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SOME ASPECTS OF NEYMAN TRIANGLES AND DELANNOY ARRAYS

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Summary. This note considers some number theoretic properties of the orthonormal Neyman polynomials which are related to Delannoy numbers and certain complex Delannoy numbers.

1 INTRODUCTION

Rayner and Best point out that “the concept of smooth goodness of fitness tests was introduced in Neyman (1937)” [22]. Goodness of fit concepts in general usually go back to Karl Pearson [20]. Rayner [21] further pointed out that Jerzy Neyman’s smooth alternative of order k to the uniform distribution on (0,1) has probability density for

$$h\ (y,\theta) = exp\ \left.\sum_{i=1} \theta_i \pi_i(y) - K(\theta)\right\}, 0 < y < 1, k = 1,2,... \tag{1.1}$$

where $K(\theta)$ is a normalising constant and the $\pi_i(y)$ are orthonormal polynomials (Freeman) related to the Legendre polynomials.

It is the purpose of this note to consider some number theoretic properties of the $\pi_i(y)$ polynomials (i = 0,1,2,3,4 in Rayner) which, for convenience, we label as Neyman polynomials. In Deveci and Shannon [9] complex-type k-Fibonacci numbers are defined and the relationships between the k-step Fibonacci numbers and the complex-type k-Fibonacci numbers are provided together with miscellaneous properties of the complex-type k-Fibonacci numbers. In addition, they studied the complex-type k-Fibonacci sequence modulo m. Finally, they obtained the period of the complex-type 2-Fibonacci sequences in the Dihedral group D_{2n}, $(n \geq 2)$.

In this paper, we define the complex-type Delannoy numbers and then give the relationships between the Delannoy numbers and the complex-type Delannoy numbers. Furthermore, we study the complex-type Delannoy sequence modulo m.

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2 NEYMAN POLYNOMIALS

Rayner elsewhere lists the first five such polynomials and we add some more in order to build up a picture of patterns. To help with this we have slightly modified some aspects of his notation as in Bera and Ghosh [3]:

$$\pi_0(y) = \sqrt{1}(1)$$

$$\pi_1(y) = \sqrt{3}(2y-1)$$

$$\pi_2(y) = \sqrt{5}(6y^2-6y+1)$$

$$\pi_3(y) = \sqrt{7}(20y^3-30y^2+12y-1)$$

$$\pi_4(y) = \sqrt{9}(70y^4-140y^3+90y^2-20y+1)$$

$$\pi_5(y) = \sqrt{11}(252y^5-630y^4+560y^3-210y^2+30y-1)$$

$$\pi_6(y) = \sqrt{13}(924y^6-2772y^5+3150y^4-1680y^3+420y^2-42y+1).$$

Blinov and Lemeshko [4] have set out corresponding Legendre polynomials as, in effect,

$$p_0(y) = \sqrt{1}(1)$$

$$p_1(y) = \sqrt{3}(2y)$$

$$p_2(y) = \sqrt{5}(6y^2-0.5)$$

$$p_3(y) = \sqrt{7}(20y^3-3y)$$

$$p_4(y) = \sqrt{9}(70y^4-15y^2+0.375).$$

3 NEYMAN TRIANGLE

We assemble the absolute values of the polynomial coefficients into a triangle, as the row sums are all unity if we include the signed values of the coefficients. The row sums are in the right-most column, and the pertinent OIES references [23] are in the bottom row.

1							1
2	1						3
6	6	1					13
20	30	12	1				63
70	140	90	20	1			321
252	630	560	210	30	1		1683
924	2772	3150	1680	420	42	1	8989
A000984	A002457	A002544	A007744	A106440	A013613	---	A001850

Table 1: Neyman triangle

The leading diagonals in this table generate the sequence {1,2,7,26,101,404,1645,…} which does not seem to be in OEIS, but the anti-diagonals can related to OEIS sequences in Table 2(a).

1	1	1	1	1	1	1	1	A000012
2	6	12	20	30	42	56	72	A002378
6	30	90	210	420	756	1260	1980	A033487
20	140	560	1680	4200	9240	18480	34320	A105939
70	630	3150	11550	34650	90090	210210	450450	70xA000581

Table 2(a): Anti-diagonals in Neyman triangle

The patterns are clearer when we express the Neyman anti-diagonals as multiples of the first element in each row, as in Table 2 (b). The leading diagonal here yields a known sequence (A005809) as do the anti-diagonals (A001519), the odd Fibonacci numbers as a bisection of the Fibonacci sequence, but we shall not pursue these here.

1 X	1	1	1	1	1	1	1	1	A000012
2 X	1	3	6	10	15	21	28	36	A000217
6 X	1	5	15	35	70	126	210	330	A000332
20 X	1	7	28	84	210	462	924	1716	A000579
70 X	1	9	45	165	495	1287	3003	6435	A000581
A0…..	00012	05408	0384	000447	53134	02299	53135	53136	

Table 2(b): Anti-diagonals in Neyman triangle

The leading diagonals in Table 2(a) generate the sequence {1,3,13,63,321,1683,8989,…} [A001850] the elements of which are the Central Delannoy numbers [2], so called because they constitute the central anti-diagonal in the infinite square Delannoy array [A008288] in Table 3. The leading anti-diagonal here is A005809.

$n\downarrow$ $m\rightarrow$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	1	1	1	1	1	1	1	1
1	1	3	5	7	9	11	13	15
2	1	5	13	25	41	61	85	113
3	1	7	25	63	129	231	377	575
4	1	9	41	129	321	681	1289	2241
5	1	11	61	231	681	1683	3653	7183
6	1	13	85	377	1289	3653	8989	19825
7	1	15	113	575	2241	7183	19825	48639

Table 3: Square Delannoy array

The leading diagonals in this array generate the Pell numbers {1,2,5,12,29,…}, and, in the sense of this paper, Alladi and Hoggatt [1] further related these numbers to Tribonacci triangles. When this array is turned clockwise through 45^0 we have the Pell triangle.

We also see regular intersections (as common elements) among the row and column sequences, which is a topic worth exploring as in Stein [24] who found it necessary to examine the intersection of Fibonacci sequences in order to answer the question of whether every member of a variety is a quasigroup given that every finite member is [25].

The Central Delannoy numbers $\{a_n\}, n \geq 0$, can be expressed as

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \tag{3.1}$$

and

$$a_n = \frac{\pi_n(2)}{\sqrt{n}} \tag{3.2}$$

in terms of the Neyman numbers, which would appear to be new. This suggests we consider in turn

$$\frac{\pi_n(3)}{\sqrt{n}} = \{1,5,37,305,2641,23525, \dots\}$$

which is A006442, the expansion of $(x^2 - 10x + 1)^{-\frac{1}{2}}$, which is also related to the Delannoy numbers. Likewise A084768 is

$$\frac{\pi_n(4)}{\sqrt{n}} = \{1,7,73,847,10321,129367,1651609, \dots\}$$

and so on.

4 THE COMPLEX-TYPE DELANNOY NUMBERS

Now we define a new sequence that we call the complex-type Delannoy sequence $\{D^i(m,n)\}$ as follows:

$$D^i(m,n) = \begin{cases} 1 & \text{if } m=0 \text{ or } n=0, \\ i \cdot D^i(m-1,n) + i \cdot D^i(m,n-1) - D^i(m-1,n-1) & \text{otherwise.} \end{cases} \tag{1}$$

Note that when $m = n = a$, the complex-type Delannoy sequence $\{D^i(m,n)\}$ is reduced to the central complex-type sequence $\{D^i(a,a)\}$.

A table for the values of the complex-type Delannoy numbers is given by below:

n↓ m→	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	1	1	1	1	1	1	1	1
1	1	2*i*-1	-3	-2*i*-1	1	2*i*-1	-3	-2*i*-1
2	1	-3	-8*i*+1	13	16*i*+1	-19	-24*i*+1	29
3	1	-2*i*-1	13	34*i*-1	-63	-98*i*-1	141	194*i*-1
4	1	1	16*i*+1	-63	-160*i*+1	321	560*i*+1	-895
5	1	2*i*-1	-19	-98*i*-1	321	802*i*-1	-1683	-3138*i*-1
6	1	-3	-24*i*+1	141	560*i*+1	-1683	-4168*i*+1	8989
7	1	-2*i*-1	29	194*i*-1	-895	-3138*i*-1	8989	22146*i*-1

Table 4: Square complex-type Delannoy numbers

From the definitions of the Delannoy numbers and the complex-type Delannoy numbers, we derive the following relations:

i. For $m,n \geq 1$

$$D^i(m,n)=\begin{cases} 2(i)^n \cdot D(m-1,n-1)-D^i(m-1,n-1), & n\equiv 1 \pmod 4, \\ 2(i)^{n+1} \cdot D(m-1,n-1)-D^i(m-1,n-1), & n\equiv 2 \pmod 4, \\ 2(i)^{n+2} \cdot D(m-1,n-1)-D^i(m-1,n-1), & n\equiv 3 \pmod 4, \\ 2(i)^{n+3} \cdot D(m-1,n-1)-D^i(m-1,n-1), & n\equiv 0 \pmod 4. \end{cases}$$

ii. For $m,n \geq 0$, $D^i(m,n)=D^i(n,m)$.

iii. For $m,n \geq 0$, $D^i(n+1,n)=D^i(n,n+1)=(-1)^n \cdot D(n,n)$.

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence.

The research on the conformity of a single term, $a_n \pmod p$, has a long history forming most known Pascal's oldest fractal problem, which was originally created by the parities of binomial coefficients $\binom{n}{k}$; see for example, [5,6,7,8,10,12,14,16,17,18,29,30]. We now extend the concept to the complex-type Delannoy numbers.

Consider the sequence

$$\{D^i(m,n)\}=\{D^i(0,n), D^i(1,n), D^i(2,n),\ldots\}$$

where n is a fixed positive integer and $m=0,1,2,\ldots$.

If we reduce the sequence $\{D^i(m,n)\}$ modulo α, taking least nonnegative residues, then we can get the repeating sequence, denoted by

$$\{D^i(m,n)(\alpha)\}=\{D^i(0,n)(\alpha), D^i(1,n)(\alpha), D^i(2,n)(\alpha),\ldots\}$$

where $D^i(u,n)(\alpha)$ is used to mean the uth element of the sequence $\{D^i(m,n)(\alpha)\}$ modulo α for the positive integer constant n.

We note here that the sequence $\{D^i(m,n)(\alpha)\}$ has the same recurrence relation as in (1).

Theorem 4.1. The sequence $\{D^i(m,n)(\alpha)\}$ is periodic.

Proof. It is clear that sequence $\{D^i(m,1)(\alpha)\}$ is a constant sequence. Since the sequence $\{D^i(m,1)(\alpha)\}$ is a constant sequence; that is, since it consists only the repetitions of a constant subsequence, we can say that the sequence $\{D^i(m,2)(\alpha)\}$ is also a periodic sequence, using the recurrence relation in the sequence $\{D^i(m,n)(\alpha)\}$. Similarly, since the sequences $\{D^i(m,1)(\alpha)\}$ and $\{D^i(m,2)(\alpha)\}$ are periodic; that is, they consist only the repetitions of constant sub-sequences, the sequence $\{D^i(m,n)(\alpha)\}$ is also periodic. By a similar idea, we get the repeating sequences

$$\{D^i(m,1)(\alpha)\},\ \{D^i(m,2)(\alpha)\},\ldots,\ \{D^i(m,n-1)(\alpha)\}$$

are periodic; that is, they consist only the repetitions of constant sub-sequences, using the recurrence relation in the sequence $\{D^i(m,n)(\alpha)\}$. Thus, this implies that the sequence $\{D^i(m,n)(\alpha)\}$ is periodic. □

Example 2.1. We have

$$\{D^i(m,3)(3)\} = \begin{Bmatrix} 1, i-1, 1, i-1, 0, i-1, 0, 2i-1, 0, 2i-1, 1, i-1, \\ 1, i-1, 1, i-1, 0, i-1, 0, 2i-1, 0, 2i-1, 1, i-1, \ldots \end{Bmatrix}$$

and its terms repeat so we get $L(D^i(m,3)(3)) = 12$, where the period of the sequence $\{D^i(m,n)(\alpha)\}$ is denoted by $L(D^i(m,n)(\alpha))$.

Conjecture 4.1. Let p be prime, let n be a fixed positive integer and $m = 0,1,2,\ldots$. If u is the smallest positive integer such that $L(D^i(m,n)(p^{u+1})) \neq L(D^i(m,n)(p^u))$, then $L(D^i(m,n)(p^v)) = p^{v-u} \cdot L(D^i(m,n)(p^u))$.

Theorem 4.2. Let α_1 and α_2 be positive integers with $\alpha_1, \alpha_2 \geq 2$, then

$$L(D^i(m,n)(lcm(\alpha_1,\alpha_2))) = lcm\left[L(D^i(m,n)(\alpha_1)), L(D^i(m,n)(\alpha_2))\right].$$

Proof. Let $lcm(\alpha_1,\alpha_2) = \alpha$. Then,

$$\begin{aligned} D^i(m,n)\left[L(D^i(m,n)(\alpha))\right] &\equiv D^i(m,n)\left[L(D^i(m,n)(\alpha))+1\right] \\ &\equiv \cdots \equiv D^i(m,n)\left[L(D^i(m,n)(\alpha))+n-1\right] \equiv 0 \pmod{\alpha} \end{aligned}$$

and

$$\begin{aligned} D^i(m,n)\left[L(D^i(m,n)(\alpha_k))\right] &\equiv D^i(m,n)\left[L(D^i(m,n)(\alpha_k))+1\right] \\ &\equiv \cdots \equiv D^i(m,n)\left[L(D^i(m,n)(\alpha_k))+n-1\right] \equiv 0 \pmod{\alpha_k} \end{aligned}$$

for $k = 1,2$. Using the least common multiple operation this implies that

$$\begin{aligned} D^i(m,n)\left[L(D^i(m,n)(\alpha))\right] &\equiv D^i(m,n)\left[L(D^i(m,n)(\alpha))+1\right] \\ &\equiv \cdots \equiv D^i(m,n)\left[L(D^i(m,n)(\alpha))+n-1\right] \equiv 0 \pmod{\alpha_k} \end{aligned}$$

for $k=1,2$. So we have $L\left(D^i(m,n)(\alpha_1)\right)\middle|L\left(D^i(m,n)(\alpha)\right)$ and $L\left(D^i(m,n)(\alpha_2)\right)\middle|L\left(D^i(m,n)(\alpha)\right)$, which means that $lcm\left[L\left(D^i(m,n)(\alpha_1)\right),L\left(D^i(m,n)(\alpha_2)\right)\right]$ divides $L\left(D^i(m,n)\left(lcm(\alpha_1,\alpha_2)\right)\right)$. We also know that

$$D^i(m,n)\left[lcm\left(L\left(D^i(m,n)(\alpha_1)\right),L\left(D^i(m,n)(\alpha_2)\right)\right)\right]\equiv D^i(m,n)\left[lcm\left(L\left(D^i(m,n)(\alpha_1)\right),L\left(D^i(m,n)(\alpha_2)\right)\right)+1\right]$$
$$\equiv\cdots\equiv D^i(m,n)\left[lcm\left(L\left(D^i(m,n)(\alpha_1)\right),L\left(D^i(m,n)(\alpha_2)\right)\right)+n-1\right]\equiv 0(\bmod\alpha_k).$$

Then,

$$D^i(m,n)\left[lcm\left(L\left(D^i(m,n)(\alpha_1)\right),L\left(D^i(m,n)(\alpha_2)\right)\right)\right]\equiv D^i(m,n)\left[lcm\left(L\left(D^i(m,n)(\alpha_1)\right),L\left(D^i(m,n)(\alpha_2)\right)\right)+1\right]$$
$$\equiv\cdots\equiv D^i(m,n)\left[lcm\left(L\left(D^i(m,n)(\alpha_1)\right),L\left(D^i(m,n)(\alpha_2)\right)\right)+n-1\right]\equiv 0(\bmod\alpha).$$

and it follows that $L\left(D^i(m,n)\left(lcm(\alpha_1,\alpha_2)\right)\right)$ divides $lcm\left[L\left(D^i(m,n)(\alpha_1)\right),L\left(D^i(m,n)(\alpha_2)\right)\right]$. Therefore, we have the following conclusions. □

Corollary 4.1. Let v and u be positive integers. If $n=2^v$, then $L\left(D^i(m,n)\left(2^u\right)\right)=2^{u-v-1}$ for $u+2\geq v$.

Corollary 4.2. Let n be a positive integer and u a positive integer such that $u\geq 2$. Then $L\left(D^i(m,n)\left(2^u\right)\right)=2^{u-1}$.

5 CONCLUDING COMMENTS

Lavers' Lemma 5 [15] suggests a way to generalize (3.1) to produce corresponding pyramids, and Horadam [13] and Subba Rao [26,27,28] contain further ideas on the study of intersections of sequences.

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