SOME CONGRUENCES INVOLVING CATALAN, PELL AND FIBONACCI NUMBERS

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Summary. In this paper, using some special numbers and combinatorial identities, we show some interesting congruences: for a prime $p > 5$,

$$\sum_{k=0}^{(p-1)/2} \frac{C_k}{8^k (k+1)} \equiv 2^{(7-p)/2} \frac{P_p}{p} - 4\left(\frac{2}{p}\right) + 8 - \frac{2^{p+2}}{p} \pmod{p},$$

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{4^k k(k+1)} \equiv -q_p(2) + \frac{5}{6} \pmod{p},$$

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{(-16)^k k(k+1)(k+2)} \equiv \frac{L_{3p}}{2^{p+2} p} - \frac{2^{p-2}}{p} + \frac{5}{2}\left(\frac{5}{p}\right) - \frac{91}{40} \pmod{p},$$

where C_n, L_n and P_n are the n th Catalan number, the n th Lucas number and the n th Pell number, respectively. $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

1 INTRODUCTION

The Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ are defined by the following recursions:

$$F_{n+1} = F_n + F_{n-1} \quad \text{and} \quad L_{n+1} = L_n + L_{n-1}, \ n > 0,$$

where $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively. If α and β are the roots of equation $x^2 - x - 1 = 0$, the Binet formulas of the sequences $\{F_n\}$ and $\{L_n\}$ have the forms

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

respectively.

The Pell sequence $\{P_n\}$ and the Pell-Lucas sequence $\{Q_n\}$ are defined recursively by

$$P_{n+1} = 2P_n + P_{n-1} \quad \text{and} \quad Q_{n+1} = 2Q_n + Q_{n-1}, \ n > 0,$$

in which $P_0 = 0$, $P_1 = 1$ and $Q_0 = Q_1 = 2$, respectively. If γ and δ are the roots of equation

$x^2 - 2x - 1 = 0$, the Binet formulas of the sequences $\{P_n\}$ and $\{Q_n\}$ have the forms

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad Q_n = \gamma^n + \delta^n,$$

respectively.

The harmonic numbers have interesting applications in many fields of mathematics, such as number theory, combinatorics, analysis and computer science. Harmonic numbers H_n are defined as for a positive integer n

$$H_n = \sum_{k=1}^{n} \frac{1}{k},$$

where $H_0 = 0$. The first few harmonic numbers are $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \ldots$

Some elementary combinatorial properties of the Catalan numbers are given in [2, 6, 8]. The Catalan numbers are given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}, \ n \in N.$$

For a prime p and an integer a with $p \nmid a$, we write the Fermat quotient $q_p(a) = (a^{p-1} - 1)/p$. For an odd prime p and an integer a, $\left(\frac{a}{p}\right)$ denotes the Legendre symbol defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } a \text{ is a quadratic residue mod } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue mod } p. \end{cases}$$

In [3], E. Lehmer showed that for prime $p > 3$,

$$H_{(p-1)/2} \equiv -2q_p(2) \ (mod\, p). \tag{1.1}$$

In [9], Z.W. Sun obtained that

$$\sum_{k=0}^{(p-1)/2} \frac{1}{m^k}\binom{2k}{k} \equiv \left(\frac{m(m-4)}{p}\right)(mod\ p),$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{1}{m^k(k+1)}\binom{2k}{k} \equiv \frac{m}{2} - \frac{m-4}{2}\left(\frac{m(m-4)}{p}\right)(mod\ p),$$

where p is an odd prime and m is any integer not divisible by p.

Let p be a fixed prime > 3. Define

$$q(x) = \frac{x^p - (x-1)^p - 1}{p} \quad \text{and} \quad G_n(x) = \sum_{k=1}^{p-1} \frac{x^k}{k^n},$$

where x is variable.

In [1], A. Granville showed that

$$q(x) \equiv - G_1(x) \pmod{p},$$

$$G_2(x) \equiv G_2(1-x) + x^p G_2(1-1/x) \pmod{p},$$

$$(q(x))^2 \equiv -2x^p G_2(x) - 2(1-x^p) G_2(1-x) \pmod{p}.$$

It was proved in [1] (the last expression on page 3) that for any integer $n > 1$,

$$\sum_{k=1}^{n-1} \binom{n-1}{k-1} \frac{(-x)^k}{k^2} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{(1-x)^k - 1}{k} + \frac{(1-x)^n - (-x)^n - 1}{n^2}. \tag{1.2}$$

In [7], Z.H. Sun obtained the following congruences: for an odd prime p and $G_n(x) \in Z_p[x]$,

$$G_2(x) \equiv \frac{1}{p} \left(\frac{1 + (x-1)^p - x^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k - 1}{k} \right) + p \sum_{r=2}^{p-1} \frac{x^r}{r^2} \sum_{s=1}^{r-1} \frac{1}{s} \pmod{p^2},$$

and for a prime $p > 3$ and $n \in N$,

$$npG_{n+1}(x) \equiv (-1)^n x^p G_n(1/x) - G_n(x) \pmod{p^2}.$$

In [10], Z.W. Sun showed that for a prime $p > 3$,

$$\sum_{k=1}^{p-1} \frac{H_k L_k}{k} \equiv 0 \pmod{p} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{H_k F_k}{k} \equiv \frac{2}{p} \sum_{k=1}^{p-1} \frac{F_k}{k} \pmod{p}.$$

In [5], H. Pan and Z.W. Sun obtained that for a prime $p > 5$,

$$\sum_{k=1}^{p-1} \frac{L_k}{k^2} \equiv 0 \pmod{p},$$

and for a prime $p \neq 2,5$,

$$\sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k} \equiv \left(\frac{p^a}{5} \right) \left(1 - 2F_{p^a - \left(\frac{p^a}{5} \right)} \right) \pmod{p^3},$$

where a is a positive integer.

In [4], S. Mattarei and R. Tauraso showed that for a prime $p > 3$,

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{x^k}{k^2} \equiv 2(G_2(\lambda) + G_2(\mu)) \pmod{p},$$

where $\lambda = \frac{1}{2}(1 + \sqrt{1-4x})$ and $\mu = \frac{1}{2}(1 - \sqrt{1-4x})$.

2 SOME CONGRUENCES INVOLVING SPECIAL NUMBERS

In this section, we will give the congruences involving some special numbers. Now, we give the following lemma for further use.

Lemma 1. Let n be any positive integer. Then

$$\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{(-x)^{k+1}}{k^2(k+1)} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{(1-x)^{k+1} - 1}{k(k+1)} + \frac{(1-x)^{n+1} - 1}{n^2(n+1)} + \frac{x}{n} H_n$$

and

$$\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{(-x)^{k+2}}{k^2(k+1)(k+2)}=\frac{1}{n}\sum_{k=1}^{n-1}\frac{(1-x)^{k+2}-1}{k(k+1)(k+2)}+\frac{(1-x)^{n+2}-1}{n^2(n+1)(n+2)}+\frac{x}{n+1}-\frac{x^2}{2n}H_n.$$

Proof. From Binomial Theorem, we have

$$\sum_{k=1}^{n-1}\frac{(1-x)^{k+1}-1}{k(k+1)}+\frac{(1-x)^{n+1}-1}{n(n+1)}+xH_n$$

$$=\sum_{k=1}^{n}\frac{(1-x)^{k+1}-1}{k(k+1)}+x\sum_{k=1}^{n}\frac{1}{k}=\sum_{k=1}^{n}\frac{(1-x)^{k+1}-1+(k+1)x}{k(k+1)}$$

$$=\sum_{k=2}^{n+1}\frac{(1-x)^{k}-1+kx}{k(k-1)}=\sum_{k=2}^{n+1}\frac{1}{k(k-1)}\sum_{j=2}^{k}\binom{k}{j}(-x)^j=\sum_{k=2}^{n+1}\frac{1}{k-1}\sum_{j=2}^{k}\frac{(-x)^j}{j}\binom{k-1}{j-1}$$

$$=\sum_{k=2}^{n+1}\sum_{j=2}^{k}\frac{(-x)^j}{j(j-1)}\binom{k-2}{j-2}=\sum_{j=2}^{n+1}\frac{(-x)^j}{j(j-1)}\sum_{k=j}^{n+1}\binom{k-2}{j-2}.$$

From the equality $\sum_{k=j}^{n+1}\binom{k-2}{j-2}=\binom{n}{j-1}$, we get

$$\sum_{k=1}^{n-1}\frac{(1-x)^{k+1}-1}{k(k+1)}+\frac{(1-x)^{n+1}-1}{n(n+1)}+xH_n$$

$$=\sum_{j=2}^{n+1}\binom{n}{j-1}\frac{(-x)^j}{j(j-1)}=\sum_{j=1}^{n}\binom{n}{j}\frac{(-x)^{j+1}}{j(j+1)}=n\sum_{j=1}^{n}\binom{n-1}{j-1}\frac{(-x)^{j+1}}{j^2(j+1)}.$$

Thus, the first identity is obtained. Similarly, we obtain other identity.

For a prime p and a rational number $x=\frac{a}{b}$ written in reduced form, $x\equiv 0\;(mod\,p)$ if and only if a is divisible by p.

Theorem 1. Let p be an odd prime. For any rational number x such that $x^{-1}\not\equiv 0(mod\,p)$,

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{k+1}x^{k+1}\equiv 1-(\lambda-\mu)^{p+1}+\frac{2^{p-1}}{p}(\lambda^p+\mu^p-1)\;(mod\,p), \tag{2.1}$$

where $\lambda=\frac{1}{2}(1+\sqrt{1-4x})$ and $\mu=\frac{1}{2}(1-\sqrt{1-4x})$.

Proof. Replacing n and x by $(p+1)/2$ and $4x$ in (1.2), respectively, we have

$$\frac{p+1}{2}\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}\frac{(-4x)^{k+1}}{(k+1)^2}=\sum_{k=1}^{(p-1)/2}\frac{(1-4x)^k}{k}+2\frac{(1-4x)^{(p+1)/2}-1}{p+1}-H_{(p-1)/2}.$$

From the congruences $\binom{(p-1)/2}{k}(-4)^k\equiv\binom{2k}{k}\ (mod\ p)$ for $k=1,...,(p-1)/2$,

$\frac{1}{p+k}\equiv\frac{1}{k}\ (mod\ p)$ for $k=1,...,p-1$ and (1.1), we write

$$-2\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{x^{k+1}}{(k+1)^2}\equiv2\big((1-4x)^{(p+1)/2}-1\big)+\sum_{k=1}^{(p-1)/2}\frac{(1-4x)^k}{k}+2q_p(2)\ (mod\ p).\qquad(2.2)$$

By congruence $\binom{p-1}{k}\equiv(-1)^k\ (mod\ p)$ for $k=1,...,p-1$, we get

$$\sum_{k=1}^{(p-1)/2}\frac{(1-4x)^k}{k}\equiv-\sum_{k=1}^{(p-1)/2}\binom{p-1}{2k-1}\frac{(1-4x)^k}{k}\qquad(2.3)$$

$$=-\frac{2}{p}\sum_{k=1}^{(p-1)/2}\binom{p}{2k}(1-4x)^k=\frac{2-\big(1+\sqrt{1-4x}\big)^p-\big(1-\sqrt{1-4x}\big)^p}{p}\ (mod\ p).$$

Substituting (2.3) into (2.2), we have

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{k+1}x^{k+1}=\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{x^{k+1}}{(k+1)^2}$$

$$\equiv1-(1-4x)^{(p+1)/2}-q_p(2)-\frac{\big(\sqrt{1-4x}-1\big)^p-\big(\sqrt{1-4x}+1\big)^p+2}{2p}$$

$$=1-(\lambda-\mu)^{p+1}+\frac{2^{p-1}}{p}\big(\lambda^p+\mu^p-1\big)\ (mod\ p).$$

Thus, this concludes the proof.

From Theorem 1, we immediately deduce the following results.

Corollary 1. Let p be an odd prime. Then

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{4^k(k+1)}\equiv4\big(1-q_p(2)\big)\ (mod\ p),$$

$$\sum_{k=0}^{(p-1)/2}\frac{(-1)^kC_k}{k+1}\equiv2^{p-1}\frac{1-L_p}{p}+5\left(\frac{5}{p}\right)-1\ (mod\ p),\qquad(2.4)$$

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{8^k(k+1)}\equiv2^{(7-p)/2}\frac{P_p}{p}-4\left(\frac{2}{p}\right)+8-\frac{2^{p+2}}{p}\ (mod\ p),$$

$$\sum_{k=0}^{(p-1)/2} \frac{C_k}{(-4)^k (k+1)} \equiv -\frac{2Q_p}{p} + \frac{2^{p+1}}{p} + 8\left(\frac{2}{p}\right) - 4 \pmod{p},$$

$$\sum_{k=0}^{(p-1)/2} \frac{C_k}{(-16)^k (k+1)} \equiv -\frac{2^{3-p}}{p} L_{3p} + \frac{2^{p+3}}{p} + 20\left(\frac{5}{p}\right) - 16 \pmod{p},$$

and for $p > 3$

$$\sum_{k=0}^{(p-1)/2} \frac{C_k}{9^k (k+1)} \equiv \frac{2^{p-1}}{p}\left(\frac{L_{2p}}{3^{p-2}} - 9\right) - 5\left(\frac{5}{p}\right) + 9 \pmod{p}.$$

Proof. For the proof (2.4), considering $x = -1$ in Theorem 1, we have

$$\sum_{k=0}^{(p-1)/2} \frac{C_k}{(-1)^{k+1}(k+1)}$$

$$\equiv 1 - 5^{(p+1)/2} + \frac{2^{p-1}}{p}\left(\left(\frac{1+\sqrt{5}}{2}\right)^p + \left(\frac{1-\sqrt{5}}{2}\right)^p - 1\right) \equiv 1 - 5\left(\frac{5}{p}\right) + \frac{2^{p-1}}{p}\left(L_p - 1\right) \pmod{p}.$$

Similarly, we obtain other five congruences of Corollary 1.

Theorem 2. Let $p > 3$ be a prime. For any rational number x such that $x^{-1} \not\equiv 0 \pmod{p}$,

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{k(k+1)} x^{k+1} \equiv \frac{2^{p-1} x}{p}\left(\lambda^p + \mu^p - 1\right) - \frac{8x+1}{12}(\lambda - \mu)^{p+1} + \frac{6x+1}{12} \pmod{p},$$

where λ and μ are defined as before.

Proof. By the first identity of Lemma 1, we have

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-x)^{k+2}}{(k+1)^2 (k+2)} = \frac{1}{n}\sum_{k=1}^{n-1} \frac{(1-x)^{k+1} - 1}{k(k+1)} + \frac{(1-x)^{n+1} - 1}{n^2(n+1)} + \frac{x}{n} H_n.$$

Replacing n and x by $(p+1)/2$ and $4x$ in the aboving sum, respectively, we have

$$\sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} \frac{(-4x)^{k+2}}{(k+1)^2 (k+2)} = \frac{2}{p+1}\sum_{k=1}^{(p-1)/2} \frac{(1-4x)^{k+1} - 1}{k(k+1)} + 8\frac{(1-4x)^{(p+3)/2} - 1}{(p+1)^2 (p+3)} + \frac{8x}{p+1} H_{(p+1)/2}.$$

From the congruences $\binom{(p-1)/2}{k}(-4)^k \equiv \binom{2k}{k} \pmod{p}$ for $k=1,\ldots,(p-1)/2$,

$\frac{1}{p+k} \equiv \frac{1}{k} \pmod{p}$ for $k=1,\ldots,p-1$ and (1.1), we write

15

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k} \frac{x^{k+2}}{(k+1)^2(k+2)}$$

$$\equiv \frac{(1-4x)-1}{8} \sum_{k=1}^{(p-1)/2} \frac{(1-4x)^k}{k} + \frac{1}{4(p+1)} - \frac{(1-4x)^{(p+1)/2}}{4(p+1)} + \frac{(1-4x)^{(p+3)/2}}{6} + x - \frac{x}{2} - xq_p(2) - \frac{1}{6} \pmod{p}.$$

From the congruences (2.3) and $\frac{1}{p+1} \equiv 1 \pmod{p}$, we have

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{k(k+1)} x^{k+1}$$

$$\equiv -\frac{x}{2p}\left(2 - \left(1+\sqrt{1-4x}\right)^p - \left(1-\sqrt{1-4x}\right)^p\right) - \frac{(1-4x)^{(p+1)/2}}{4} + \frac{(1-4x)^{(p+3)/2}}{6} + \frac{x}{2} - xq_p(2) - \frac{5}{2}$$

$$= -\frac{x}{2p}\left(2^p - \left(1+\sqrt{1-4x}\right)^p - \left(1-\sqrt{1-4x}\right)^p\right) - \frac{(1-4x)^{(p+1)/2}}{4} + \frac{(1-4x)^{(p+3)/2}}{6} + \frac{x}{2} - \frac{5}{2} \pmod{p}.$$

Thus, the desired result is obtained.

From the above equation, we get the assertion of the theorem.

As an immediate consequence of Theorem 2, we obtain the following result.

Corollary 2. Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{4^k k(k+1)} \equiv -q_p(2) + \frac{5}{6} \pmod{p},$$

$$\sum_{k=1}^{(p+1)/2} \frac{(-1)^k C_{k-1}}{k(k+1)} \equiv 2^{p-1} \frac{L_p-1}{p} - \frac{35}{12}\left(\frac{5}{p}\right) + \frac{5}{12} \pmod{p},$$

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{8^k k(k+1)} \equiv \frac{P_p}{2^{(p-1)/2}p} - \frac{2^{p-1}}{p} - \frac{4}{3}\left(\frac{1}{2}\right)^{(p+1)/2} + \frac{7}{6} \pmod{p},$$

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{9^k k(k+1)} \equiv \frac{2^{p-1}}{p}\left(\frac{L_{2p}}{3^p} - 1\right) - \frac{85}{108}\left(\frac{5}{p}\right) + \frac{5}{4} \pmod{p},$$

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{(-4)^k k(k+1)} \equiv \frac{Q_p - 2^p}{2p} - \frac{2}{3}\left(\frac{2}{p}\right) + \frac{1}{6} \pmod{p},$$

and

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{(-16)^k k(k+1)} \equiv \frac{L_{3p}}{2^{p+1}p} - \frac{2^{p-1}}{p} + \frac{5}{6}\left(\frac{5}{p}\right) - \frac{5}{6} \pmod{p}.$$

Theorem 3. Let $p > 5$ be a prime. For any rational number x such that $x^{-1} \not\equiv 0 \pmod{p}$,

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{k(k+1)(k+2)} x^{k+2} \equiv \frac{2^{p-2}}{p} x^2 \left(\lambda^p + \mu^p - 1\right) - \frac{1}{60} (\lambda - \mu)^{p+5}$$

$$+ \frac{5}{96} (\lambda - \mu)^{p+3} - \frac{1}{32} (\lambda - \mu)^{p+1} + \frac{1}{8} x^2 + \frac{1}{12} x - \frac{1}{240} \pmod{p},$$

where λ and μ are defined as before.

As an direct consequence of Theorem 3, we obtain the following result.

Corollary 3. Let $p > 5$ be a prime. Then

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{4^k k(k+1)(k+2)} \equiv -\frac{1}{2} q_p(2) + \frac{47}{120} \pmod{p},$$

$$\sum_{k=1}^{(p+1)/2} \frac{(-1)^k C_{k-1}}{k(k+1)(k+2)} \equiv 2^{p-2} \frac{L_p - 1}{p} - \frac{15}{16} \left(\frac{5}{p}\right) + \frac{3}{80} \pmod{p},$$

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{8^k k(k+1)(k+2)} \equiv \frac{P_p}{2^{(p+1)/2} p} - \frac{2^{p-2}}{p} - \frac{3}{5} \left(\frac{1}{2}\right)^{(p+1)/2} + \frac{21}{40} \pmod{p},$$

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{9^k k(k+1)(k+2)} \equiv \frac{2^{p-2}}{p} \left(\frac{L_{2p}}{3^p} - 1\right) - \frac{29}{48} \left(\frac{5}{9}\right)^{(p+1)/2} + \frac{43}{80} \pmod{p},$$

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{(-4)^k k(k+1)(k+2)} \equiv \frac{Q_p - 2^p}{4p} + \frac{1}{5} \left(\frac{2}{p}\right) - \frac{11}{40} \pmod{p},$$

and

$$\sum_{k=1}^{(p+1)/2} \frac{C_{k-1}}{(-16)^k k(k+1)(k+2)} \equiv \frac{L_{3p}}{2^{p+2} p} - \frac{2^{p-2}}{p} + \frac{5}{2} \left(\frac{5}{p}\right) - \frac{91}{40} \pmod{p}.$$

3. CONCLUSION

Using combinatorial identities, some interesting congruences are investigated on the sums which include Catalan numbers. It is conceivable to extend our results using the useful technics or methods. A first question on the extension of these congruences can be viewed as generalizations on using the q-Binomial coefficients instead of the binomial coefficients. A second question on such extensions of these congruences can be observed that for an odd prime p and any rational number x such that $x^{-1} \not\equiv 0 \pmod{p}$, the congruences of the sums with decreasing factorials including Catalan numbers $\sum_{k=1}^{(p-1)/2} \frac{C_{k-1}}{k^{\underline{m}}} x^k \pmod{p}$.

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