includes nonlinear and dispersion terms, the competition of which determines the behavior of the solution. The solution of equation (1) is represented in the form of a moving soliton.

The soliton is a stationary unipolar pulse traveling in the positive direction of the X axis with a speed Q

$$u(x,t) = \frac{A}{\mathrm{ch}\left[(x - x_0 - Qt)/\delta\right]^2}, \tag{2}$$

where $A = 3Q/\alpha$ is the amplitude and $\delta = \sqrt{4\beta/Q}$ is the half-width (at the level of $0.42A$) of the soliton. The analytical representation of the soliton will be used to test the computational method.

To complete the formulation of the Cauchy problem, it is necessary to set the boundary conditions. As the initial condition, the grid representation of the soliton (2) is specified. Equation (1) requires the setting of three boundary conditions on two boundaries of the final computational domain.

At the left boundary: $\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = 0$, At the right boundary $u|_{x=L} = 0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$. (3)

3 FINITE-DIFFERENCE APPROXIMATION

In the space of variables $\Omega_{(x,t)}$, we construct a difference grid uniform in x

$$\omega_{\Delta x}^{\Delta t} = \left\{ (x_m, t^k):\ \ x_{m+1} = x_m + \Delta x,\ \ t^{k+1} = t^k + \Delta t^k,\ \ m = 0,\dots,M,\ \ k = 0,\dots,K \right\}, \tag{4}$$

on which the grid function is defined $u_m^k \equiv u(x_m, t^k)$.

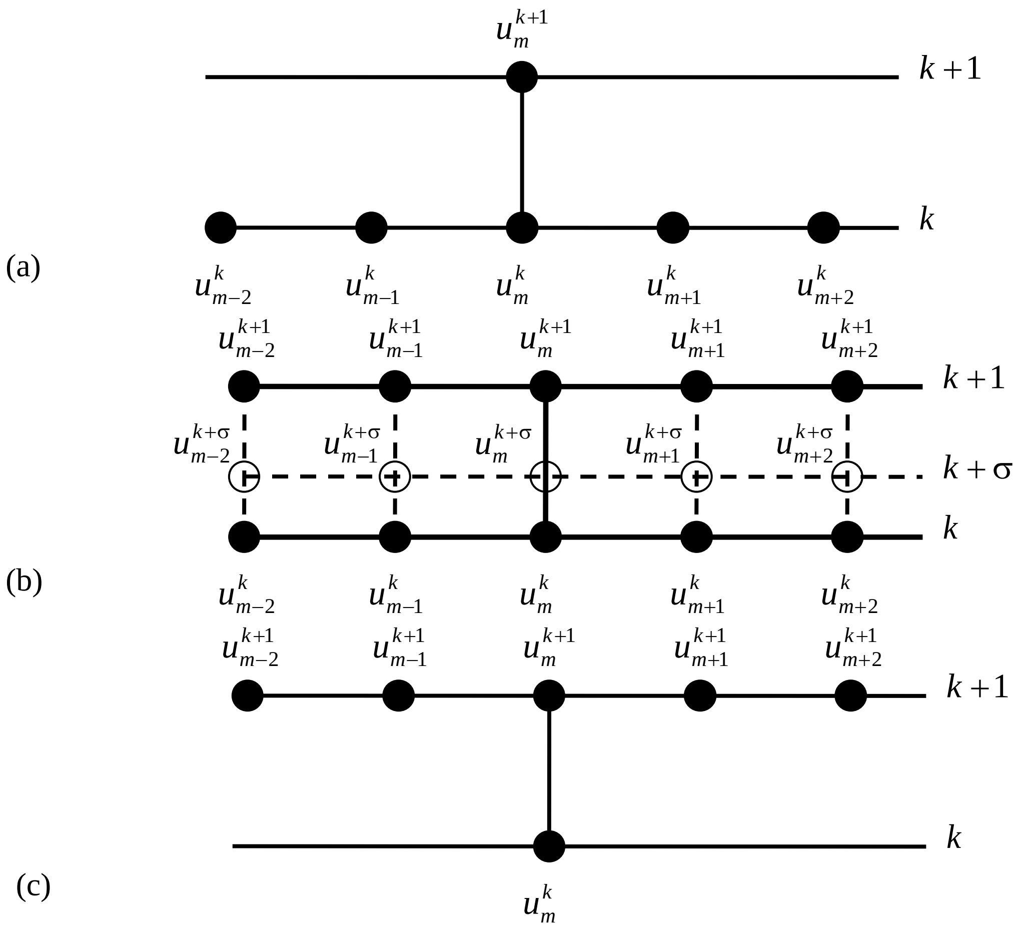


Fig. 1. Grid patterns, implemented by the scheme (5), for different values of σ. (a) σ=0, i.e. $u^{k+\sigma} = u^k$ (explicit scheme); (b) $0<\sigma<1$, empty circles are fictional nodes at the intermediate time layer $(k+\sigma)$; (c) σ=1, i.e. $u^{k+\sigma} = u^{k+1}$.

Using (4), we construct a family of finite-difference schemes for the approximation of the equation (1):

$$\frac{u_m^{k+1}-u_m^k}{\Delta t}+\frac{\alpha}{\Delta x}\left(\frac{(u_{m+1}^{k+\sigma}+u_m^{k+\sigma})^2}{8}-\frac{(u_m^{k+\sigma}+u_{m-1}^{k+\sigma})^2}{8}\right)+\beta\frac{u_{m+2}^{k+\sigma}-2u_{m+1}^{k+\sigma}+2u_{m-1}^{k+\sigma}-u_{m-2}^{k+\sigma}}{2\Delta x^3}=0 \qquad (5)$$

where $u_m^{k+\sigma}=(1-\sigma)u_m^k+\sigma u_m^{k+1}$, σ is the weight coefficient, which values σ ϵ [0 – 1] determine the degree of "implicitness" of the difference scheme. The value σ = 0 corresponds to a completely explcit scheme, while σ = 1 – to a completely implicit one.

In all finite-difference schemes (5), the second order of approximation in spatial x is implemented for the derivatives of both the 1st and the 3rd order. With respect to the time variable t for all values σ≠0.5, the schemes (5) have the 1st order of approximation. For, σ = 0.5, the expression (5) is an implicit difference scheme of the Crank-Nichols type with the second order of approximation in time. Fig. 1. shows grid patterns for different values of σ.

4 COMPUTATIONAL ALGORITHM

The difference approximation (5), applied to the interior points of the computational domain, generates a system of nonlinear equations with respect to quantities $\tilde{u}$ on a new time layer. This system is solved at each time step by the Newton iterative method, for which the procedure of its linearization is performed, after which it is transformed into a linear system of equations with a 5-diagonal (band) matrix. In the only case of σ=0, the matrix degenerates into the identity one, and the scheme becomes explicit.

5 COMPUTATIONAL EXPERIMENT

For numerical testing, the following parameters of the equation $\alpha=6,\ \beta=1$ and soliton (2) presented in Fig. 2(a) were used: $x_0=10,\ Q=4\ \Rightarrow\ A=2,\ \delta=1$;

In the computational domain with the size $L=420$, a computational grid was constructed containing $M=2100$ intervals with a spatial step size $\Delta x=0.2$. The total width of the soliton 2δ contained 10 intervals.

For implicit schemes σ≠0, a mechanism for automatic time step selection was implemented, based on the following parameters: the maximum allowable number of iterations at each time step was 3-4, the criterion of convergence of the iterative process includes the relative and absolute errors, the values of which were taken equal to 10^{-9}. For an explicit scheme (σ=0), the time step is discussed below.

Figure 2(b) shows the solution using the Crank-Nichols-type scheme (σ=0.5) at the time *t*=100 when the soliton has moved from the initial position to a distance of 400δ. The transfer speed determined from the numerical solution turned out to be 1.1% less than the analytical value (i.e., the lag was about 4δ). In this case, the amplitude of the soliton fluctuates around the average value with a standard deviation of 0.3%, and the average value itself is only 0.011% greater than the analytical one (Fig. 4(a)). That is, it can be assumed that the numerical solution preserves the amplitude of the soliton with good accuracy during the entire calculation process.

The time step during the entire computational process fluctuated with a small amplitude around a constant value (Fig. 4(b)). These small fluctuations were associated with the organization of the automatic step selection mechanism.

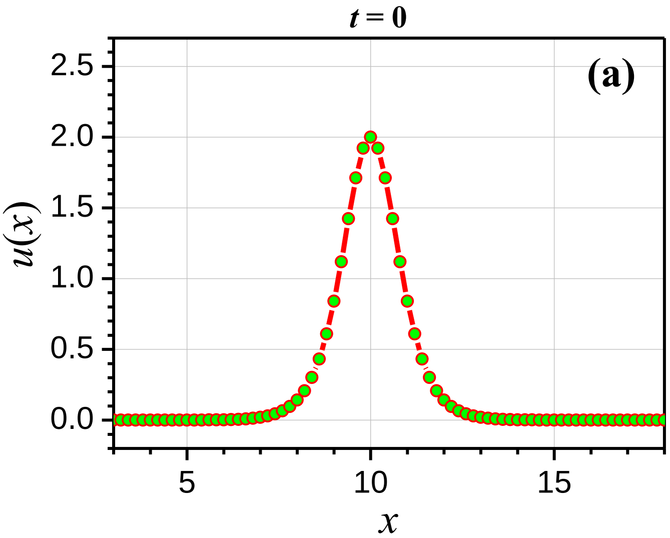
(a) 

Fig.2 (a). Spatial profile of soliton at the initial time t=0.

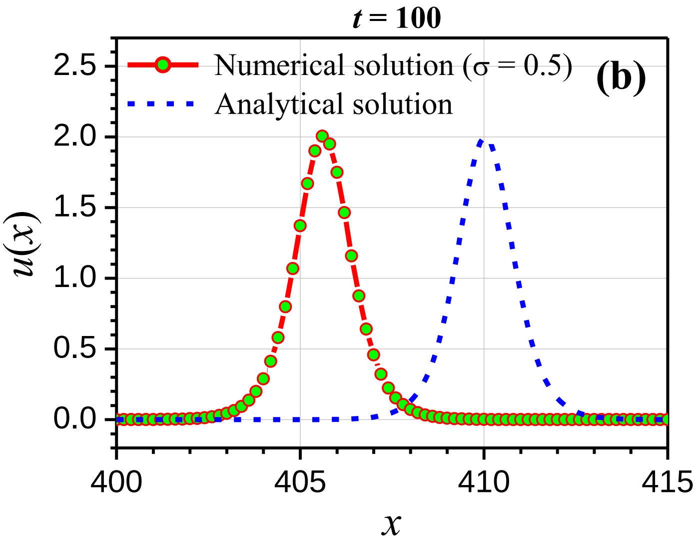
(b) 

Fig.2 (b). Comparison of the numerical solution obtained using a scheme of Crank-Nichols type (σ=0.5) with the analytical one at the time t=100.

The solution using a completely implicit scheme (σ=1) leads to significantly worse results (Fig. 3). This is due to the high schematic viscosity, which in this case is the reason for a strong drop in the amplitude of the soliton with time, which in turn decreases its velocity (Fig. 4(a)). In this case, as the amplitude decreases, the integration step increases (Fig.4(b)). However, at the initial moments of time, when the amplitude is not yet very different from the initial value, the time step turns out to be about 2 times less than for the Crank-Nichols type scheme.

Figure 5 shows a numerical solution using an explicit difference scheme (σ=0). Figure (a) shows the solution with a time step $\Delta t = 0.0001$. By the moment of time t=3, a loss of stability occurs and further calculation becomes impossible. Note that a decrease in the integration step pushes further in time the moment of stability loss.

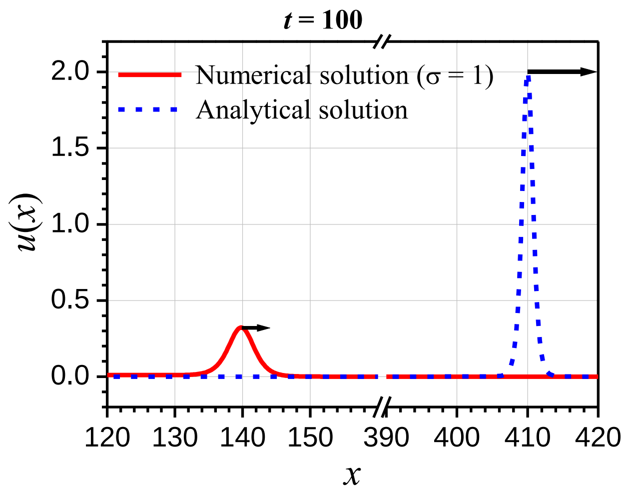


Fig.3. Comparison of numerical solution using a completely implicit scheme ($\sigma=1$) with the analytical one at the time t=100.

(a)

(b)

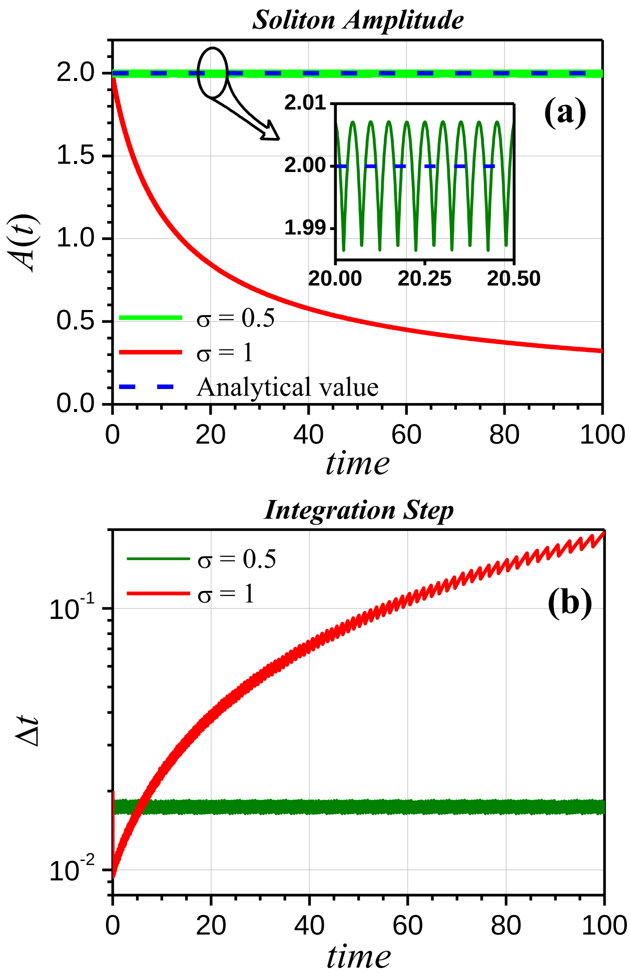


Fig.4. The soliton amplitude versus time (a) and automatically chosen integration step (b) for two implicit schemes: $\sigma=0.5$ and 1.

Thus, decreasing the step by a factor of 2 to $\Delta t = 5\cdot 10^{-5}$ leads to the fact that the destruction of the solution occurs at about time *t*=6. Figure (b) shows the loss stability at the time of *t*=30 when integrating with a step $\Delta t = 1\cdot 10^{-5}$. According to the results obtained in theoretical works [24, 25], explicit two-layer difference schemes for the KdV equation are absolutely unstable. Our results are in complete agreement with this conclusion.

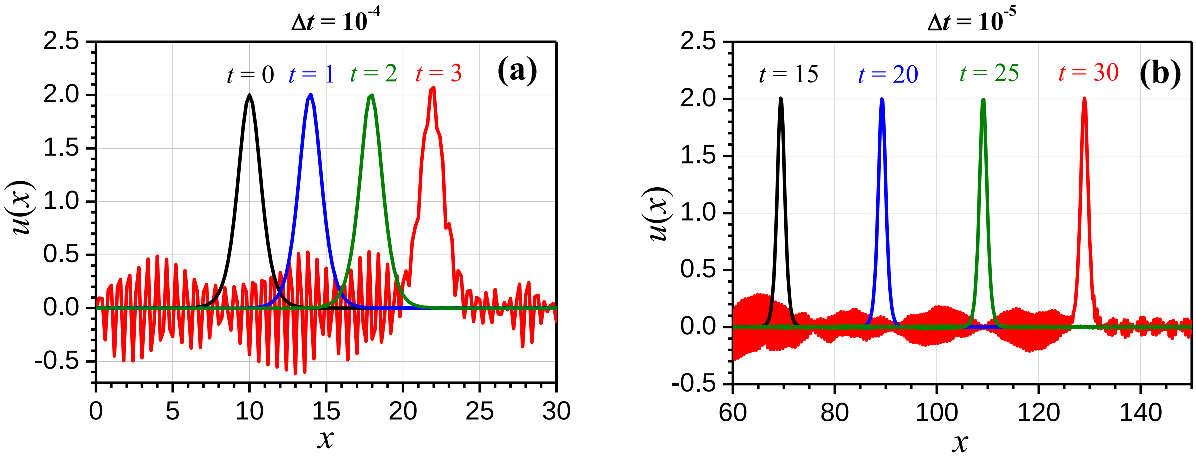


Fig .5. Numerical solution using an explicit difference scheme (*σ=0*) with different integeration steps (by an order)

6 ANALYTICAL STUDY

When using the finite difference method, it is not the original partial differential equation that is solved numerically, but a modified equation called the differential approximation of the difference scheme [33–35]. The right side of this approximation is the approximation error and is equal to the difference between the original partial differential equation and its finite-difference analogue. Investigation of the right-hand sides of differential approximations makes it possible to establish the predominant contribution to the approximation error of the highest derivatives and the related properties of difference schemes such as dissipation and dispersion. It is known that if the main term in the expression for the approximation error contains derivatives of an even order, then the dominant properties of difference schemes will be dissipative, and if derivatives of an odd order, then the dominant properties will be dispersive.

Let us analyze the family of schemes (5) using the method of the modified equation [33-35]. To do this, first we replace the sought function $u(x,t)$ by $f(x,t) = \alpha\cdot u(x,t)$. This allows one to get rid of the coefficient α as in both the original equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\left(\frac{f^2}{2}\right) + \beta\frac{\partial^3 f}{\partial x^3} = 0 \ , \tag{6}$$

and in the finite-difference approximation

$$\frac{f_m^{k+1} - f_m^k}{\Delta t} + \frac{1}{\Delta x}\left(\frac{(f_{m+1}^{k+\sigma} + f_m^{k+\sigma})^2}{8} - \frac{(f_m^{k+\sigma} + f_{m-1}^{k+\sigma})^2}{8}\right) + \beta\frac{f_{m+2}^{k+\sigma} - 2f_{m+1}^{k+\sigma} + 2f_{m-1}^{k+\sigma} - f_{m-2}^{k+\sigma}}{2\Delta x^3} = 0 \tag{7}$$

That is, the factor α is simply the scaling factor for the ordinate.

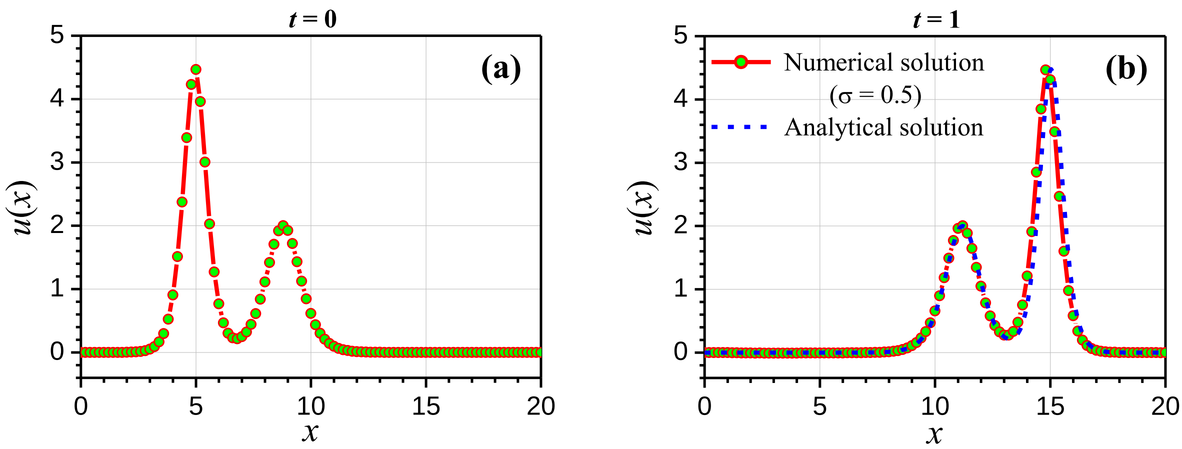


Fig.6. Numerical solution of the problem of collision of solitons: (a) initial condition, (b) comparison with the analytical solution at the time t=1.

Next, we expand the two-dimensional function $f(x,t)$ in a Taylor series at a point $(x_m, t^{k+1/2})$ and substitute it into scheme (7). Leaving the terms in the resulting expression not higher than the second order of smallness, we can write

$$f'_t + f \cdot f'_x + \beta f'''_{xxx} = -\left[f \cdot f^{(\mathrm{II})}_{xt} + f'_t f'_x + \beta f^{(\mathrm{IV})}_{xxxt}\right]\left(\sigma - \frac{1}{2}\right)\Delta t -$$
$$-\left[\frac{1}{8} f \cdot f^{(\mathrm{III})}_{xtt} + f'_t f^{(\mathrm{II})}_{xt}\left(\sigma - \frac{1}{2}\right)^2 + \frac{1}{8} f''_{tt} f'_x + \frac{f'''_{ttt}}{24} + \frac{\beta}{8} f^{(\mathrm{V})}_{xxxtt}\right]\Delta t^2 - \left[\frac{2}{3} f \cdot f'''_{xxx} + f'_x f''_{xx} + \beta f^{(\mathrm{V})}_{xxxxx}\right]\frac{\Delta x^2}{4} \tag{8}$$

In this expression (8), to simplify notation, the indices m and k + 1/2 for the function f and all its derivatives are omitted. We focus our attention on the first-order term in Δt. Using the original equation (6), we get rid of the time derivatives:

$$\begin{aligned} f'_t &= -f \cdot f'_x - \beta f'''_{xxx} \\ f''_{xt} &= \frac{\partial}{\partial x}\left(-f \cdot f'_x - \beta f'''_{xxx}\right) = -(f'_x)^2 - f \cdot f''_{xx} - \beta f^{(\mathrm{IV})}_{xxxx} \\ f^{(\mathrm{IV})}_{xxxt} &= -3(f''_{xx})^2 - 4 f'_x f'''_{xxx} - f \cdot f^{(\mathrm{IV})}_{xxxx} - \beta f^{(\mathrm{VI})}_{xxxxxx} \end{aligned} \tag{9}$$

In addition, for simplicity, we use the boundedness of the function *f* and all its derivatives in the domain of definition, and we replace the coefficients of the second-order terms by the constants K_1 K_2. As a result, we finally get:

$$f'_t + f \cdot f'_x + \beta f'''_{xxx} = K_1 \Delta x^2 + K_2 \Delta t^2 +$$
$$+\Delta t\left(\sigma - \frac{1}{2}\right)\left[\left\{f^2 f''_{xx} + 2\beta f \cdot f^{(\mathrm{IV})}_{xxxx} + \beta^2 f^{(\mathrm{VI})}_{xxxxxx} + 3\beta (f''_{xx})^2\right\} + \left\{2 f \cdot (f'_x)^2 + 5\beta f'_x f'''_{xxx}\right\}\right] \tag{10}$$

Now let's analyze the resulting expression (10). In the term of the 1st order with respect to Δt, the square brackets contain two groups of terms: in the 1st curly bracket there are even derivatives with respect to x, in the 2nd - odd ones. I.e. the 1st curly bracket corresponds to the schematic viscosity, and the 2nd - to the schematic dispersion. In this case, the sign of the coefficient in front of the dispersion terms is insignificant, while in front of the diffusion terms it is very important. This sign is determined by the difference (σ-1/2).

When σ>1/2, the coefficient in front of the first-order diffusion terms is positive, and scheme (7) implements equation (6) with an additional viscosity proportional to the first power of the time step. This provides, on the one hand, stable behavior in the calculation process, on the other hand, distortion of the solution; over time, the initial perturbation is smeared out. It is this effect that we observed in the numerical solution using a completely implicit scheme (see Fig. 3).

When σ<1/2, the coefficient in front of the first-order diffusion terms is negative. That is, equation (10) gets negative viscosity. This means that scheme (7) becomes absolutely unstable. And since the absolute value of the coefficient is still proportional to the 1st power of the time step, the destruction of the solution occurs the earlier, the larger the time step is. This is precisely what we observed when experimenting with an explicit difference scheme (see Fig. 5).

In addition, now the assertion of theoretical works [24, 25] about the absolute instability of explicit two-layer difference schemes for the KdV equation can be extended for the family of schemes (7): all schemes (7) are absolutely unstable for σ<1/2, i.e. with “any prevalence of explicitness”.

The highlighted value of σ is 1/2. For this single value, the first-order term in (10) vanishes, and thus scheme (7) receives the second-order approximation in both variables, $O(\Delta t^2 + \Delta x^2)$. In addition, the effects of the scheme viscosity and 1st order dispersion are nullified. It is precisely because of this that, in a numerical solution on a somewhat coarse grid, it was possible to obtain the transport of a soliton with practically no distortions over considerable distances (see Fig. 2).

So, scheme (5) with σ=1/2 showed the best results in modeling of the problem of the soliton transfer.

7 VERIFICATION OF THE APPROXIMATION ORDER

Using the example of the problem of collision of solitons, we numerically verify the order of approximation of scheme (5) at σ=0.5. For equation (1) with parameters $\alpha = 6,\ \beta = 1$, there is the following analytical solution to this problem [36]:

$$U_{exact}(x,t) = \frac{22.5\,\mathrm{ch}[x-4t-8]^2 + 10\,\mathrm{sh}[1.5(x-9t-5.5)]^2}{\{\mathrm{ch}[1.5(x-9t-5.5)]\,\mathrm{ch}[x-4t-8] + 2\,\mathrm{ch}[0.5(x-19t-0.5)]\}^2} \tag{11}$$

We will use this solution to set the initial condition (Fig.6a) and then estimate the error of the numerical solution. For this, we use the following definitions of the error of the numerical solution at the time t^k:

$$\begin{aligned} &1)\ \varepsilon_{L2} = \sqrt{\frac{1}{M+1}\sum_{m=0}^{M}\left[u_m^k - U_{exact}(x_m,t^k)\right]^2} \\ &2)\ \varepsilon_C = \max_{0\le m\le M}\left(\left|u_m^k - U_{exact}(x_m,t^k)\right|\right) \end{aligned} \tag{12}$$

At the first stage, we choose the size of the spatial step $\Delta x_{(I)} = 0.2$, as in the previous problem on the transfer of a soliton. We solve the problem in the computational domain $x \in [0, 20]$ in the time interval $t \in [0,1]$ (Fig.6b) with automatic selection of the time step in order to determine the maximum allowable time step. It turned out $\Delta t_{\max} \approx 4 \cdot 10^{-3}$. Then we make calculations by disabling the automatic selection mechanism and decreasing the time step until the errors at the end of the calculation $t_{end} = 1$, calculated by formulas (12), stop decreasing. This means that in the approximation error

$$\varepsilon(\Delta t, \Delta x) = E_{\Delta t}(\Delta t) + E_{\Delta x}(\Delta x) \tag{13}$$

the part related to the time step became negligible compared to the part related to the space step: $E_{\Delta t}(\Delta t_0) << E_{\Delta x}(\Delta x_{(I)})$. It turned out for our problem $\Delta t_0 \approx 2 \cdot 10^{-6}$. Wherein

$$\begin{aligned} &1)\, E_{L2}(\Delta x_{(I)}) = 0.21359560124 \\ &2)\, E_C(\Delta x_{(I)}) = 0.89613301323 \end{aligned} \tag{14}$$

We do the calculation again, leaving the step Δt_0 unchanged, but reducing the step in space by 2 times:. We get

$$\begin{aligned} &1)\, E_{L2}(\Delta x_{(I)}/2) = 0.05356903839 \\ &2)\, E_C(\Delta x_{(I)}/2) = 0.22397163990 \end{aligned}$$

Thus, with good accuracy

$$\frac{E_{L2}(\Delta x_{(I)})}{E_{L2}(\Delta x_{(I)}/2)} = 3.99, \quad \frac{E_C(\Delta x_{(I)})}{E_C(\Delta x_{(I)}/2)} = 4.00$$

That is, with a 2-fold decrease in the spatial step, the associated approximation error decreased by a factor of 4. This means that the scheme has a 2nd order of approximation in space and an error $O(\Delta x^2)$.

To determine the order of approximation in time, we find the ratio:

$$\frac{E_{\Delta t}(\Delta t_{\max})}{E_{\Delta t}(\Delta t_{\max}/2)} = \frac{\varepsilon(\Delta t_{\max}, \Delta x_{(I)}) - E_{\Delta x}(\Delta x_{(I)})}{\varepsilon(\Delta t_{\max}/2, \Delta x_{(I)}) - E_{\Delta x}(\Delta x_{(I)})} \tag{15}$$

As the spatial part of the error $E_{\Delta x}(\Delta x_{(I)})$, we can use the previously obtained values (14). Let us calculate the total error again:

$$1)\ \varepsilon_{L2}(\Delta t_{\max}, \Delta x_{(I)}) = 0.21931557047, \quad \varepsilon_{L2}(\Delta t_{\max}/2, \Delta x_{(I)}) = 0.2150291787$$

$$2)\ \varepsilon_C(\Delta t_{\max}, \Delta x_{(I)}) = 0.91958522470, \quad \varepsilon_C(\Delta t_{\max}/2, \Delta x_{(I)}) = 0.90208673036$$

Substituting all values into (15) we obtain

$$\frac{E_{L2}(\Delta t_{\max})}{E_{L2}(\Delta t_{\max}/2)} = 3.99, \quad \frac{E_C(\Delta t_{\max})}{E_C(\Delta t_{\max}/2)} = 3.94$$

This means that the scheme has a 2nd order of approximation in time and an error $O(\Delta t^2)$.

8 CONCLUSIONS

In this paper, we investigated a family of weighted two-layer difference schemes for the Korteweg-de Vries equation on an Eulerian difference grid.

It is shown numerically that the best results are obtained using an explicit-implicit difference scheme of the Crank-Nichols type of the second order of approximation $O(\Delta t^2 + \Delta x^2)$. This scheme is capable of stably reproducing a stationary solution with good accuracy for a long time. The second order of approximation in both variables is numerically confirmed by the example of the problem of collision of solitons.

A completely implicit two-layer scheme of the 1st order in time and 2nd in space $O(\Delta t + \Delta x^2)$, although absolutely stable, nevertheless, due to the high scheme viscosity, significantly distorts the solution.

The calculation with the use of an explicit two-layer scheme has never been completed. There always came a moment of loss of stability, even with a very small time step. Although, up to this point, the solution was quite acceptable.

An analytical study of the family of finite-difference schemes (5) using the modified equation method fully confirmed the results of numerical experiments. The analysis of the structure of the approximation error for a family of two-layer finite-difference schemes made it possible to explicitly show the reasons for the success of explicitly implicit Crank– Nichols type schemes with $O(\Delta t + \Delta x^2)$ and the absolute instability of the family of schemes (5) in the case of "prevalence of explicitness" with a parameter σ<0.5. High scheme viscosity of absolutely stable fully implicit two-layer schemes of the 1st order $O(\Delta t + \Delta x^2)$ indicate the need to improve the accuracy of the space-time approximation.

An important advantage of the considered schemes is their simplicity and transparency of the basic mathematical constructions.

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SURFACE RUNOFF MODEL IN GERA SOFTWARE: PARALLEL IMPLEMENTATION AND SURFACE-SUBSURFACE COUPLING

V. KRAMARENKO[1,2] AND K.NOVIKOV[3*]

[1] Nuclear Safety Institute of the Russian Academy of Sciences, Moscow, Russia.
[2] Sechenov University, Moscow, Russia.
[3] Marchuk Institute of Numerical Mathematics of the Russian Academy of Sciences, Moscow, Russia.
*Corresponding author. E-mail: konst.novikov@gmail.com

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Summary. In this article we present a mathematical model used for surface runoff simulation in GeRa software. The model is based on diffusive wave approximation for the shallow water equations with Manning formula for flow velocity estimation. It is implemented using INMOST software platform for parallel mathematical modeling. Parallel efficiency of the model implementation is adressed for some widely used verification benchmarks. We also present surface-subsurface coupling approach used in GeRa software and discuss practical aspects of the nonlinear solver.

1 INTRODUCTION

Surface water is one of the key components of the hydrologic budget of the watershed. Thus computational efficiency of the surface runoff model implementation as well as effective surface-subsurface coupling become of great concern to hydrologic modeling software developers. Considering multiprocessor architecture of the modern computers it is natural to use distributed approach for mathematical models implementation.

Surface runoff model in GeRa software [1] based on 2D diffusive wave approximation of the shallow water equations [2] coupled with subsurface flow model based on 3D Richards equation with consideration of fluid and medium compressibility [3] is implemented numerically using finite volume discretization method with two-point flux approximation and Newton iterations as a nonlinear solver. Surface and subsurface models are coupled by first-order exchange flux [4], [5].

Recently, a large number of different highly efficient computational codes for groundwater modelling have appeared [6][7][8][9] and parallelization of different aspects of this process remains challenging [10], [11], [12], [13]. The GeRa software was developed taking into account the necessity of massive parallel calculations [14]. Now this code is used for high-performance modelling of real objects [15]. Parallelization of surface flow modelling unit is required for the integration with the rest part of the GeRa software. Coupled surface-subsurface model parallelization is carried out using INMOST platform for distributed mathematical modeling [16]. Moreover, feature set of INMOST includes tools for automatic differentiation for residual vector and jacobian matrix construction for nonlinear solver.

In this article, we address parallel efficiency of the surface runoff GeRa model in conjunction with groundwater flow model. The serial version of the model was previously discussed in [17]. Here we address the parallel implementation of the model. Coupled model is tested and verified using benchmarks presented in [4]. The solution obtained using GeRa software is compared to numerical results of other surface-subsurface simulators such as ATS [11], GEOtop [17], [18], HGS [19], Parflow [20], InHM [21], [22], An and Yu model [23],

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OpenGeoSys [24], [25], Cast3M [26], CATHY [27], MIKE-SHE [28].

We also address numerical issues caused by discontinuous surface-subsurface flux close to zero surface water levels.

2 SURFACE RUNOFF MATHEMATICAL MODEL

Let's consider a domain $\Omega \in \mathbb{R}^3$ with boundary $\partial\Omega = \Gamma_s \cup \Gamma_g$, where Γ_s is surface boundary and Γ_g is subsurface boundary. Ω corresponds to a geological domain with Γ_s being the land surface. Surface runoff model is applied in the two-dimensional domain Γ_s. In GeRa the model is based on diffusive wave approximation of shallow water equations and Manning formula for friction slopes [19]:

$$\frac{\partial h_s}{\partial t} - \nabla \cdot \left(K_s \nabla H_s \right) = q - q_{ss}, \tag{1}$$

where

$$K_s = \frac{h_s^{5/3}}{\nu\sqrt{\left|\nabla H_s\right|}} \tag{2}$$

and $h_s = h_s(x,t)$ is the unknown surface water depth, $H_s(x,t) = h_s(x,t) + z(x)$, ν is the Manning's roughness coefficient, q is the precipitation rate, q_{ss} is the surface-subsurface flux density. We refer to K_s as a surface conductivity coefficient.

Two types of boundary conditions are considered on the boundary $\partial\Gamma_s$. The first one is critical depth boundary condition [19]:

$$-K_s \nabla H_s \cdot \mathbf{n} = \sqrt{g h_s^3}, \tag{3}$$

and the second one is homogeneous Neumann boundary condition:

$$-K_s \nabla H_s \cdot \mathbf{n} = 0, \tag{4}$$

where $\mathbf{n}$ is outward unit normal vector, g is the gravity acceleration.

To model groundwater flow we use modified Richards equation for variably saturated media with consideration of fluid and medium compressibility in domain Ω [3]:

$$\frac{\partial \theta(h_g)}{\partial t} + Ss_{stor} \frac{\partial h_g}{\partial t} - \nabla \cdot K_g \nabla (h_g + z) = 0, \tag{5}$$

where θ is the water content, h_g is the pressure head, $S = S(h_g) = \dfrac{\theta}{\theta_s}$ is the saturation, s_{stor} is the specific storage, $K_g = K(h_g)$ is the hydraulic conductivity, θ_s is the maximum (saturated) water content.

Water content θ is associated with pressure head by van Genuchten model [29]:

$$\theta = \begin{cases} \theta_r + \dfrac{\theta_s - \theta_r}{\left(1 + \left|\alpha h_g\right|^n\right)^m}, h_g < 0, \\ \theta_s, h_g \geq 0, \end{cases} \tag{6}$$

where θ_r is residual water content, α and n are model parameters, $m = 1 - 1/n$. Hydraulic conductivity is approximated using Mualem's model [30]:

$$K_g = K_{sat} S_e^{0.5} \left(1 - \left(1 - S_e^{\frac{1}{m}}\right)^m\right)^2, \tag{7}$$

where K_{sat} is saturated conductivity, $S_e = \dfrac{\theta - \theta_r}{\theta_s - \theta_r}$ is the effective saturation.

The following Neumann type boundary conditions are set on $\partial\Omega$:

$$-K_g(h_g(x,t) + z)\nabla h_g(x,t)\mathbf{n} = \begin{cases} q_{ss}, x \in \Gamma_s, \\ 0, x \in \Gamma_g. \end{cases} \tag{8}$$

Here $\boldsymbol{n}$ is an outward normal vector to the boundary.Thus on Γ_s the flux is defined by surface-subsurface water interaction and is zero on the rest of the boundary.

Surface-subsurface coupling approach is based on first-order exchange coefficient [31] (i.e. flux density is proportional to difference between surface water depth and subsurface pressure head):

$$q_{ss}(h_s, h_g) = \begin{cases} \dfrac{K_{ss}}{d}(h_s - \overline{h_g}) \text{ for } (h_s > 0, -\infty < \overline{h_g} < +\infty) \text{ and } (h_s = 0, \overline{h_g} > 0), \\ 0 \text{ for } (h_s = 0, \overline{h_g} \leq 0), \end{cases} \tag{9}$$

where K_{ss} is bottom sediments conductivity, d is the bottom sediments layer thickness, $\overline{h_g}$ is the limited groundwater pressure head. The latter is determined by the following formula with small positive ϵ_g:

$$\overline{h_g} = \frac{1}{2}\left(h_g - d + \sqrt{(h_g + d)^2 + \epsilon_g^2} - \epsilon_g\right). \tag{10}$$

This expression is used to provide nonlinear solver convergence and smoothly approximate the following value:

$$\widehat{h_g} = \max\{h_g, -d\}. \tag{11}$$

As one can get negative h_s during the nonlinear solver iterations, definition of expression (10) should be extended for $h_s < 0$:

$$q_{ss}(h_s,h_g)=\begin{cases}\dfrac{K_{ss}}{d}(h_s-\overline{h_g}) \text{ for } (h_s>0,-\infty<\overline{h_g}<+\infty) \text{ and } (h_s\le 0,\overline{h_g}>h_s),\\ 0 \text{ for } (h_s\le 0,\overline{h_g}\le h_s).\end{cases} \quad (12)$$

This definition means that water flow cannot have a downward direction (from surface to subsurface) when there is no water on the surface.

Equation (12) is discontinuous with respect to both arguments for $h_s=0, h_g<0$. The discontinuity may result in nonlinear solver oscillations near the surface-subsurface flux discontinuity line. To overcome this problem, we use modified formula (smoothed) for surface-subsurface flux. To provide further details we first decompose the range of h_s and h_g into 3 subdomains (see fig. 1). We use the domain *A* as an interface between domains *B* and *C* and smooth the flux function in it. The following expression is used for surface-subsurface flux which is continuously differentiable with respect to both arguments for $h_s\ge 0$ except the square domain $0\le h_s\le\varepsilon, 0\le h_g\le\varepsilon$, where it is discontinuous along the $h_s=h_g$ segment,

$$q_{ss}(h_s,h_g)=\begin{cases}\dfrac{K_{ss}}{d}(h_s-\overline{h_g}),(h_s,h_g)\in C,\\ \dfrac{K_{ss}}{d}\left(-\dfrac{1}{\varepsilon^2}h_s^3+\dfrac{1}{\varepsilon}h_s^2+(h_g-\varepsilon)\dfrac{1-\cos\dfrac{\pi h_s}{\varepsilon}}{2}\right),(h_s,h_g)\in A,\\ 0,(h_s,h_g)\in B.\end{cases} \quad (13)$$

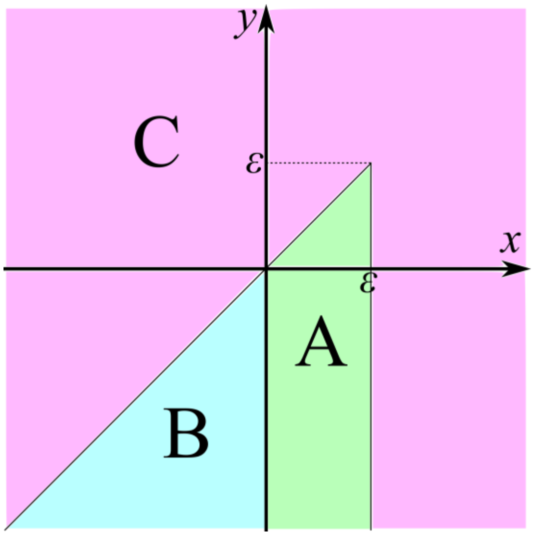

Figure 1. Decomposition of h_s and h_g range

3 NUMERICAL SOLUTION

The system to be solved is composed of coupled surface flow equations (1) and subsurface flow equations (5). We use finite volume method and implicit Euler scheme to discretize model equations. Newton-Raphson method with relaxation is applied to solve the nonlinear problem in GeRa. Surface mesh on Γ_S is obtained as trace of 3D mesh in $\mathbf{\Omega}$.

To define the residual on every Newton-Raphson iteration l we decompose it into three parts. Consider first the residual $R_{s,i}^{l,n}$ of the surface flow equation in i-th cell E_i of surface mesh in Γ_s at n-th timestep:

$$R_{s,i}^{l,n} = R_{acc,s,i}^{l,n} + R_{flow,s,i}^{l,n} + R_{ss,s,i}^{l,n}. \tag{14}$$

Here accumulation term $R_{acc,s,i}^{l,n}$ corresponds to time derivative and precipitation sources, flow term $R_{flow,s,i}^{l,n}$ corresponds to water flow inside the computational domain (i.e. Γ_s for surface runoff) and surface-subsurface term $R_{ss,s,i}^{l,n}$ corresponds to surface-subsurface flux. The same approach is applied to calculate groundwater flow equation residual:

$$R_{g,i}^{l,n} = R_{acc,g,i}^{l,n} + R_{flow,g,i}^{l,n} + R_{ss,g,i}^{l,n}. \tag{15}$$

Note that $R_{s,i}^{l,n}$ and $R_{g,i}^{l,n}$ are functions of both surface water depth and groundwater pressure head as surface-subsurface flux depends on both of these variables and we use fully implicit scheme.

The combination of two vectors $R_{s,i}^{l,n}$ and $R_{g,i}^{l,n}$ is a residual vector for Newton–Raphson method.

3.1 Discretization of surface runoff model

Accumulation term corresponding to time derivative and source term (precipitation) can be written as follows:

$$R_{acc,s,i}^{l,n} = S_i \frac{h_{s,i}^{l,n} - h_{s,i}^{n-1}}{\Delta t^n} - S_i q_i^n, \tag{16}$$

where S_i is the area of E_i, n is the time step index, $h_{s,i}^{l,n}$ is the surface water depth at l-th Newton–Raphson iteration in E_i, Δt^n is the time increment, q_i^n is the precipitation rate (or other sources) in E_i.

Flow term corresponds to water flow on the surface domain. Using linear two-point flux approximation, we get the following expression:

$$R_{flow,s,i}^{l,n} = \sum_{e_{ij} \in \partial E_i} K_{s,ij}^{l,n} \frac{h_{s,j}^n - h_{s,i}^n}{\left|c_i c_j\right|} \left|l_{ij}\right|, \tag{17}$$

where summation is over surface mesh cells neighboring to E_i through edges, $K_{s,ij}^{l,n}$ is discretization of surface conductivity on a common edge e_{ij} of cells E_i and E_j, $|c_i c_j|$ is the distance between E_i and E_j cells' centers, $|l_{ij}|$ is the length of e_{ij}.

$$K_{s,ij}^{l,n} = \begin{cases} \dfrac{\left(h_{s,ij}^{l,n}\right)^{5/3}}{\nu\left(\left(\nabla_x H_{s,ij}^{l,n}\right)^2 + \left(\nabla_y H_{s,ij}^{l,n}\right)^2\right)^{1/4}}, h_{s,ij}^{l,n} \geq 0, \\ 0, h_{s,ij}^{l,n} < 0, \end{cases} \tag{18}$$

where $h_{s,ij}^{l,n}$ is approximation of h_s on e_{ij} at l-th Newton iteration at n-th timestep, $\left[\nabla_x H_{s,ij}^{l,n}, \nabla_y H_{s,ij}^{l,n}\right]$ is approximation of ∇H_s on e_{ij} at l-th nonlinear iteration at n-th time step. For negative $h_{s,ij}^{l,n}$, $K_{s,ij}^{l,n}$ is assumed to be equal to zero.

We use upwind approximation for the numerator of (18):

$$h_{s,ij}^{l,n} = \begin{cases} h_{s,i}^{l,n}, H_{s,i}^{l,n} \geq H_{s,j}^{l,n} \\ h_{s,j}^{l,n}, H_{s,i}^{l,n} < H_{s,j}^{l,n}, \end{cases} \tag{19}$$

where z_i and z_j are z-coordinates of E_i and E_j centers respectively. We also use upwind approximation for the denominator of (18). Assume, that E_k is the upwind cell, i.e.

$$E_k = \begin{cases} E_i, H_{s,i}^{l,n} \geq H_{s,j}^{l,n}, \\ E_j, H_{s,i}^{l,n} < H_{s,j}^{l,n}. \end{cases} \tag{20}$$

For a cell E_k consider two sets of cells. Σ_k^{edge} is a set of surface mesh cells neighboring to E_k over an edge, Σ_k^{node} is a set of surface mesh cells neighboring to E_k over a node. For example shown in fig. 2 $\Sigma_j^{edge} = \left\{i, j_1^{edge}, j_2^{edge}\right\}, \Sigma_j^{node} = \Sigma_j^{edge} \cup \left\{j_1^{node}, j_2^{node}, j_3^{node}, ..., j_9^{node}\right\}$.

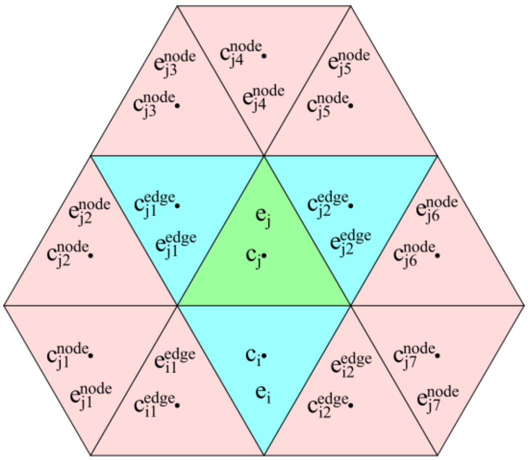

Figure 2. Illustration of E_j^{edge} and E_j^{node} sets for j-th cell, E_j^{edge} consists of cyan-colored cells, E_j^{node} consists of cyan and pink-colored cells

For each element of Σ_k^{edge} we consider the following equation based on Taylor series:

$$H_{s,\alpha}^{l,n} = H_{s,k}^{l,n} + (x_\alpha - x_k)\nabla_x H_{s,k}^{l,n} + (y_\alpha - y_k)\nabla_y H_{s,k}^{l,n} \tag{21}$$

where α is an element of Σ_k^{edge}, x_γ, y_γ are x, y coordinates of γ-th cell, $H_{s,k}^{l,n} = h_{s,k}^{l,n} + z_k$. Equations (21) for each $\alpha \in \Sigma_k^{edge}$ compose a linear system of equations for unknown $H_{s,x,k}^{l,n}$ and $H_{s,y,k}^{l,n}$. In case if this system is underdetermined we use Σ_k^{node} set of cells instead of Σ_k^{edge}.

Consider the following matrix and vector:

$$A = \begin{bmatrix} x_{\alpha_1} - x_k & y_{\alpha_1} - y_k \\ x_{\alpha_2} - x_k & y_{\alpha_2} - y_k \\ x_{\alpha_3} - x_k & y_{\alpha_3} - y_k \\ \dots & \dots \\ x_{\alpha_m} - x_k & y_{\alpha_m} - y_k \end{bmatrix}, \qquad b = \begin{bmatrix} H_{s,\alpha_1}^{l,n} - H_{s,k}^{l,n} \\ H_{s,\alpha_2}^{l,n} - H_{s,k}^{l,n} \\ H_{s,\alpha_3}^{l,n} - H_{s,k}^{l,n} \\ \dots \\ H_{s,\alpha_m}^{l,n} - H_{s,k}^{l,n} \end{bmatrix}, \tag{22}$$

where $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m$ are elements of Σ_k^{edge} or Σ_k^{node} (depending on whether linear system $Ax = b$ is underdetermined or not for Σ_k^{edge}).

The gradient $\left[\nabla_x H_{s,ij}^{l,n}, \nabla_y H_{s,ij}^{l,n}\right]$ is defined as $\underset{x}{arg\,min} \| Ax - b \|^2$ (linear least squares problem solution):

$$\left[\nabla_x H_{s,ij}^{l,n}, \nabla_y H_{s,ij}^{l,n}\right]^T = \left(A^T A\right)^{-1} A^T b. \tag{23}$$

3.2 Discretization of groundwater flow model

Again, consider separate components of nonlinear residual.

$$R_{acc,g,i}^{l,n} = V_i \left(\frac{\theta(h_{g,i}^{l,n}) - \theta(h_{g,i}^{n-1})}{\Delta t^n} + S(h_{g,i}^{l,n}) s_{stor} \frac{h_{g,i}^{l,n} - h_{g,i}^{n-1}}{\Delta t^n} \right), \tag{24}$$

where V_i is the volume of i-th subsurface cell of 3d mesh, $h_{g,i}^{l,n}$ is the groundwater pressure head at l-th nonlinear iteration at n-th time step in this cell.

$$R_{flow,g,i}^{l,n} = \sum_j K_g(h_{g,ij}^{l,n}) \frac{h_{g,j}^{l,n} - h_{g,i}^{l,n}}{\left|c_i c_j\right|} S_{ij}, \tag{25}$$

where summation is over cells of subsurface mesh neighboring to i-th cell through a face, $\left|c_i c_j\right|$ is the distance between centers of i-th and j-th cells, S_{ij} is the area of a common face of these cells, $K_g(h_g)$ is defined by (7), $h_{g,ij}^{l,n}$ is upwind pressure head defined by $h_{g,ij}^{l,n} = max\{h_{g,i}^{l,n}, h_{g,j}^{l,n}\}$.

3.3 Discretization of surface-subsurface flux

Consider residual term $R_{ss,s,i}^{l,n}$. We define this residual term as follows (domains A, B and C are depicted on fig. 1):

$$R_{ss,s,i}^{l,n} = S_i \begin{cases} \frac{K_{ss}}{d}(h_{s,i}^{l,n} - \overline{h_{g,i}^{l,n}}), (h_{s,i}^{l,n}, \overline{h_{g,i}^{l,n}}) \in C, \\ -\frac{K_{ss}}{d}\left(-\frac{1}{\varepsilon^2}\left(h_{s,i}^{l,n}\right)^3 + \frac{1}{\varepsilon}\left(h_{s,i}^{l,n}\right)^2 + (\overline{h_{g,i}^{l,n}} - \varepsilon)\frac{1-\cos\frac{\pi h_{s,i}^{l,n}}{\varepsilon}}{2} \right), (h_{s,i}^{l,n}, \overline{h_{g,i}^{l,n}}) \in A, \\ 0, (h_{s,i}^{l,n}, \overline{h_{g,i}^{l,n}}) \in B. \end{cases} \tag{26}$$

The groundwater counterpart of this term can be defined by $R_{ss,g,i}^{l,n} = -R_{ss,s,i_s}^{l,n}$, where i is an index of top level 3d subsurface mesh cell, which has i_s-th 2d surface mesh cell as one of its faces.

4 NUMERICAL EXPERIMENTS

Three numerical experiments are considered. In the first one no groundwater flow is modelled as we verify simple surface runoff model without coupling. The following two experiments are devoted to coupled surface-subsurface simulation. Coupled numerical experiments are then examined for parallel implementation efficiency.

4.1 Surface runoff

In this numerical experiment, we model surface runoff without coupling with groundwater. Numerical solution is compared to analytical solution of kinematic wave equation. Note that assumptions of diffusive wave approximations differ from kinematic wave. However, we propose to compare diffusive and kinematic wave approximation solutions due to the following arguments. First, analytical solutions for diffusive wave equation presented in papers are obtained using additional strong assumptions [33], [34]. Second, both approximations are formulated for the same original shallow water equations, thus approximate the same model.

Ground surface is a $200\text{m} \times 100\text{m}$ rectangle tilted with slope equal to 0.01 along the longest side. However, we add artificial river banks to prevent water outflow from the lateral sides of the domain. Geometry of the domain is illustrated by fig. 3. Rainfall intensity is equal to 5×10^{-6} m/s for the first 15000 seconds of the experiment and 0 for the next 15000 seconds of the experiment. Overall experiment duration is 30000 seconds. Manning roughness coefficient is $\nu = 0.05\,\text{s/m}^{1/3}$. Comparison of the numerical results for linear discharge density through the outlet with the analytical solution is depicted on fig. 4 (linear discharge density is equal to the discharge divided by the outlet length, which is equal to 100 m). As one can see on the figure numerical results are close to the analytical solution, however some qualitative difference remains. The latter may be caused by the slight diffusive and kinematic wave model disagreement.

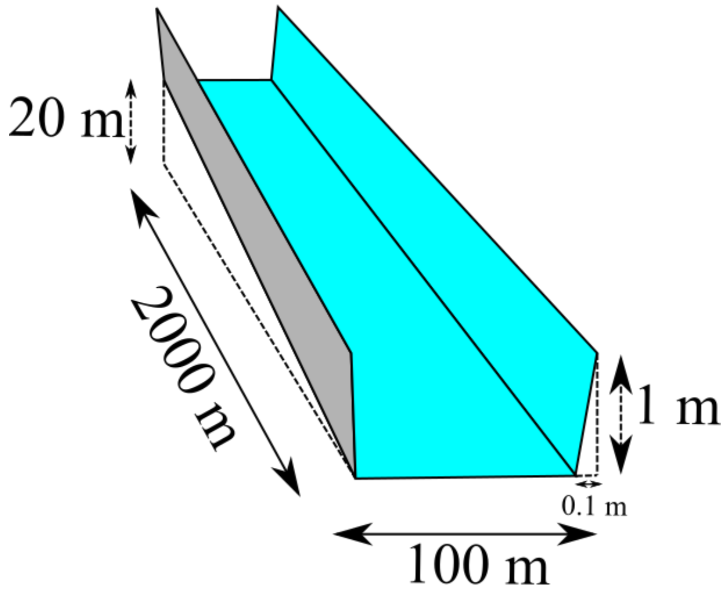

Figure 3. Geometry of tilted v-catchment numerical experiment domain

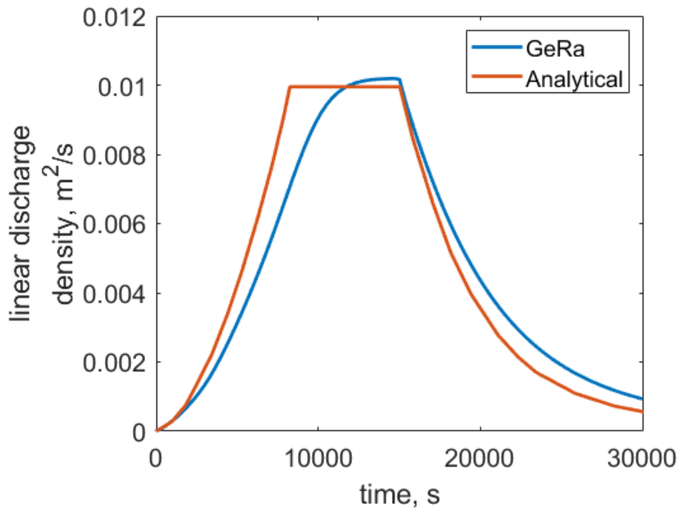

Figure 4. Water discharge dynamics for tilted v-catchment numerical experiment

4.2 Tilted v-catchment with subsurface

This numerical experiment as well as corresponding other simulators' numerical results are described in [4]. Ground surface is a 10m wide channel with parallel walls with banks tilted in x and y directions. Slope in y direction is constant and equal to 0.02, slope in x direction is zero for the channel, and 0.05 for channel banks. Bottom of the domain has the same geometry as ground surface and is located 5m below the surface (see fig. 5 for domain geometry scheme). Two different precipitation scenarios were modeled: no rainfall during the

120 hours of experiment in the first scenario, 20 hours of rainfall with precipitation rate 0.1 m/h and 100 hours of recession in the second scenario. Authors of [4] suggest to use zero surface water depth and vertically hydrostatic initial conditions with water table 2 m below the ground surface as initial conditions. Boundary conditions are critical depth boundary condition for surface layer and no-flux boundary conditions for subsurface.

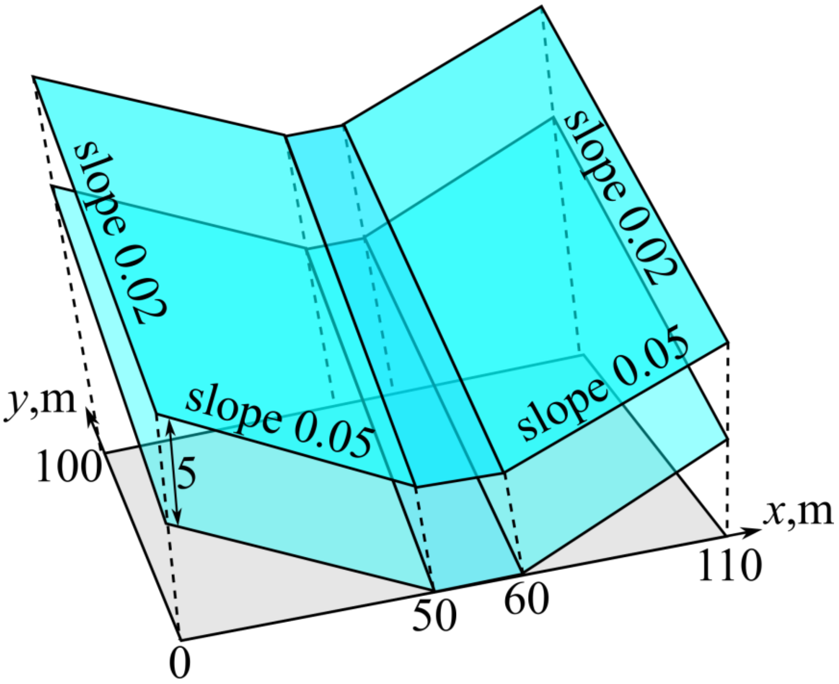

Figure 5. Geometry of tilted v-catchment numerical experiment domain

The following model parameters were used [4]:

- $\nu = 1.74\times10^{-3}\,\text{h/m}^{1/3}$ for the channel and $\nu = 1.74\times10^{-4}$ $\text{h/m}^{1/3}$ elsewhere,
- K_{sat} = 10 m/h,
- n = 2 and α = 6 m^{-1},
- $\theta_r = 0.08, \theta_s = 0.4$,
- $s_{stor} = 10^{-5}$ m^{-1},
- precipitation rate: 0 for 120 h for the first scenario, 0.1 m/h for the first 20 h and 0 afterwards for the second scenario.

One of the simulators considered in [4] uses first-order exchange as a coupling method (HGS simulator), however there is no exact value defined for proportionality coefficient for the surface-subsurface flux in this paper. Therefore, bottom sediment parameters were estimated for GeRa to fit the results of other simulators. For this numerical experiment, we used $K_{ss} = 20$ m/day and $d = 0.2$ m.

Using these model parameters, we simulated the test case and obtained water dynamics for the surface and subsurface layers. Comparison discharge rate through the outlet obtained by GeRa code with other simulator results is presented in fig. 6 for the first scenario and fig. 7 for the second scenario. As one can see from the figures Gera software produces the solution close to the other simulators results. Absolute values of GeRa solution lie between other simulators.

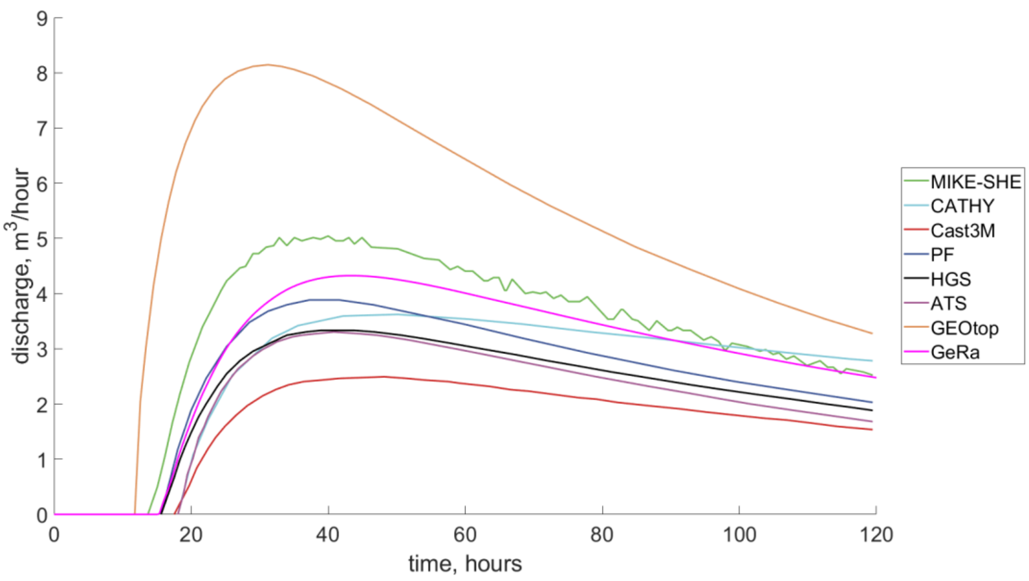

Figure 6. Water discharge dynamics for the first scenario of the tilted v-catchment benchmark

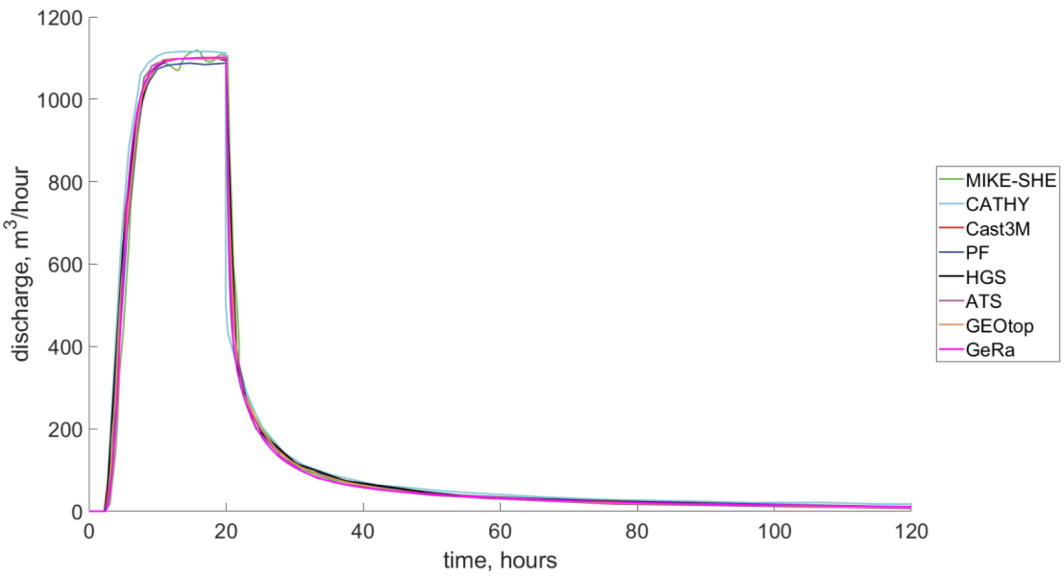

Figure 7. Water discharge dynamics for the second scenario of the tilted v-catchment benchmark

4.2 Borden benchmark

Field study was originally presented by Abdul and Gillham [35], [36] where outlet discharge has been measured for 100 minutes of the experiment. The experiment site is approximately 18 m wide and 90m long. The exact surface geometry is described by Digital Elevation Model of the terrain [4] and is depicted in fig. 8. We considered a region with relief level less than 3.02 m as a channel domain and the rest of the surface as channel banks. The subsurface computational domain is bounded by $z = 0$ plane at the bottom. Numerical experiment is implemented and described in [4], [20], [23].

We used the following model parameters:

- $\nu = 0.03$ s/m$^{1/3}$ for the channel and $\nu = 0.3$ s/m$^{1/3}$ elsewhere [20], [23],
- K_{sat} = 0.036 m/h [4],
- n = 6 and α = 1.9 m^{-1} [4],
- $\theta_r = 0.067, \theta_s = 0.37$ [4],
- $s_{stor} = 10^{-6}$ m^{-1},
- precipitation rate: 0.02 m/h for the first 50 minutes, 0 for the last 50 minutes.

For this numerical experiment $K_{ss} = 0.47$ m/day and $d = 0.2$ m were used. Note that authors of [4] used constant value for Manning's roughness for the whole domain, while different values for the channel and the channel banks are used in [20], [23].

Zero water level on the surface and hydrostatic initial conditions with water table at $z =$ 2.78 m were used as initial conditions. Boundary conditions are critical depth boundary for the surface layer and no-flux boundary conditions for the subsurface.

Comparison between GeRa numerical discharge rate, other simulator discharge rate and experimental data is depicted in fig. 9. As one can see from the figure GeRa results are close to the experimental discharge rate. Moreover GeRa results agree with other simulators under consideration.

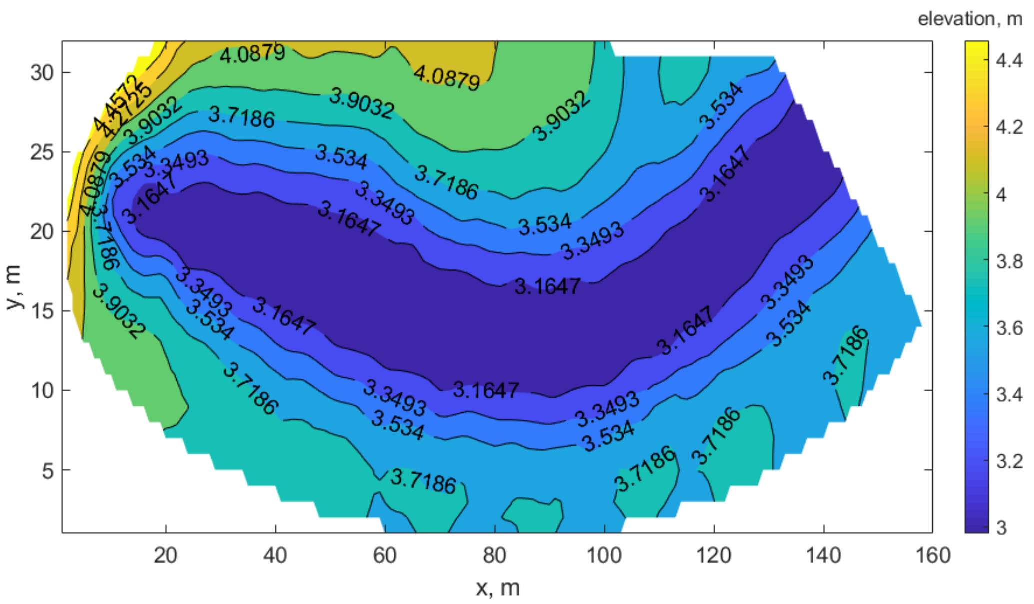

Figure 8. Borden benchmark surface elevation described by Digital Elevation Model with 0.5m resolution [4]

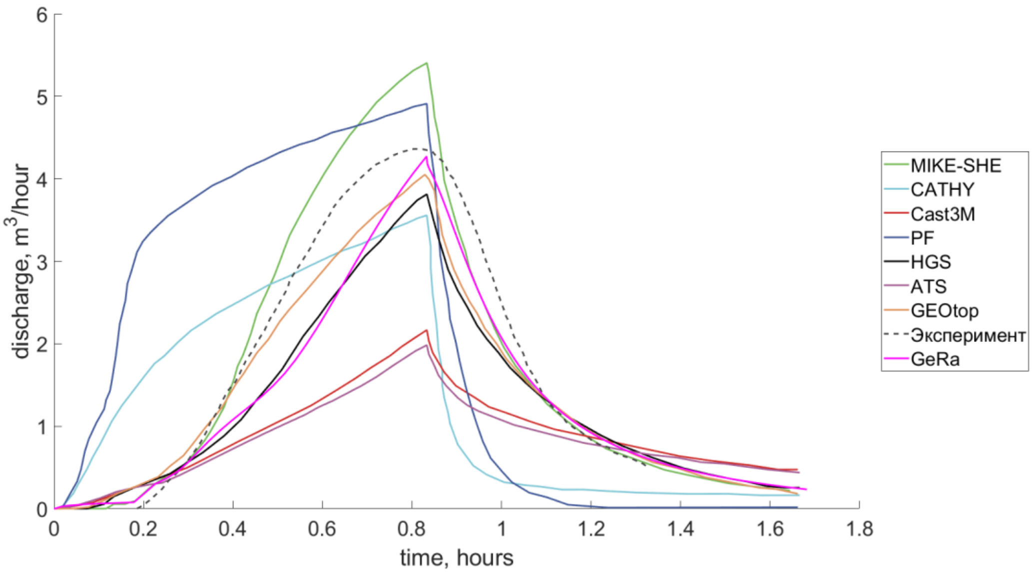

Figure 9. Water discharge dynamics for Borden benchmark

4.3 Parallel numerical experiments

In this section, we consider parallel efficiency of the coupled surface-subsurface model implemented in GeRa. Parallelization is implemented with MPI technology used in INMOST. For the numerical experiments presented in the section we dramatically refined the meshes for the experiments described previously. During Newton iteration, we need to solve a system of linear equations. Note, that in case of convergence failure we refine the time step. Maximum number of Newton iterations before the time step refinement is one of nonlinear solver parameters. To solve the linear systems of equations obtained on each Newton iteration we use PETSc package [37], namely BiCGStab solver with Schwartz preconditioner. On each processor ILU(k) preconditioner is used. For the mesh cell distribution between processors ParMETIS package is used [38].

All experiments are performed on INM RAS cluster [39] using the computational nodes of the x12core segment:

- Compute Node Arbyte Alkazar+ R2Q50
- 24 cores (two 12-core Intel Xeon E5-2670v3@2.30GHz processors or Intel Xeon Silver 4214@2.20GHz);
- RAM: 64 GB;
- Operating system: SUSE Linux Enterprise Server 15 SP2;
- Network: Mellanox Infiniband.

Due to node configuration we consider not 1, 2, 4,…,2^n cores, but 3, 6,…, $3*2^n$ cores to measure parallel efficiency.

For the tilted v-catchment numerical experiment, (first precipitation scenario is considered) mesh size is 285750 cells. For Newton iterations, nonlinear problem parameters are the following:

- initial time step is 0.001 days;
- maximum number of nonlinear iterations before time step reduction is 40;

- stopping criterion is residual reduction by 10^{-3} factor.

For linear system solution PETSc parameters are the following:

- Schwartz overlap between processors is 1;
- ILU factor level for each processor is 1;
- stopping criterion is initial residual reduction by factor 10^{-9};

In the Borden experiment, mesh size is 278140 cells. For Newton iterations, nonlinear problem parameters are the following:

- initial time step is 0.001 days;
- maximum number of nonlinear iterations before time step reduction is 300;
- stopping criterion is residual reduction by factor 10^{-4}.

For linear system solutions PETSc parameters are the following:

- Schwartz overlap between processors is 3;
- ILU factor level for each processor is 3;
- Stopping criterion is initial residual reduction by factor $10^{-9.}$

Results are shown in the Table 1 for the tilted v-catchment experiment and Table 2 for the Borden experiment. For each number of processors (first column) we list total solution time of the experiment (second column), acceleration (third column) and efficiency of parallelization (fourth column). Acceleration is the ratio between total solution times for current number of processors and for the baseline number of processors. The baseline number is equal to 3 for tilted v-catchment benchmark and 12 for Borden benchmark (we do not use serial computation on a single processor due to long computational time for the refined mesh). Parallelization efficiency is the ratio between solution times for current number of processors and for two times smaller number of processors. In other words, efficiency value shows a speedup for one step of processors number increasing.

Number of processors	Solution time	Acceleration	Efficiency
3	29456	1.0	-
6	19521	1.5	1.5
12	11039	2.7	1.8
24	5081	5.8	2.2
48	2645	11.1	1.9
96	1267	23.4	2.1
192	838	35.1	1.5

Table 1. Parallel efficiency results for the tilted v-catchment experiment

Number of processors	Solution time	Acceleration	Efficiency
12	92684	1.0	-
24	51997	1.8	1.8
48	28083	3.3	1.8
96	15138	6.1	1.9
192	9111	10.17	1.7

Table 2. Parallel efficiency results for the Borden experiment

Both experiments demonstrate good scalability and parallel efficiency of coupled surface-subsurface water simulations. Maximum speedup is 35 times for tilted v-catchment experiment on 192 cores (theoretical maximum is 64 times). Both experiments also demonstrate fair efficiency. Average efficiency is more than 1.6 for both experiments and in some cases hyper linear speedup is observed.

5 CONCLUSIONS

Surface runoff model implemented in GeRa software package is described in the article. Verification benchmarks previously applied to the serial implementation of the model in [17] were used here to demonstrate validity of the parallel version of the model itself as well as coupled surface-subsurface model. We also used these benchmarks to assess parallelization efficiency of the coupled model. Numerical experiments show good scalability of the implementation. The acceleration for the parallel implementation is up to 35 times for 192 processors for the tilted v-catchment benchmark relative to the baseline time obtained for 3 processors.

We also suggested surface-subsurface flux smoothing approach in order to prevent nonlinear solver oscillations.

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MODERN MISSION DESIGN WITH THE HIGH INCLINED ORBIT FORMATION USING GRAVITY ASSISTS: MAIN METHODS

G.K. BOROVIN[1*], A.V. GRUSHEVSKII[1], A.G. TUCHIN[1], AND D.A. TUCHIN[1]

[1] Keldysh Institute of applied mathematics RAS

Moscow, Russia

* Corresponding author. E-mail: borovin@keldysh.ru

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Summary. An effective space exploration is impossible without gravity assists (GA) using. Their application relaxes the constraints imposed on the space mission scenarios by the characteristic velocity budgets being realized at the current stage of development of space technology. A significant change in the inclinations of operational spacecraft (SC) orbits in flight aimed at studying the inner heliosphere from out-of ecliptic positions (the ESA "Solar Orbiter" mission, Russian "Interheliozond") is needed to accomplish some prospective space missions. Low-cost tours for the high inclined orbit formation in the Solar system with use of gravity assists near its planets (Earth and Venus) with the full ephemeris using are considered. The limited dynamic possibilities of using gravity maneuvers require their repeated performance. Based on the formalization of the search for the GA- timetables with subsequent adaptive involvement of a large number of options, a high-precision algorithm for synthesizing chains of increasing gravity assists was built. Its use leads to a significant inclination change of the research SC's orbit without significant fuel consumption during a reasonable flight time.

1 INTRODUCTION

An effective space exploration is impossible without gravity assists (GA) using. Their application relaxes the constraints imposed on the space mission scenarios by the characteristic velocity budgets being realized at the current stage of development of space technology. A significant change in the inclinations of operational spacecraft (SC) orbits in flight aimed at studying the inner heliosphere from out of ecliptic positions (the ESA "Solar Orbiter" mission, the Russian "Interheliozond" project, etc.) is needed to accomplish some prospective space missions. Low-cost tours for the high inclined orbit formation in the Solar system with use of gravity assists near its planets (Earth and Venus) with the full ephemeris using are considered. The limited dynamic possibilities of using gravity maneuvers require their repeated performance. Relevance of regular creation of optimum scenarios — sequences of cranking passing of celestial bodies and solution of conditions of their execution is obvious. The technology for synthesizing such scenarios is complicated by the necessity of their 3D design with allowance made for precise ephemeris models. The formalism is based on two basic factors for designing high-inclination orbits. The first factor is geometric restrictions on the maximum possible inclination of the SC's orbit, which is achievable depending on the relative value of the excess vector of the SC's hyperbolic velocity (the

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asymptotic velocity of the SC relative to the planet) compared to the average orbital velocity of the planet for any flight sequence. The second factor is dynamic restrictions on the maximum angle of rotation of the asymptotic velocity vector of the SC during a single gravity assist. Dynamic restrictions also depend on the value of the asymptotic velocity of the SC and the gravitational parameters of the planet. A joint analysis of the factors presented makes it possible to draw a conclusion about the dynamic nature of the planned space mission, which, however, will require further clarification. Based on the formalization of the search for the such scenarios with subsequent adaptive involvement of a large number of options, a high-precision algorithm for synthesizing chains of increasing gravity assists was built. Its use leads to a significant change in the inclination of the research SC's orbit without significant fuel consumption during a reasonable flight time.

In previous works [1-3], the authors conducted a comparative analysis of various modern astrodynamics developments [4-8], affecting the 3D spatial implementation of gravity assists, in order to clarify the possibility of their application, taking into account the exact ephemeris. In [3] a refined analytical formula of the inclination changes of the SC orbit were obtained for one-pass 3D GA obtained and the results of calculations by using the parameters changes of the orbital inclination of the SC relative the Solar system planets and their satellites. In this paper, the main focus is on the development of algorithms for the synthesis of multi-pass GA chains, the use of which leads to a significant increase in the inclination of the SC orbit above the ecliptic plane.

In this paper, we generalize the formulas of [1] for the inclination of the SC orbit and its changes during gravitational maneuvers near the planet to the overall case of elliptical orbits of not only the SC, but also the planet. The geometric interpretation of the obtained formulas is presented. Expressions are found for the coordinates of the inclination pole on the invariant sphere of the asymptotic velocity of the SC. The procedure for GA chains reaching the inclination pole to achieve the geometrically acceptable maximum inclination of the SC orbit during multi-pass gravitational maneuvers is analytically studied.

2 GEOMETRIC RESTRICTIONS OCCURRED DURING GRAVITY ASSISTS PERFORMING

Using the results of [1, 2], we introduce spherical coordinates to describe the invariant sphere of the position of the ends of the CS's asymptotic velocity vector $\mathbf{V}_\infty$ of the during gravity assists: radius V_∞ and angles ρ, σ. The angle ρ is the angle between the vector $\mathbf{V}_{\infty,out}$ obtained after the gravity assist and its projection on the orbital plane, and the angle σ is the angle between this projection and the orbital velocity vector of the planet $\mathbf{V}_{pl}$.

The restrictions on changing the inclination i of the SC's orbit when performing GA with a selected planet ("solo" GA) can be interpreted as geometric and dynamic [1, 2].

Geometric restrictions define the maximum value of i for any number of solo GA, which for the case (1) is given by the dimensionless asymptotic velocity of the KA v_∞ [1-3, 8-12]:

$$\sin i_{\max} = v_\infty , \tag{1}$$

$$v_\infty = V_\infty / V_{pl} . \tag{2}$$

It is easy to see that from (1) and (2) follows a restriction for the maximum possible inclination:

$$i_{\max} < \pi/2\,. \tag{3}$$

3 GRAVITY ASSISTS DYNAMIC RESTRICTIONS

Dynamic restrictions of the SC's orbit inclination changing are determined by the magnitude of the planet's gravitational field and the minimum allowable flyby distance near it. For the rotation angle φ of the SC's asymptotic velocity vector after a single-pass GA, is valid the formula [4]:

$$\sin\frac{\varphi}{2} = \frac{\mu}{\mu + r_\pi V_\infty^2} \tag{4}$$

where μ – gravitational parameter of the flyby body, r_π – the distance of the pericenter of the SC''s flyby hyperbola, which cannot be less than the radius of the partner planet R_{pl}. The position of the "inclination pole" - the extremum point $\mathrm{T_{Pole}} : i = i_{\max}$ on the V_∞-sphere is schematically shown in Fig. 1. the Sequence of any solo GA in order to increase the inclination of the SC's orbit (we will call them "increasing chains") should be as close as possible to the point $\mathrm{T_{Pole}}$.

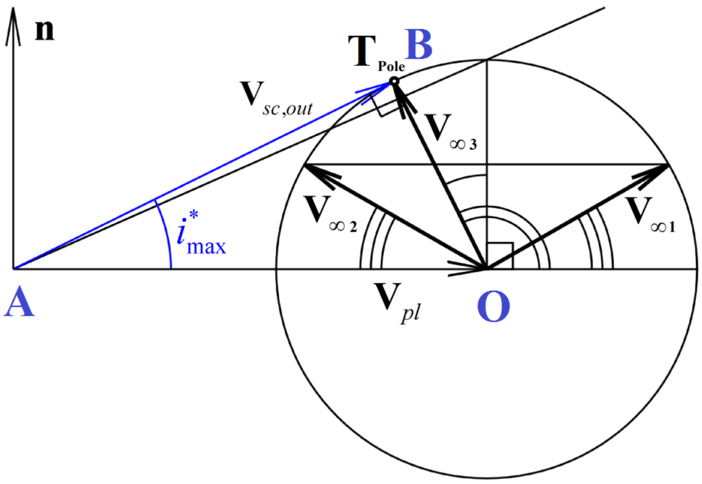

Fig. 1. The inclination pole $\mathrm{T_{Pole}}$ - the inclination extremum point location on the V_∞ - sphere in case $\sin\sigma = 0$

The overall meaning of dynamic constraints for GA is that the end of the output vector $\mathbf{V}_{\infty,out}$ for a single-pass GA does not go beyond the spherical region ("spherical cap") $S_{c\varphi}$. This region is the intersection of a sphere and a solid angle formed by a cone with a solution angle of 2φ, the axis of which is the vector of the input (before GA) SC's asymptotic velocity $\mathbf{V}_{\infty,in}$ (Fig. 2). The base of the spherical cup is obviously a circle K_φ of radius $r_\infty = V_\infty \sin\varphi$.

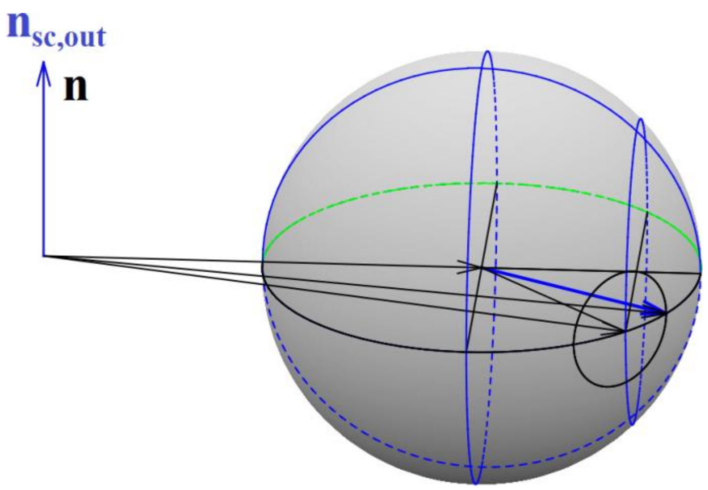

Fig. 2. The end of the output vector $\mathbf{V}_{\infty,\mathrm{out}}$ for the single-pass GA does not extend beyond the spherical region $S_{c\varphi}$ ("spherical cap)

4 OVEERALL CASE OF THE PLANET'S ELLIPTIC ORBIT

Let's obtain a generalization of the results [1] for the case of elliptical orbits of the planet and SC. We then use the formula for the tangent of the desired angle of inclination i between two planes with normals $\mathbf{n},\mathbf{n}_{sc}$ under the condition (3):

$$\mathrm{tg}\, i = \frac{|\mathbf{n}\times\mathbf{n}_{\mathrm{sc}}|}{\mathbf{n}\cdot\mathbf{n}_{\mathrm{sc}}} \tag{5}$$

Let γ is the trajectory angle (the angle between the velocity vector of a planet and the" virtual " vector of the circular orbital velocity of that planet at the same point). For a circular orbit [1] γ=0. We introduce the right triple of Cartesian coordinates (X,Y,Z) so that the axis X it is directed along the velocity vector of the planet, and Y is orthogonal to its orbital plane. Then $\mathbf{V}_{pl}=(V_{pl},0,0)$, $\mathbf{r}_{pl}=(r_{pl}\sin\gamma, r_{pl}\cos\gamma, 0)$, and

$$\mathbf{n}=\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\mathrm{pl}}=(0,\ 0,\ -r_{\mathrm{pl}}V_{\mathrm{pl}}\cos\gamma),$$

$$\mathbf{n}_{\mathrm{sc}}=\mathbf{r}_{\mathrm{pl}}\times\left(\mathbf{V}_{\mathrm{pl}}+\mathbf{V}_{\infty}\right),$$

$$\mathbf{n}\times\mathbf{n}_{\mathrm{sc}}=\left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\mathrm{pl}}\right)\times\left(\mathbf{r}_{\mathrm{pl}}\times\left(\mathbf{V}_{\mathrm{pl}}+\mathbf{V}_{\infty}\right)\right)=$$
$$=\left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\mathrm{pl}}\right)\times\left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\mathrm{pl}}\right)+\left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\mathrm{pl}}\right)\times\left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\infty}\right)=\left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\mathrm{pl}}\right)\times\left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\infty}\right).$$

Let's use the identity: $(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})=\mathbf{a}(\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}))$, , which in this case will mean:

$$\left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\mathrm{pl}}\right)\times\left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\infty}\right)=\mathbf{r}_{\mathrm{pl}}\left(\mathbf{r}_{\mathrm{pl}}\cdot\left(\mathbf{V}_{\mathrm{pl}}\times\mathbf{V}_{\infty}\right)\right).$$

Writing $\mathbf{V}_{\infty}$ as:

$$\left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\mathrm{pl}}\right)\times\left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\infty}\right)=\mathbf{r}_{\mathrm{pl}}\left(\mathbf{r}_{\mathrm{pl}}\cdot\left(\mathbf{V}_{\mathrm{pl}}\times\mathbf{V}_{\infty}\right)\right) \tag{6}$$

we find

$$\mathbf{V}_{\mathrm{pl}} \times \mathbf{V}_{\infty} = (0,\ -V_{\mathrm{pl}}V_{\infty}\sin\rho,\ V_{\mathrm{pl}}V_{\infty}\cos\rho\sin\sigma) = (0,\ -V_{\mathrm{pl}}V_{\infty z},\ V_{\mathrm{pl}}V_{\infty y}) \qquad (7)$$

and

$$\mathbf{n}\times\mathbf{n}_{\mathrm{sc}} = \mathbf{r}_{\mathrm{pl}}\left(\mathbf{r}_{\mathrm{pl}}\cdot\left(\mathbf{V}_{\mathrm{pl}}\times\mathbf{V}_{\infty}\right)\right) = -\mathbf{r}_{\mathrm{pl}}\cdot r_{\mathrm{pl}}V_{\mathrm{pl}}V_{\infty z}\cos\gamma \qquad (8)$$

This implies an important statement [1].

Statement. The SC's orbit normal vector $\mathbf{n}_{sc}$ after GA will always be orthogonal the radius vector of the planet $\mathbf{r}_{pl}$, that is, for any GA, $\mathbf{n}_{sc}$ it always remains in the plane that is orthogonal $\mathbf{r}_{pl}$.

Expression (7) can be represented in coordinate form. We calculate for such a representation $\mathbf{n}\cdot\mathbf{n}_{\mathrm{sc}}$:

$$\begin{aligned}\mathbf{n}\cdot\mathbf{n}_{\mathrm{sc}} &= \left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\mathrm{pl}}\right)\cdot\left(\mathbf{r}_{\mathrm{pl}}\times\left(\mathbf{V}_{\mathrm{pl}}+\mathbf{V}_{\infty}\right)\right) = \left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\mathrm{pl}}\right)^2 + \left(\mathbf{r}_{\mathrm{pl}}\times\mathbf{V}_{\mathrm{pl}}\right)\cdot\left(\mathbf{r}_{\mathrm{pl}}\cdot\mathbf{V}_{\infty}\right) = \\ &= r_{\mathrm{pl}}^2\cdot V_{\mathrm{pl}}^2\cdot\cos^2\gamma + r_{\mathrm{pl}}^2\left(\mathbf{V}_{\mathrm{pl}},\mathbf{V}_{\infty}\right) - \left(\mathbf{r}_{\mathrm{pl}}\cdot\mathbf{V}_{\mathrm{pl}}\right)\cdot\left(\mathbf{r}_{\mathrm{pl}}\cdot\mathbf{V}_{\infty}\right)\end{aligned} \qquad (9)$$

Using relations

$$\begin{aligned}&\mathbf{V}_{\mathrm{pl}}\cdot\mathbf{V}_{\infty} = V_{\mathrm{pl}}V_{\infty x},\ \mathbf{r}_{\mathrm{pl}}\cdot\mathbf{V}_{\mathrm{pl}} = r_{\mathrm{pl}}V_{\mathrm{pl}}\sin\gamma, \\ &\mathbf{r}_{\mathrm{pl}}\cdot\mathbf{V}_{\infty} = r_{\mathrm{pl}}V_{\infty x}\sin\gamma + r_{\mathrm{pl}}V_{\infty y}\cos\gamma\end{aligned} \qquad (10)$$

we can find:

$$\begin{aligned}\mathbf{n}\cdot\mathbf{n}_{\mathrm{sc}} &= r_{\mathrm{pl}}^2\cdot V_{\mathrm{pl}}^2\cdot\cos^2\gamma + r_{\mathrm{pl}}^2V_{\mathrm{pl}}V_{\infty x} - r_{\mathrm{pl}}^2V_{\mathrm{pl}}\sin\gamma\left(V_{\infty x}\sin\gamma + V_{\infty y}\cos\gamma\right) = \\ &= r_{\mathrm{pl}}^2\cdot V_{\mathrm{pl}}^2\cdot\cos^2\gamma + r_{\mathrm{pl}}^2V_{\mathrm{pl}}V_{\infty x}\cos^2\gamma - r_{\mathrm{pl}}^2V_{\mathrm{pl}}V_{\infty y}\sin\gamma\cos\gamma.\end{aligned} \qquad (11)$$

As a result, we get the formula for the tangent of the angle of inclination:

$$\mathrm{tg}\, i = \frac{V_{\infty z}}{V_{\mathrm{pl}}\cos\gamma + V_{\infty x}\cos\gamma - V_{\infty y}\sin\gamma}, \qquad (12)$$

$$\mathrm{tg}\, i = \frac{V_{\infty}\sin\rho}{V_{\mathrm{pl}}\cos\gamma + V_{\infty}\cos\rho\cos(\gamma+\sigma)}. \qquad (13)$$

The obtained formula (13) can be considered as a functional relation $tg\ i(\rho,\sigma)$.

Note that the parameter γ, as a parameter of the planet's orbit, it does not depend on the point of GA. It follows from (13) and (1) that the maximum of the function $tg\ i$ is reached at a certain pole of inclination on the V_{∞} - sphere $\sigma=\sigma^*, \rho=\rho^*$, for which is properly:

$$\cos(\gamma+\sigma^*) = -1,\ \sigma^* = \pi-\gamma, \qquad (14)$$

$$\cos\rho^* = \frac{V_{\infty}/V_{pl}}{\cos\gamma} = \frac{v_{\infty}}{\cos\gamma} = \sin i_{\max}.$$

For the case of a circular orbit of the planet, the relation will be fulfilled $\sigma^*=\pi$.

5 BASIC ANGLES OF GRAVITY ASSISTS MANEUVERS FOR PLANETS

We compare the dependence of the maximum angle of rotation of the asymptotic velocity vector of the SC $\varphi_{\max}$, substituting the condition in (4) $r_{\pi}=R_{pl}$, and geometrically acceptable inclination of the formed orbit of the spacecraft (2) for the planets of the Solar system.

Comment. Comparison of (2) and Fig. 1 [4], formula (1.2.10), shows that in [4] the value of $V_{sc,out}$ approximately replaced in the denominator by V_{pl}, so that:

$$\sin i \approx \frac{V_\infty}{V_{\rm pl}} \sin\varphi_{\max} = v_\infty \sin\varphi_{\max} . \tag{15}$$

Expression (15) in some cases , for not very large values of v_∞, can serve as a satisfactory approximation of $\sin i$ [1, 2, 3].

The graphs $\varphi_{\max}$ relative V_∞ for the terrestrial planets and Jupiter are shown in Fig. 3. They show that the maximum rotation angles are reached at near-zero values V_∞. However, the value of $i_{\max}$ in this case, according to (4), it is close to zero.

GA "efficiency" appears only when increasing V_∞ to the values that provide the required value for a space mission $i_{\max}$, but at the same time the value $\varphi_{\max}$, which is demonstrated in this graph. The bold line denotes the model value of the demanded design inclination angle $i_{\max} = \pi/6$. The vertical, lowered from the point of its intersection with the graph of the function of the maximum inclination of the planet, shows the corresponding value of the rotation angle $\varphi_{\max}$ of the SC's asymptotic velocity vector on one single GA.

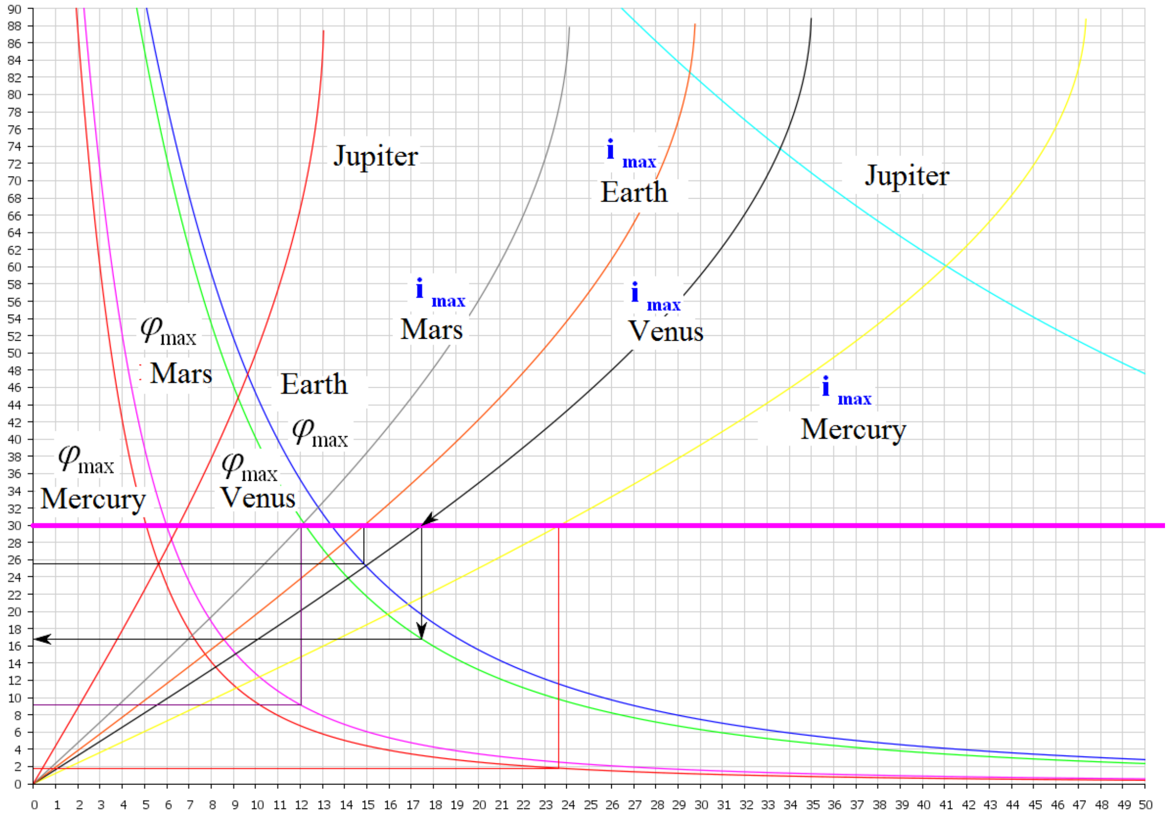

Fig. 3. The dependence of $\varphi_{\max}$ and $i_{\max}$ for the planets of the earth group and Jupiter (in degrees) on the value of the dimensionless asymptotic velocity V_∞ (in km/s)

6 SEARCHING ALGORITHMS FOR THE GRAVITY ASSISTS CHAINS THAT INCREASE THE SC'S ORBIT INCLINATION

The chain GA in the case of their solo execution can be represented as the group of automorphisms V_∞ - sphere. Climbing on the V_∞ - sphere is required to the target point close to the pole of its inclination $\mathrm{T}_{\mathrm{Pole}}$. Arrival to the $\mathrm{T}_{\mathrm{Pole}}$. will mean reaching the geometrically maximum possible inclination of the SC's orbit (1).

According to [1], for missions with $v_\infty = 1/2$ the pole $\mathrm{T}_{\mathrm{Pole}}$ will be localized at the intersection of latitude $\rho = 60^\circ$ and the longitudinal plane σ=0 (Fig. 4).

When conducting an increasing chain of "solo" GA near a fixed planet the V_∞ - sphere is an invariant. Therefore, if necessary, it is possible to solve the problem of constructing a connected route from the starting point of the first GA Λ_0 (contained in the spherical cap $S_{c\varphi,1}$) up to the point of the inclination pole $\mathrm{T}_{\mathrm{Pole}}$ with a sufficiently large number of connecting (in the limiting case, touching) local spherical caps $S_{c\varphi,1}, S_{c\varphi,2}, ..., S_{c\varphi,N}$ [2], (Fig. 4, green circles).

The maps must overlap. Each $S_{c\varphi,2},, S_{c\varphi,N-1}$ must contain at least the two points of different resonance lines between the orbital periods of planets and SC (GA output), which provides the construction of a new SC to the planet after a short time.

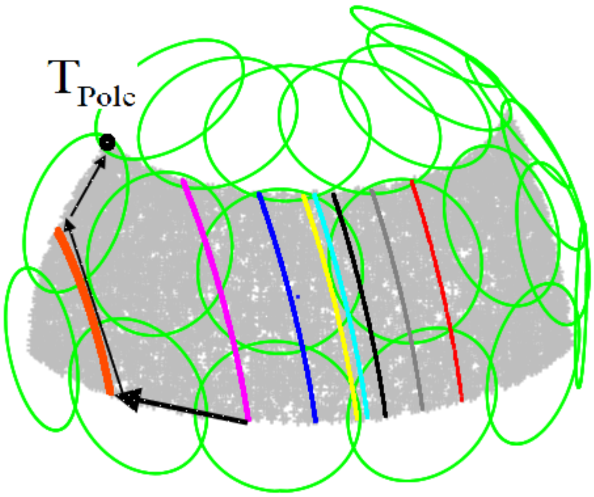

Fig. 4. For missions of any class $i_{\max} = i_{required}$, the inclination pole will be at latitude $\rho = \pi - i_{required}$. Isolines of the resonant ratios between the SC's orbital periods and the planet of the following types are plotted: 1:2, 3:4, 1:1 (blue line), 5:4, 4:3, 3:2, 2:1, 3:1

7 CONCLUSIONS

The formulas obtained in [1-3] for the orbital inclination of the spacecraft and its change as a result of gravity assist maneuvers (GM) around the planet are generalized for the case when not only the spacecraft's orbit, but also the planet's orbit is elliptical. The geometric interpretation of the formulas is described. Formulas are obtained for the coordinates of the inclination pole on the invariant sphere of the asymptotic velocity of the spacecraft. A comparative analysis of the results and modern descriptions of spatial 3D GMs was carried out [4-8]. Some cases are described when A. Labunsky's approximation [4] is acceptable [1-3].

A comparative analysis of the results and modern descriptions of spatial 3D GM is carried out [4-8]. The procedure for constructing a GM chains leading to the inclination pole to achieve a geometrically acceptable maximum inclination of the spacecraft's orbit is analytically investigated.

The characteristic "working" size of the spherical region of the elementary GM on the surface of the sphere is determined. An algorithm for searching for ballistic scenarios is presented, which reduces to constructing a finite simply connected chain GM from the initial GM to the inclination pole, covered with spherical caps on the resonant lines of the invariant sphere. These chains can go either along the resonant isolines, or jumping between them. As a result, a formalized structure of GMS that increase the inclination of the SC's orbit is synthesized, which allows automating the process of adaptive synthesis of phase beams of the corresponding optimal trajectories consisting of millions of variants.

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ANALYTIC APPROXIMATION OF THE DEBYE FUNCTION

K. V. KHISHCHENKO[1,2,3*]

[1] Joint Institute for High Temperatures of the Russian Academy of Sciences
Izhorskaya 13 Bldg 2, 125412 Moscow, Russia

[2] Moscow Institute of Physics and Technology
Institutskiy Pereulok 9, 141701 Dolgoprudny, Moscow Region, Russia

[3] South Ural State University
Lenin Avenue 76, 454080 Chelyabinsk, Russia

*Corresponding author. E-mail: konst@ihed.ras.ru

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Summary. An expression in a closed form is proposed for the approximation of the Debye function used in thermodynamic models of solids. This expression defines an analytic function that has the same limiting behavior as the Debye function at low and high temperatures. The approximation gives the maximum relative deviation from the value of the Debye function less than 0.001. The proposed expression can be useful in the equations of state of solids in a wide temperature range.

1 INTRODUCTION

The Debye model [1] was proposed for description of thermodynamic behavior of materials in a wide range of temperatures. It represents the phonon contribution to equations of state of solids as an interpolation between limiting cases of low and high temperatures [2–25]. Equations of state of matter are necessary for analysis and numerical simulation of physical phenomena under extreme conditions of high temperatures and high pressures [19, 26–40].

Analytic expressions of thermodynamic potentials within the Debye model contain the Debye function in a form of integral [1],

$$D(x) = \frac{3}{x^3} \int_0^x \frac{t^3 \mathrm{d}t}{\mathrm{e}^t - 1}, \tag{1}$$

$x > 0$, which cannot be expressed in elementary functions. Despite of that this integral can be written as analytic expression with infinite series [1, 3, 41, 42] or special functions (polylogarithms and the Riemann zeta function) [43], closed-form expressions approximating the Debye function are interesting for practical use in thermodynamic calculations. Many works are devoted to elaboration of simple approximations of the Debye functions with different ac-

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curacy [14, 44–50]. All of such approximations can be sorted as piecewise continuously differentiable functions [44, 45, 47] and smooth (in particular, analytic) functions [14, 46, 48–50].

In the present work, an expression is proposed approximating the Debye function in a closed form of analytic function. Some results of calculations are presented illustrating the accuracy of this approximation.

2 MODEL OF THERMODYNAMIC PROPERTIES OF SOLIDS

Thermodynamic potential the Helmholtz free energy is traditionally taken as a basis of equation of state model. This potential can be presented as a sum of three parts:

$$F(V,T) = F_{\mathrm{c}}(V) + F_{\mathrm{a}}(V,T) + F_{\mathrm{e}}(V,T), \tag{2}$$

those are a portion of energy corresponding to zero temperature $T = 0$ (F_{c}) and thermal contributions of ions and electrons (F_{a} and F_{e}, respectively). Here V is the specific volume; T is the temperature. Considering thermal contribution of ions in solids, one can take into account portions of energy of acoustical (F_{ac}) and optical ($F_{\mathrm{o}\alpha}$) modes of ions vibrations. For the unit cell of the crystal structure with ν particles, this contribution is as follows:

$$F_{\mathrm{a}}(V,T) = F_{\mathrm{ac}}(V,T) + \sum_{\alpha=1}^{3(\nu-1)} F_{\mathrm{o}\alpha}(V,T). \tag{3}$$

The energy of acoustic vibration modes is usually considered within the framework of the Debye model [1]:

$$F_{\mathrm{ac}}(V,T) = \frac{RT}{\nu}[3\ln[1-\exp(-\theta_{\mathrm{ac}}/T)] - D(\theta_{\mathrm{ac}}/T)]. \tag{4}$$

Optical mode contributions are commonly considered in terms of the Einstein model [51]:

$$F_{\mathrm{o}\alpha}(V,T) = \frac{RT}{\nu}\ln[1-\exp(-\theta_{\mathrm{ac}}/T)]. \tag{5}$$

Here, θ_{ac} and $\theta_{\mathrm{o}\alpha}$ are the characteristic temperatures of the acoustical and optical modes of ions vibrations.

The first and second derivatives of the Helmholtz free energy with respect to temperature determine the entropy and isochoric heat capacity of a substance:

$$S = -(\partial F/\partial T)_V, \tag{6}$$

$$C_V = T(\partial S/\partial T)_V. \tag{7}$$

Consequently, the first and second derivatives of the Debye function appear in the entropy and the isochoric heat capacity of solids. In particular, if the characteristic temperature θ_{ac} does not depend on temperature, one can obtain the following expressions for the contributions of acoustic modes:

$$S_{ac} = -\frac{R}{\nu}\left[3\ln[1-\exp(-x)] - \frac{3x}{e^x-1} - D(x) + xD'(x)\right], \tag{8}$$

$$C_{V\,ac} = \frac{R}{\nu}\left[\frac{3x^2e^x}{(e^x-1)^2} + x^2D''(x)\right], \tag{9}$$

where $x = \theta_{ac}/T$.

Sometimes, it is convenient to take into account the properties of the Debye function:

$$D'(x) = \frac{3}{e^x-1} - \frac{3}{x}D(x), \tag{10}$$

$$D''(x) = -\frac{3e^x}{(e^x-1)^2} - \frac{3}{x}\left[\frac{3}{e^x-1} - \frac{4}{x}D(x)\right]. \tag{11}$$

Then the entropy and specific heat capacity are related to the value of the Debye function:

$$S_{ac} = -\frac{R}{\nu}[3\ln[1-\exp(-x)] - 4D(x)], \tag{12}$$

$$C_{V\,ac} = \frac{3R}{\nu}\left[4D(x) - \frac{3x}{e^x-1}\right]. \tag{13}$$

However, relations (10) and (11) may not be valid for approximation functions used instead of the Debye function. Then using equations (12) and (13) will lead to some inaccuracy.

3 INFINITE-SERIES FORMS OF THE DEBYE FUNCTION

Following Debye [1], one can rewrite integral in equation (1) and obtain

$$D(x) = \frac{3}{x^3}\int\limits_0^\infty \frac{t^3\mathrm{d}t}{e^t-1} - \frac{3}{x^3}\int\limits_x^\infty \frac{t^3\mathrm{d}t}{e^t-1}. \tag{14}$$

The first integral in equation (14) has known value $\pi^4/15$ [1, 52]; the last integral in equation (14) can be simplified using the Taylor series

$$\frac{1}{1-y} = \sum_{k=0}^{\infty} y^k, \tag{15}$$

which is convergent for $|y| < 1$, and integrated by parts:

$$\int_x^\infty \frac{t^3 \mathrm{d}t}{\mathrm{e}^t - 1} = \int_x^\infty t^3 \mathrm{e}^{-t} \sum_{k=0}^\infty \mathrm{e}^{-kt} \mathrm{d}t = \sum_{k=1}^\infty \int_x^\infty t^3 \mathrm{e}^{-kt} \mathrm{d}t = \sum_{k=1}^\infty \left(\frac{x^3}{k} + \frac{3x^2}{k^2} + \frac{6x}{k^3} + \frac{6}{k^4} \right) \mathrm{e}^{-kt}. \tag{16}$$

So, one obtains

$$D(x) = \frac{\pi^4}{5x^3} - 3 \sum_{k=1}^\infty \frac{1}{k} \left(1 + \frac{3}{kx} + \frac{6}{k^2 x^2} + \frac{6}{k^3 x^3} \right) \mathrm{e}^{-kt} \tag{17}$$

for $x > 0$.

At high temperatures, one can use the following relation [53]:

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^\infty \frac{B_n}{n!} t^n, \tag{18}$$

which is convergent for $|t| < 2\pi$. Here, B_n are the Bernoulli numbers [53]. One obtains

$$\int_0^x \frac{t^3 \mathrm{d}t}{\mathrm{e}^t - 1} = \int_0^x t^2 \sum_{n=0}^\infty \frac{B_n}{n!} t^n \mathrm{d}t = \sum_{n=0}^\infty \frac{B_n}{n!} \int_0^x t^{n+2} \mathrm{d}t = \sum_{n=0}^\infty \frac{B_n}{n!} \frac{x^{n+3}}{n+3}. \tag{19}$$

So,

$$D(x) = \sum_{n=0}^\infty \frac{3B_n}{(n+3)n!} x^n \tag{20}$$

for $|x| < 2\pi$.

4 APPROXIMATION FORM

Truncated series (17) is normally used as the basis of approximation of the Debye function at low temperatures.

In this work, a similar form of approximation function is proposed:

$$K_{LM}(x) = A_{03} x^{-3} - \sum_{l=1}^L \left(A_{l0} + A_{l1} x^{-1} + A_{l2} x^{-2} + A_{l3} x^{-3} \right) \mathrm{e}^{-lx}. \tag{21}$$

Evidently, the value of first-term coefficient

$$A_{03} = \frac{\pi^4}{5} \tag{22}$$

secures the same limiting behavior of the function $K_{LM}(x)$ and all its derivatives as the Debye function $D(x)$ and its derivatives for $x \to \infty$.

Using the Taylor series

$$\mathrm{e}^y = \sum_{n=0}^{\infty} \frac{1}{n!} y^n, \tag{23}$$

one can evaluate limiting behavior of the function $K_{LM}(x)$ for $x \to 0$:

$$K_{LM}(x) = A_{03}x^{-3} - \sum_{n=0}^{\infty} \sum_{l=1}^{L} \frac{1}{n!} (-l)^n \big(A_{l0}x^n + A_{l1}x^{n-1} + A_{l2}x^{n-2} + A_{l3}x^{n-3}\big). \tag{24}$$

Representing series (24) in the form

$$K_{LM}(x) = \sum_{m=-3}^{\infty} C_m x^m, \tag{25}$$

one can easily obtain sequence of relations between coefficients of equations (24) and (25):

$$C_{-3} = A_{03} - \sum_{l=1}^{L} A_{l3}, \tag{26}$$

$$C_{-2} = -\sum_{l=1}^{L} \big(A_{l2} - lA_{l3}\big), \tag{27}$$

$$C_{-1} = -\frac{1}{2} \sum_{l=1}^{L} \big(2A_{l1} - 2lA_{l2} + l^2 A_{l3}\big) \tag{28}$$

and

$$C_m = -\sum_{l=1}^{L} \frac{(-l)^m}{(m+3)!} \big((m+1)(m+2)(m+3)A_{l0} - (m+2)(m+3)lA_{l1} + (m+3)l^2 A_{l2} - l^3 A_{l3}\big) \tag{29}$$

for $m \geqslant 0$.

Comparing form (25) with series (20), one can formulate conditions of coincidence of limiting behavior of the function $K_{LM}(x)$ and its derivatives up to order $M \leqslant 4(L-1)$ for $x \to 0$:

$$A_{03} - \sum_{l=1}^{L} A_{l3} = 0, \tag{30}$$

$$-\sum_{l=1}^{L} \left(A_{l2} - lA_{l3}\right) = 0, \tag{31}$$

$$-\sum_{l=1}^{L} \left(2A_{l1} - 2lA_{l2} + l^2 A_{l3}\right) = 0 \tag{32}$$

and

$$-\sum_{l=1}^{L} (-l)^m \big((m+1)(m+2)(m+3)A_{l0} - (m+2)(m+3)lA_{l1} + (m+3)l^2 A_{l2} - l^3 A_{l3}\big)$$
$$= 3(m+1)(m+2)B_m \tag{33}$$

for $0 \leqslant m \leqslant M$.

Solving this system of $M+4$ equations, one obtains $M+4$ coefficients (A_{L3}, A_{L2}, A_{L1}, A_{L0}, $A_{(L-1)3}$ and so on, if any) of the function $K_{LM}(x)$ (21) with the same limiting behavior as the Debye function $D(x)$ for $x \to 0$ up to order M of derivatives.

Values of the rest of coefficients A_{10}, A_{11}, A_{12}, A_{13}, A_{20} and so on (if any) can be naturally taken from series (17):

$$A_{l0} = \frac{3}{l},\ A_{l1} = \frac{9}{l^2},\ A_{l2} = \frac{18}{l^3},\ A_{l3} = \frac{18}{l^4}. \tag{34}$$

5 APPROXIMATION WITH L = 1

For $L = 1$, the approximation function $K_{LM}(x)$ has the only variant $K_{10}(x)$ with 4 coefficients A_{li} with $l = 1$, $i = 0$, 1, 2 and 3:

$$A_{10} = \frac{\pi^4}{30} - 1,\ A_{11} = \frac{\pi^4}{10},\ A_{12} = \frac{\pi^4}{5},\ A_{13} = \frac{\pi^4}{5}. \tag{35}$$

Calculated values of the function $K_{10}(x)$ and its first and second derivatives $K'_{10}(x)$ and $K''_{10}(x)$ are shown in figures 1–3 in comparison with the values of the Debye function $D(x)$ and its derivatives $D'(x)$ and $D''(x)$. In addition, the relative deviations of the function $K_{10}(x)$ and its derivatives $K'_{10}(x)$ and $K''_{10}(x)$ from the reference function $D(x)$ and its derivatives $D'(x)$ and $D''(x)$ are presented in figures 1(b), 2(b) and 3(b), respectively. The reference values of $D(x)$, $D'(x)$ and $D''(x)$ were calculated using truncated series (17) with $k \leqslant 12$ for $x > 3.34$ and (20) with $n \leqslant 66$ for $x \leqslant 3.34$.

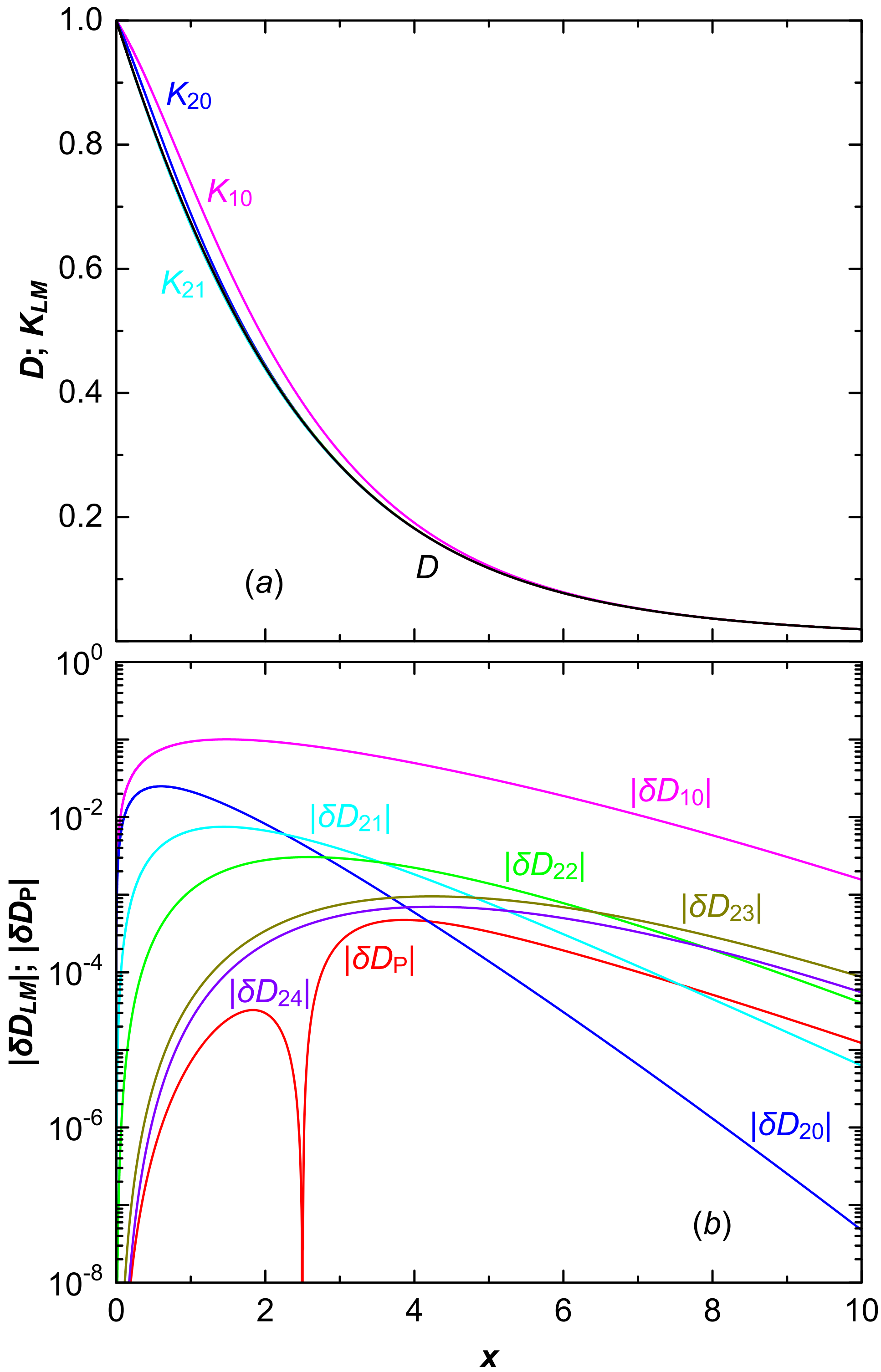

Figure 1: (a) The Debye function $D(x)$ and the approximation functions $K_{LM}(x)$. (b) The absolute values of the relative deviations of the functions $K_{LM}(x)$ from the reference function $D(x)$, $|\delta D_{LM}(x)| = |1 - K_{LM}(x)/D(x)|$; $|\delta D_{\mathrm{P}}(x)| = |1 - D_{\mathrm{P}}(x)/D(x)|$, where $D_{\mathrm{P}}(x)$ is the approximation function by Prut [47].

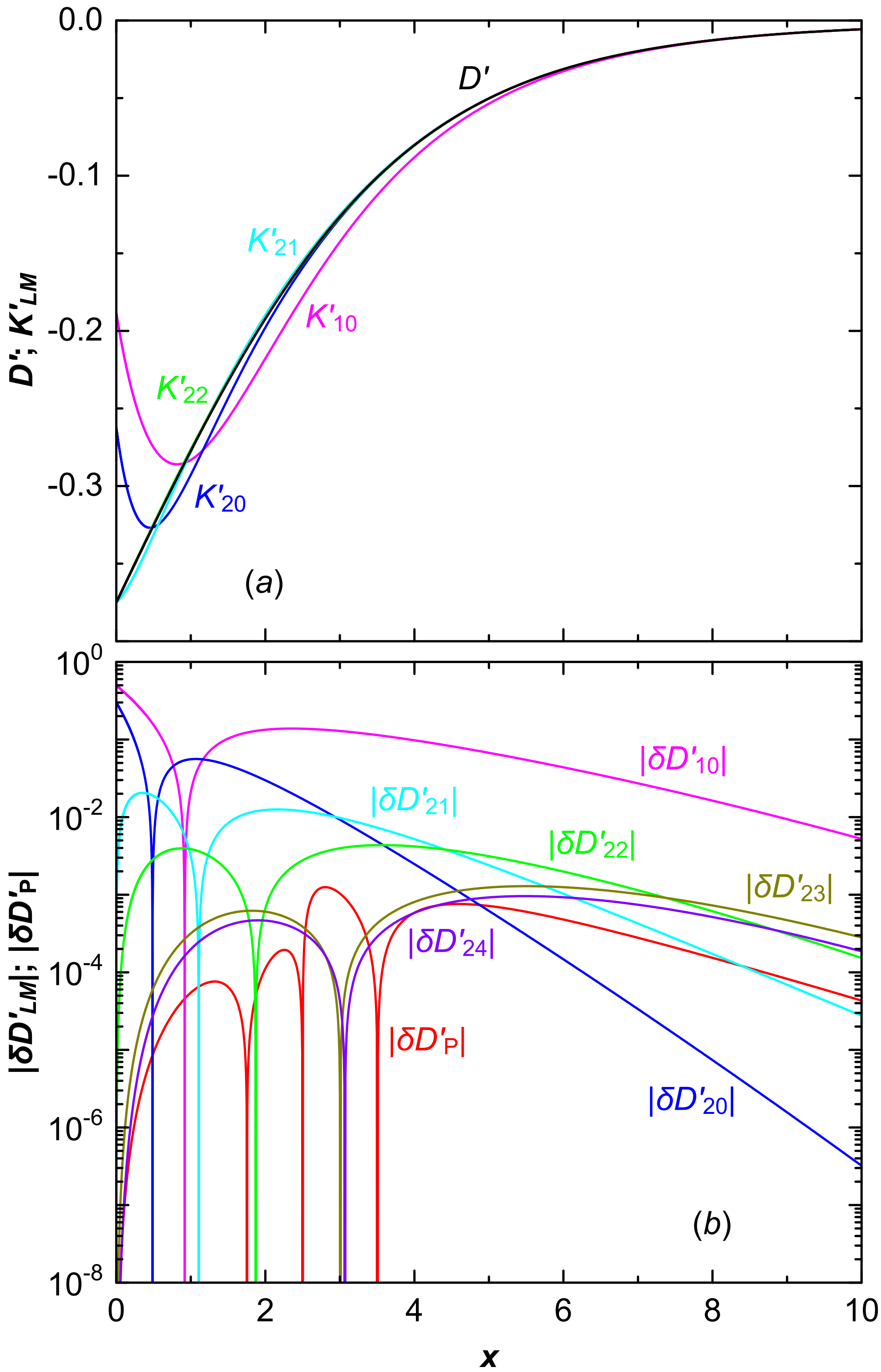

Figure 2: (a) The first derivative with respect to x of the Debye function, $D'(x)$, and the approximation functions, $K'_{LM}(x)$. (b) The absolute values of the relative deviations of the derivatives $K'_{LM}(x)$ and $D'_{\mathrm{P}}(x)$ from the reference derivative $D'(x)$, $|\delta D'_{LM}(x)| = |1 - K'_{LM}(x)/D'(x)|$ and $|\delta D'_{\mathrm{P}}(x)| = |1 - D'_{\mathrm{P}}(x)/D'(x)|$.

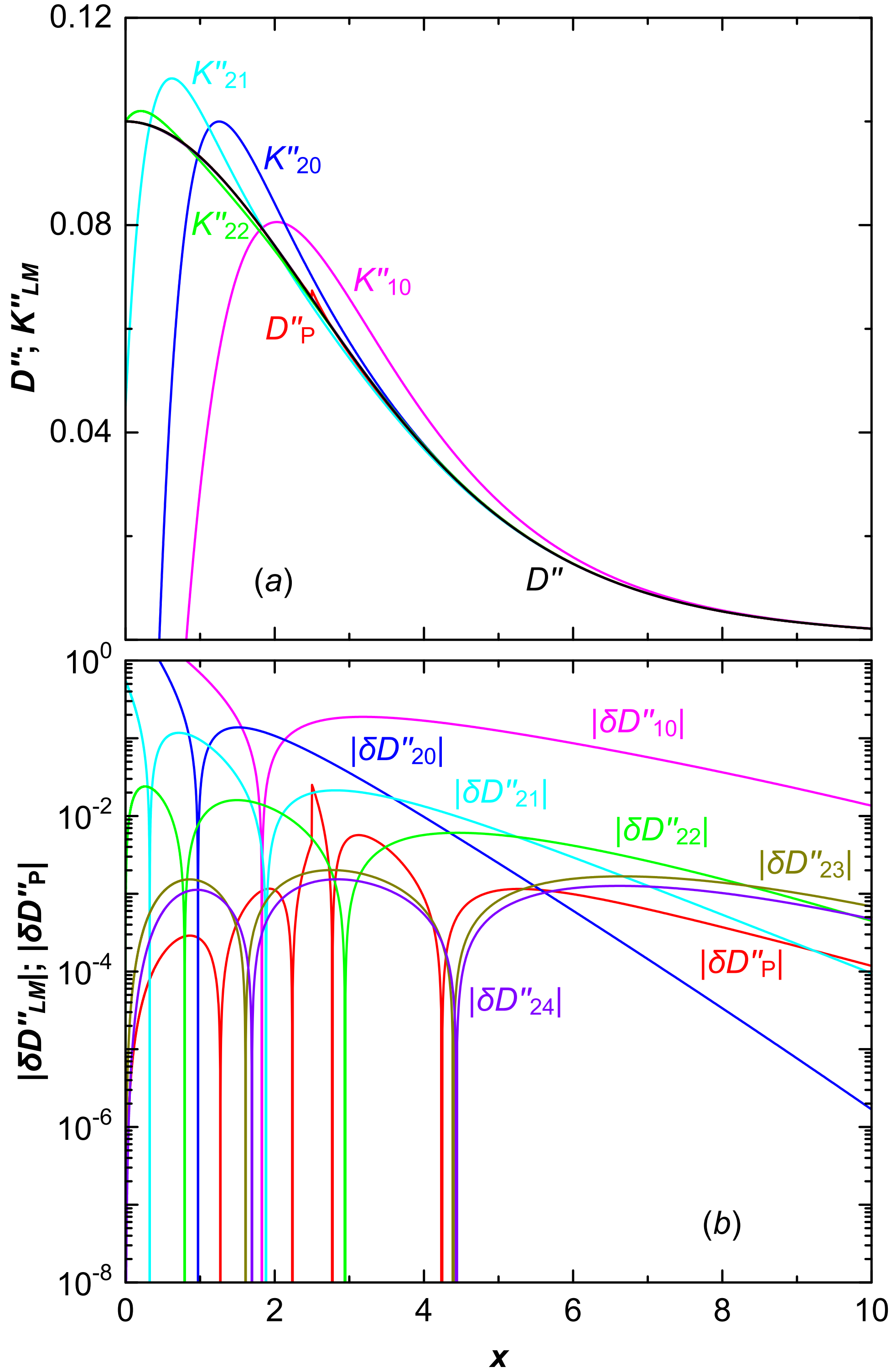

Figure 3: (a) The second derivative with respect to x of the Debye function, $D''(x)$, and the approximation functions, $K''_{LM}(x)$ and $D''_{\mathrm{P}}(x)$. (b) The absolute values of the relative deviations of the derivatives $K''_{LM}(x)$ and $D''_{\mathrm{P}}(x)$ from the reference derivative $D''(x)$, $|\delta D''_{LM}(x)| = |1 - K''_{LM}(x)/D''(x)|$ and $|\delta D''_{\mathrm{P}}(x)| = |1 - D''_{\mathrm{P}}(x)/D''(x)|$.

The maximum absolute values of the relative deviations in the region $x > 0$ are approximately 0.1 for $K_{10}(x)$, 0.5 for $K'_{10}(x)$ and 4.0 for $K''_{10}(x)$. Note that the relative deviation of the third derivative $K'''_{10}(x)$ from the reference $D'''(x)$ grows at $x \to 0$ in inverse proportion to x.

The ratios of the isochoric heat capacity $C_V(x)$ to its value in the high-temperature limit ($x \to 0$), $C_\mathrm{h} = 3R/\nu$, from equation (9) with the approximation derivative $K''_{10}(x)$ and from equation (13) with the approximation function $K_{10}(x)$ are shown in figure 4 in comparison with the reference dependence $C_V(x)/C_\mathrm{h}$ that is obtained using equation (9) with the reference $D''(x)$.

The maximum absolute value of the relative deviation from the reference dependence $C_V(x)$ is approximately 0.07 for the case of equation (9) with $K''_{10}(x)$ and 0.3 for the case of equation (13) with $K_{10}(x)$, as one can see in figure 4(*b*).

6 APPROXIMATIONS WITH L = 2

For $L = 2$, the approximation function $K_{LM}(x)$ has 5 variants $K_{2M}(x)$ for $M = 0$, 1, 2, 3 and 4 with 8 coefficients A_{li} at $l = 1$ and 2, $i = 0$, 1, 2 and 3:

$$A_{10} = 3 \text{ (for } M = 0,\ 1,\ 2 \text{ and } 3) \text{ or } A_{10} = \frac{8\pi^4}{15} - 49 \text{ (for } M = 4), \tag{36}$$

$$A_{11} = 9 \text{ (for } M = 0,\ 1 \text{ and } 2) \text{ or } A_{11} = \frac{8\pi^4}{5} - \frac{219}{2} - 12A_{10} \text{ (for } M = 3 \text{ and } 4), \tag{37}$$

$$A_{12} = 18 \text{ (for } M = 0 \text{ and } 1) \text{ or } A_{12} = \frac{16\pi^4}{5} - 117 - 36A_{10} - 8A_{11} \text{ (for } M = 2,\ 3 \text{ and } 4), \tag{38}$$

$$A_{13} = 18 \text{ (for } M = 0) \text{ or } A_{13} = \frac{16\pi^4}{5} - 39 - 24A_{10} - 12A_{11} - 4A_{12} \text{ (for } M = 1,\ 2,\ 3 \text{ and } 4), \tag{39}$$

$$A_{20} = \frac{4\pi^4}{15} - 1 - A_{10} - A_{11} - \frac{1}{2}A_{12} - \frac{1}{6}A_{13}, \tag{40}$$

$$A_{21} = \frac{2\pi^4}{5} - A_{11} - A_{12} - \frac{1}{2}A_{13}, \tag{41}$$

$$A_{22} = \frac{2\pi^4}{5} - A_{12} - A_{13}, \tag{42}$$

$$A_{23} = \frac{\pi^4}{5} - A_{13}. \tag{43}$$

Calculated values of the functions $K_{2M}(x)$ and their first and second derivatives $K'_{2M}(x)$ and $K''_{2M}(x)$ are shown in figures 1–3. One can see that the functions $K_{20}(x)$ and $K_{21}(x)$, as well as the function $K_{10}(x)$, are easily distinguishable from the reference function $D(x)$ in figure 1(*a*).

The derivatives $K'_{2M}(x)$ and $K''_{2M}(x)$ with $M = 0$, 1 and 2, as well as the derivatives $K'_{10}(x)$ and $K''_{10}(x)$, are also easily distinguishable from the reference derivatives $D'(x)$ and $D''(x)$ in figures 2(*a*) and 3(*a*).

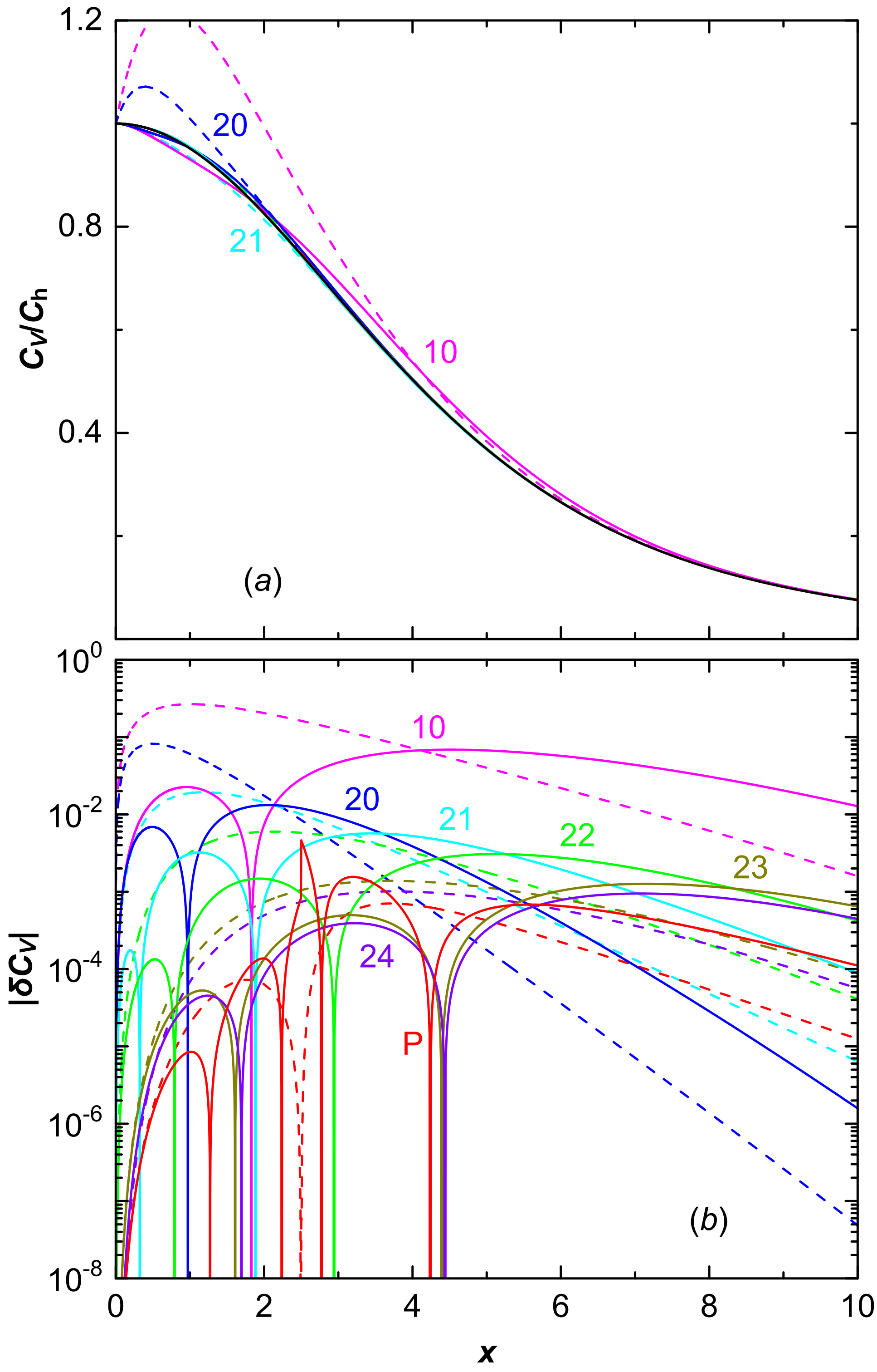

Figure 4: (a) The reference (black line) and approximation ratios $C_V(x)/C_\mathrm{h}$ (colored lines) and (b) the absolute values of the relative deviations $|\delta C_{V\mathrm{A}}(x)|$ of the approximation dependences (A $= LM$ and P) from the reference $C_V(x)$: solid lines—equation (9) with $K''_{LM}(x)$ and $D''_\mathrm{P}(x)$; dashed lines—equation (13) with $K_{LM}(x)$ and $D_\mathrm{P}(x)$.

A	$\lvert\delta D_{\mathrm{A}}\rvert_{\mathrm{m}}$	$\lvert\delta D'_{\mathrm{A}}\rvert_{\mathrm{m}}$	$\lvert\delta D''_{\mathrm{A}}\rvert_{\mathrm{m}}$	$\lvert\delta D'''_{\mathrm{A}}\rvert_{\mathrm{m}}$	$\lvert\delta C_{V\mathrm{A}(9)}\rvert_{\mathrm{m}}$	$\lvert\delta C_{V\mathrm{A}(13)}\rvert_{\mathrm{m}}$
10	1.00×10^{-1}	4.98×10^{-1}	$3.99\times10^{+0}$	∞	6.87×10^{-2}	2.66×10^{-1}
20	2.49×10^{-2}	3.01×10^{-1}	$4.61\times10^{+0}$	∞	1.32×10^{-2}	8.20×10^{-2}
21	7.50×10^{-3}	2.05×10^{-2}	5.48×10^{-1}	∞	5.68×10^{-3}	1.94×10^{-2}
22	3.05×10^{-3}	4.36×10^{-3}	2.40×10^{-2}	∞	3.07×10^{-3}	6.00×10^{-3}
23	9.51×10^{-4}	1.29×10^{-3}	2.02×10^{-3}	9.70×10^{-2}	1.27×10^{-3}	1.38×10^{-3}
24	6.99×10^{-4}	9.59×10^{-4}	1.53×10^{-3}	3.69×10^{-2}	9.48×10^{-4}	1.01×10^{-3}
P	4.73×10^{-4}	1.25×10^{-3}	2.52×10^{-2}	5.42×10^{-1}	4.62×10^{-3}	7.06×10^{-4}

Table 1: Maximum absolute values of the relative deviations $\delta D_{\mathrm{A}}(x)$, $\delta D'_{\mathrm{A}}(x)$, $\delta D''_{\mathrm{A}}(x)$, $\delta D'''_{\mathrm{A}}(x)$ and $\delta C_{V\mathrm{A}}(x)$ for $x > 0$, where the cases A $= LM$ and P correspond to the approximation functions K_{LM} (21) and D_{P} [47]; the last two columns correspond to the use of equations (9) and (13), respectively.

The remaining functions $K_{2M}(x)$ and derivatives $K'_{2M}(x)$ and $K''_{2M}(x)$ almost coincide with the reference dependences $D(x)$, $D'(x)$ and $D''(x)$ in figures 1(*a*), 2(*a*) and 3(*a*).

The absolute values of the relative deviations of the function $K_{2M}(x)$ and their derivatives $K'_{2M}(x)$ and $K''_{2M}(x)$ from the reference function $D(x)$ and its derivatives $D'(x)$ and $D''(x)$ are presented in figures 1(*b*), 2(*b*) and 3(*b*), respectively. One can see that the maxima of these absolute values for $L = 2$ are less than the corresponding maxima for $K_{10}(x)$, $K'_{10}(x)$ and $K''_{10}(x)$. These maxima at $L = 2$ decrease monotonically with increasing M (table 1).

In the best case for $L = 2$, the maximum absolute values of the relative deviations for $x > 0$ are approximately 0.0007 for $K_{24}(x)$, 0.001 for $K'_{24}(x)$, 0.002 for $K''_{24}(x)$ and 0.04 for $K'''_{24}(x)$.

Note that, for $L = 2$ and $M = 0$, 1 and 2, as well as for $L = 1$, the relative deviations of the third derivatives $K'''_{2M}(x)$ from the reference derivative $D'''(x)$ grow at $x \to 0$ in inverse proportion to x.

The ratios of the isochoric heat capacity $C_V(x)/C_{\mathrm{h}}$ from equation (9) with the approximation derivatives $K''_{2M}(x)$ and from equation (13) with the approximation functions $K_{2M}(x)$ are shown in figure 4. One can see that, for $L = 2$ and $M = 0$ and 1, as well as for $L = 1$, these ratios are easily distinguishable from the reference dependence $C_V(x)/C_{\mathrm{h}}$ in figure 4(*a*). For $L = 2$ and $M = 2$, the dependence $C_V(x)/C_{\mathrm{h}}$ can be distinguished in the case of the use of equation (13) with the approximation function $K_{22}(x)$. The dependences $C_V(x)/C_{\mathrm{h}}$ for the remaining cases of $L = 2$ and $M = 2$, 3 and 4 almost coincide with the corresponding reference dependence in figure 4(*a*).

As one can see in figure 4(*b*) and table 1, the maximum absolute values of the relative deviations $\lvert\delta C_{VLM}\rvert_{\mathrm{m}}$ (for $x > 0$) decrease monotonically with increasing L from 1 to 2 and with increasing M from 0 to 4. Moreover, using equation (13) gives a higher relative deviation than using equation (9). In the best case for $L=2$ and $M=4$, the maximum absolute value of the relative deviation $|\delta C_{VLM}|_\mathrm{m}$ in the region $x>0$ is approximately 0.0009 using equation (9).

For comparison, the results of using the piecewise continuously differentiable approximation function $D_\mathrm{P}(x)$ [47] are presented in figures 1–4 and table 1. Unlike analytic functions $K_{LM}(x)$, using equation (13) with $D_\mathrm{P}(x)$ gives a lower relative deviation than using equation (9) with discontinuous derivative $D''_\mathrm{P}(x)$. Despite the slightly lower values of the maximum deviations $|\delta D_\mathrm{P}|_\mathrm{m} \approx 0.0005$ and $|\delta C_{V\mathrm{P}(13)}|_\mathrm{m} \approx 0.0007$, the use of the analytic approximation function $K_{24}(x)$ seems preferable in thermodynamic models.

7 CONCLUSIONS

Thus, a family of analytic functions $K_{LM}(x)$ is proposed that approximates the Debye function $D(x)$, $x>0$, in closed form. Among these functions with $L=1$ and 2, the case of $K_{24}(x)$ for $x>0$ gives the lowest maximum relative deviations of the function and its first and second derivatives from the reference function $D(x)$ (less than 0.0007) and its derivatives $D'(x)$ (less than 0.001) and $D''(x)$ (less than 0.002), as well as the lowest maximum relative deviation for the value of the isochoric heat capacity (less than 0.001). The proposed expressions can be useful in modeling the equations of state for solids in a wide range of temperatures and densities.

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TO THE 80th ANNIVERSARY FROM THE BIRTH OF A.A. SAMOKHIN, DOCTOR OF PHYSICAL AND MATHEMATICAL SCIENCES, CHIEF RESEARCHER OF THE PROKHOROV GENERAL PHYSICS INSTITUTE OF THE RUSSIAN ACADEMY OF SCIENCES

V.I. MAZHUKIN[1*], Ž. PAVIĆEVIĆ[2,3], O.N. KOROLEVA[1], A.V. MAZHUKIN[1]

[1] Keldysh Institute of Applied Mathematics of the RAS, Moscow, Russia.

[2]Faculty of Natural Sciences and Mathematics, University of Montenegro, Podgorica, Montenegro

[3]National Research Nuclear University MEPhI, Moscow, Russia.

* Corresponding author. E-mail: vim@modhef.ru

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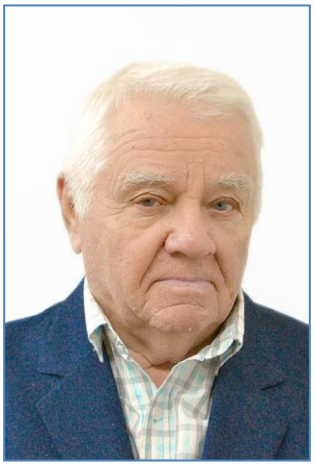
Summary. The article is dedicated to the 80th anniversary of the birth of the Soviet and Russian theoretical physicist, Doctor of Physical and Mathematical Sciences A.A. Samokhin, Chief Researcher of the Theoretical Department of the Institute of Prokhorov General Physics Institute of the RAS, a regular contributor to Mathematica Montisnigri and a long-term active participant in the international scientific seminar "Mathematical Models and Modeling in Laser-Plasma Processes and Advanced Scientific Technologies" (LPpM3), one of the founders of which is Mathematica Montisnigri.

11 december 2020 marks the 80th anniversary of the birth of Alexander Alexandrovich Samokhin, Doctor of Physics and Mathematics, Soviet and Russian physicist. Samokhin's scientific life is associated with the P.N. Lebedev Physical Institute of the RAS (FIAN), and later with the Prokhorov General Physics Institute of the RAS (GPI RAS), where he still works as the chief researcher of the theoretical department. "Physicist from God" - such a description was given to A.A. Samokhin in the book [1] “Events and People. (1948-2010)" written by an outstanding scientist in the field of plasma physics, twice laureate of the USSR State Prizes, laureate of the M.V. Lomonosov Moscow State University, Doctor of Physical and Mathematical Sciences, Professor Anri Amvrosyevich Rukhadze, with whom A.A. Samokhin had long-term cooperation. Aleksandr Aleksandrovich is a regular contributor to Mathematica Montisnigri and a long-term active participant in the international scientific seminar "Mathematical Models and Modeling in Laser-Plasma Processes and Advanced Scientific Technologies" (LPpM3, Montenegro), one of the founders of which is Mathematica Montisnigri. An active life position has led to the combination of scientific work with public discussion of scientific community problems. A. Samokhin is the chairman of the trade union



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of the Institute, a long-term member of the All-Russian Trade Union of RAS Workers, and also one of the organizers of the "Society of Scientists", created in 2012.

Alexander Alexandrovich was born in Rostov-on-Don and graduated from school there in 1957. He spent several war years outside the place of his birth, finding himself with his mother in the occupation near Kotelnikovo, known from the history of the Battle of Stalingrad.

The rapid development and widespread popularization of science in the USSR, characteristic of the post-war period (after the victory in the war with Nazi Germany in 1945), led A.A. Samokhin to the Faculty of Physics of M.V. Lomonosov Moscow State University, after which he entered the postgraduate course of the N.N. Semenov Institute of Chemical Physic of the AS of USSR. In his postgraduate studies, he is engaged in various approaches to the theoretical description of the response of a concentrated paramagnetic spin system in solids to an alternating magnetic field, showing additional interest in general problems of the nonequilibrium behavior of macroscopic systems. His first works [2-4] attracted the attention of the staff of the team headed by the famous scientist I. Prigozhin. The end of the postgraduate study in 1966 coincided with the time of the "Prokhorov recruitment" of young employees at the P.N. Lebedev Physical Institute of the RAS (FIAN), including A.A. Samokhin. Soon after, Samokhin defended his Ph.D. thesis on the theory of nonlinear response of a spin system in solids. Several years after defending his Ph.D. thesis, A.A. Samokhin. continued to actively deal with general issues of nonequilibrium statistical mechanics, returning to them later as well [5-7].

At the same time, he began to study new issues in the theory of the interaction of intense electromagnetic radiation with atoms, molecules and condensed media. During this study the circle of coauthors also expanded, including colleagues from the Moscow Engineering Physics Institute (MEPhI), M.V. Lomonosov Moscow State University, Baikov Institute of Metallurgy and Materials Science of the RAS (IMET) and other organizations.

The results of the work of A.A. Samokhin on the effect of laser radiation on individual quantum systems were published, in particular, in [8-10] and other articles, while his studies of the interaction of radiation with condensed media began with the question of the instability of the irradiated surface of a transparent liquid due to the ponderomotive effect [11]. Further theoretical and experimental studies, concerning, in the main, the effect of radiation on absorbing condensed media, are fairly fully reflected (until the mid-1980s) in three large articles from Proceedings of the Institute of General Physics Academy of Sciences of the USSR [12-14]. Articles [12,13] are actually material for A.A. Samokhin doctoral dissertation, which was defended in 2000.

In [12, 14], in particular, the efficiency of studying laser-induced fast phase transformations by registering the acoustic disturbances arising in this case was demonstrated and, using a model example [15], it was shown how the solution of the evaporation Stefan problem can change when taking into account the dependence of the state of the irradiated liquid on pressure.

In the same period, A.A. Samokhin started the fruitful scientific cooperation with the students of the outstanding Soviet and Russian mathematician, academician of the Academy of Sciences of USSR and the Russian Academy of Sciences, the founder of the Soviet and Russian schools of mathematical modeling, A.A. Samarsky from the Keldysh Institute of Applied Mathematics of the RAS. The publications of the first decades of this collaboration, in which the continuum and model-kinetic approaches were used, dealt with various nonequilibrium effects in laser ablation and were partially reflected in [16-22]. In subsequent theoretical works, in addition to the methods mentioned above, the molecular dynamic modeling was also used [23-32].

The results of these works, as well as of the other studies [33–37], made it possible to formulate a number of important conclusions on nonequilibrium laser ablation processes, among which one can single out the first formulated fundamental physical problem of determining the equilibrium physical characteristics of a substance, in particular, the parameters of its critical point, by the results of experiments in nonequilibrium conditions. Attention is drawn to the groundlessness of the widely used extension of the results of describing the decay of a metastable liquid at low overheating to the near-spinodal region, where the emerging nuclei of a new phase can no longer be considered independent. Under conditions of laser ablation, an important role is also played by the spatial inhomogeneity of temperature, which significantly affects the dynamics of the decay of a highly superheated metastable liquid, and possible abrupt changes in the electromagnetic properties of a substance such as a metal-insulator transition. When modeling the gas-dynamic boundary conditions at the evaporation front, the influence of the features of their dependence on the Mach number on the problem of the morphological stability of the evaporation front was established.

In addition to work on the main direction of his activity, AA Samokhin also paid attention to other issues, in particular, related to the manifestations of misunderstandings and misconduct in science, which are discussed in [1,9,12,13, 38-40].

We wish Alexander Alexandrovich Samokhin good health, long and fruitful scientific and social activities.

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К 80-ЛЕТИЮ СО ДНЯ РОЖДЕНИЯ ДОКТОРА ФИЗИКО-МАТЕМАТИЧЕСКИХ НАУК, ГЛАВНОГО НАУЧНОГО СОТРУДНИКА ИНСТИТУТА ОБЩЕЙ ФИЗИКИ ИМ. А.М. ПРОХОРОВА РАН А.А. САМОХИНА

В.И. МАЖУКИН[1*], ЖАРКО ПАВИЧЕВИЧ[2,3], О.Н. КОРОЛЕВА[1], А.В. МАЖУКИН[1]

[1] Институт прикладной математики им. М.В. Келдыша РАН, Москва, Россия

[2] Факультет естественных наук и математики, Университет Черногории, Подгорица, Черногория

[3]Национальный исследовательский ядерный университет МИФИ, Москва, Россия

*Ответственный автор. E-mail: vim@modhef.ru

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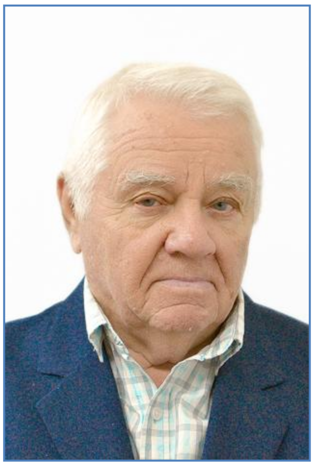
Ключевые слова: задача Стефана, фазовые превращения, лазерная абляция, критические параметры, оптоакустика, математическое моделирование.

Аннотация. Статья посвящена 80-летию со дня рождения советского и российского физика-теоретика, доктора физико-математических наук А.А. Самохина, главного научного сотрудника теоретического отдела Института общей физики им. А.М. Прохорова РАН, постоянного автора журнала Mathematica Montisnigri и многолетнего активного участника международного научного семинара «Математические модели и моделирование в лазерно-плазменных процессах и передовых научных технологиях» (LPpM3), одним из учредителей которого является журнал Mathematica Montisnigri.



11 декабря 2020 года исполнилось 80 лет со дня рождения Александра Александровича Самохина, доктора физико-математических наук, советского и российского физика. Научная жизнь Самохина связана с Физическим институтом им. П.Н. Лебедева АН СССР (ФИАН), а впоследствии и с Институтом общей физики им. А.М. Прохорова РАН (ИОФ РАН), в котором он и по сей день трудится в качестве главного научного сотрудника теоретического отдела. «Физик от бога» - такую характеристику дал А.А. Самохину в своей книге [1] «События и люди. (1948-2010 годы)» выдающийся ученый в области физики плазмы, дважды лауреат Государственных премий СССР, лауреат премии имени М.В. Ломоносова МГУ, доктор физико-математических наук, профессор Анри Амвросьевич Рухадзе, с которым А.А.Самохина связывало

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Key words and Phrases: Stefan problems, Phase changes, Laser ablation, Critical parameters, Optoacoustics, Mathematical modeling.

долговременное сотрудничество. Александр Александрович является постоянным автором журнала Mathematica Montisnigri и многолетним активным участником международного научного семинара «Математические модели и моделирование в лазерно-плазменных процессах и передовых научных технологиях» (LPpM3, Черногория), одним из учредителей которого является журнал Mathematica Montisnigri. Активная жизненная позиция привела к совмещению научной работы с общественной. Самохин А.А является председателем профсоюзной организации Института, многолетним членом Всероссийского профсоюза работников РАН, а также одним из организаторов «Общества научных работников», созданного в 2012 году.

Родился Александр Александрович в Ростове-на-Дону и там же окончил школу в 1957 г. Несколько военных лет он провел вне места своего рождения, оказавшиьсь вместе с матерью в оккупации близ Котельниково, известного по истории Сталинградской битвы.

Бурное развитие и широкая популяризация науки в СССР, характерная для послевоенного времени (после победы в войне с фашистской Германией в 1945 году), привели А.А. Самохина на физический факультет МГУ им. М.В. Ломоносова, после окончания которого, он поступает в аспирантуру Института химической физики им. Н.Н. Семенова АН СССР. В аспирантуре занимается различными подходами к теоретическому описанию отклика концентрированной парамагнитной спиновой системы твердого тела на переменное магнитное поле, проявляя дополнительный интерес к общим проблемам неравновесного поведения макроскопических систем. Его первые работы [2-4], привлекли внимание сотрудников коллектива, руководимого известным ученым И. Пригожиным. Окончание аспирантуры в 1966 г. совпало со временем очередного «Прохоровского набора» молодых сотрудников в Физический институт им. П.Н. Лебедева Академии Наук СССР, в числе которых оказался и Самохин А.А. Вскоре после этого Самохин защитил кандидатскую диссертацию по теории нелинейного отклика спиновой системы твердого тела. Несколько лет после защиты кандидатской диссертации Самохин А.А. продолжал активно заниматься общими вопросами неравновесной статистической механики, возвращаясь к ним и в более поздние времена [5-7].

В это же время он приступил к исследованию новых вопросов теории взаимодействия интенсивного электромагнитного излучения с атомами, молекулами и конденсированными средами. При этом расширялся и круг соавторов, в число которых входили также коллеги из Московского инженерно-физического института (МИФИ), МГУ им. М.В. Ломоносова, Института металлургии им. А.А. Байкова АН СССР (ИМЕТ) и других организаций.

Результаты работ Самохина А.А. по воздействию лазерного излучения на отдельные квантовые системы публиковались, частности, в [8-10] и других статьях, а его исследования взаимодействия излучения с конденсированными средами начались с вопроса о неустойчивости облучаемой поверхности прозрачной жидкости за счет пондеромоторного эффекта [11]. Дальнейшие теоретические и экспериментальные исследования, касающиеся, в основном, воздействия излучения на поглощающие конденсированные среды, достаточно полно отражены (до середины 80-х годов) в трех больших статьях сборника Труды ИОФАН [12-14]. Статьи [12,13] представляют собой фактически материал докторской диссертации, защита которой состоялась в 2000г.

В работах [12,14] была, в частности, продемонстрирована эффективность исследования лазерно-индуцированных быстрых фазовых превращений путем регистрации возникающих при этом акустических возмущений и на модельном примере [15] показано, как может изменяться решение испарительной задачи Стефана при учете зависимости состояния облучаемой жидкости от давления.

В тот же период началось, продолжающееся до настоящего времени, плодотворное научное сотрудничество А.А. Самохина с учениками выдающегося советского и российского математика, академика Академии Наук СССР и Российской Академии Наук, основоположника советской и российской школы математического моделирования, А.А. Самарского из Института прикладной математики им. М.В. Келдыша РАН. Публикации первых десятилетий этого сотрудничества, в которых использовались континуальные и модельно-кинетические подходы, касались различных неравновесных эффектов при лазерной абляции и частично отражены в [16-22]. В последующих теоретических работах в качестве основного подхода, кроме уже упомянутого выше, использовался метод молекулярно-динамического моделирования [23-32].

Результаты этих работ, а также других исследований [33-37], позволили сформулировать ряд важных выводов по неравновесным процессам лазерной абляции, в числе которых можно выделить впервые сформулированную фундаментальную физическую проблему определения равновесных физических характеристик вещества, в частности, параметров его критической точки, по результатам экспериментов в неравновесных условиях. Обращено внимание на необоснованность широко используемого распространения результатов описания распада метастабильной жидкости при малых перегревах на околоспинодальную область, где возникающие зародыши новой фазы уже нельзя считать независимыми. В условиях лазерной абляции важную роль играет также пространственная неоднородность температуры, существенно влияющая на динамику распада сильно перегретой метастабильной жидкости, и возможные резкие изменения электромагнитных свойств вещества типа перехода металл-диэлектрик. При моделировании газодинамических граничных условий на фронте испарения было установлено влияние особенностей их зависимости от числа Маха на проблему морфологической устойчивости испарительного фронта.

Кроме работ по основному направлению своей деятельности, А.А.Самохин уделял также внимания и другим вопросам, в частности, связанным с проявлениями недоразумений и недобросовестности в науке, которые обсуждаются в [1,9,12,13, 38-40].

Пожелаем Александру Александровичу Самохину крепкого здоровья, долгой и плодотворной научной и общественной деятельности.

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