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# NOTE ON A THEOREM OF ZEHNXIAG ZHANG

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**Summary.** A sequence  of strictly positive integers is said to be primitive if none of its terms divides the others. In this paper, we give a new proof of a result,  conjectured by P. Erdős and Z. Zhang in 1993, on a primitive sequence whose the number of the prime factors of the termes counted with multiplicity is at most 4. The objective of this proof is to improve the complexity, which helps to prove this conjecture.

## 1. INTRODUCTION

A sequence A of strictly positive integers is said to be primitive if none of its terms divides the others. We define the degree of A by $deg(\mathrm{A}) = max\{\Omega(a)\ a \in \mathrm{A}\}$ where $\Omega(a)$ is the number of prime factors of $a$ counted with multiplicity, we take $deg(A) = 0$ if $A = \{1\}$ or $\emptyset$. Erdős [2] showed that for a primitive set $A$, $\sum_{a\in A}\frac{1}{a\log a} < \infty$. Later in [3], Erdős asked if is true that for any primitive sequence $A$,

$$\sum_{a\in A,\ a\leq n}\frac{1}{a\log a} \leq \sum_{p\in P,\ p\leq n}\frac{1}{p\log p}\ for\ n > 1,$$

where $P$ denotes the set of prime numbers. After a few years, Zhang [5], proved the following:

**Theorem**. For any primitive sequence $A$ whose the number of the prime factors of the termes counted with multiplicity is at most 4, we have

$$\sum_{a\in A,\ a\leq n}\frac{1}{a\log a} \leq \sum_{p\in P,\ p\leq n}\frac{1}{p\log p}\ for\ n > 1.$$

In our work, by using the new estimations of the n-th prime number, we simplify the complexity (the number $N = 20000$ decreased to 95). Throughout the paper we denotes by $p_m$ the m-th prime number and we put $f(A) = \sum_{a\in A}\frac{1}{a\log a}$ where, $f(A) = 0$ if $deg(A) = 0$.

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For a primitive sequence $A$ and $m \geq 1$, we pose

$$A_m = \{a \in A, \text{ the prime factors of } a \text{ are} \geq p_m\},$$
$$A'_m = \{a \in A_m, p_m \mid a\},$$
$$A''_m = \left\{\frac{a}{p_m} : a \in A'_m\right\}.$$

Clearly, the union $A = \bigcup_{m\geq 1} A'_m$ is disjoint and $deg(A''_m) < deg(A)$ when $A$ is finit. Our method based on the fact that a primitive sequence $A$ does not contain simultaneously $p_1$ and ${p_1}^4$.

## 2. MAIN RESULTS

We need the following lemmas.

**Lemma 2.1** Let $n > 1$ be an integer, put $F(n) = \log n + \log\log n - 1$ then

$$p_n \geq nF(n), for\ n \geq 2 \text{ [1]} \quad (1)$$
$$p_n > n(\log(nF(n)) - \alpha), \quad for\ n \geq 3 \quad (2)$$
$$p_n \leq n(F(n) + \beta), \quad for\ n \geq 95 \quad (3)$$

where $\alpha = 1.127$ and $\beta = 0.305$.

**Proof.** Consider the function $g$ defined on $\mathbb{N}$ by

$$n \mapsto \mathrm{g(n)} = \frac{\mathrm{p_n}}{\mathrm{n}} - \log(\mathrm{nF(n)}) \quad \text{for} \quad \mathrm{n} \geq 3,$$

then according to (1), we have $g(n) \geq h(n)$ where

$$h(n) = -1 - \log\left(1 + \frac{loglogn - 1}{logn}\right),$$

the study of the real function $x \mapsto h(x)$ $(x \geq 3)$ gives us $h(x) \geq h\left(e^{e^2}\right) > -\alpha$, then $g(n) > -\alpha$, which is equivalent to

$$p_n > n(\log(nF(n)) - \alpha), \ for\ n \geq 3.$$

A computer calculation shows that for $95 \leq n < 7022$, we have

$$p_n \leq n(F(n) + \beta),$$

and on the other hand we have $p_n \leq n(\log n + \log\log n - 0.9385)$ where $n \geq 7022$ [4], therefore the inequality (3) is verified for $n \geq 95$. This completes the proof.

**Lemma 2.2** For $m \geq 1$ and $j \in \{1, 2, 3\}$, we have

$$\sum_{i \geq \max(m, j-1)} \frac{1}{p_i(k_j + \log p_i)} < \frac{1}{k_{j-1} + \log p_m},$$

where $k_0 = 0.023, k_1 = 0.3157, k_2 = 0.901$ *and* $k_3 = 2.079$.

**Proof.** Put $N = 95, C = 0.0713,$

$$\begin{array}{lll} u_1 = 0.09435, & u_2 = 0.387, & u_3 = 0.9723 \\ v_1 = 0, & v_2 = 0, & v_3 = -0.0074. \end{array}$$

It is clear that for $m \geq N$ and $j \in \{1, 2, 3\}$ we have $\max(m, j-1) = m$ and

$$\begin{aligned} & C \geq -\log\big(F(m)\big) + \log(1 + \frac{1}{m}) + \log(F(m+1) + \beta) \\ & C \leq u_j - k_{j-1}, \\ & v_j = \alpha - k_j + 2u_j - 1. \end{aligned} \tag{4}$$

Now Put

$$h_j(m) = \sum_{i \geq \max(m, j-1)} \frac{1}{p_i(k_j + \log p_i)}.$$

By (1) and (2) we have, for $m \geq N$ and $j \in \{1, 2, 3\}$,

$$p_i(k_j + \log p_i) > i\big(\log\big(iF(i)\big) - \alpha\big)(\, k_j + \log\big(iF(i)\big)\big),$$

Since $x7!$ $\log(xF(x))$ increases for $x > N$, we have

$$h_j(m+1) < \int_m^{\infty} \frac{dt}{t\big(\log\big(tF(t)\big) - \alpha\big)(\,\log\big(tF(t)\big) + k_j\big)},$$

use the change of variable $x = \log t$, we obtain

$$h_j(m+1) < \int_{\log m}^{\infty} \frac{dt}{(\mathrm{L(x)} - \alpha)\big(\mathrm{L(x)} + k_j\big)}, where\ \mathrm{L(x)} = \log(\mathrm{e^x F(\, e^x)}).$$

Since, for $x > \log N$,

$$\frac{1}{L'(x)} < \Big(1 - \frac{1}{L(x) - 1}\Big),$$

then

$$h_j(m+1) < \int_{\log m}^{\infty} \frac{\left(1-\frac{1}{L(x)-1}\right)L'(x)dx}{(\mathrm{L(x)}-\alpha)\left(\mathrm{L(x)}+k_j\right)},$$

by setting $y = L(x)$ and $y_m = L(\log m)$ we get

$$h_j(m+1) < \int_{y_m}^{\infty} \frac{(\mathrm{y}-2)dy}{(\mathrm{y}-1)(\mathrm{y}-\alpha)\left(\mathrm{y}+k_j\right)}.$$

For $m \geq N$ and $j \in \{1, 2, 3\}$ we put

$$g_j(m) = \frac{1}{k_{j-1}+\log p_m},$$

then according to (3) and (4) we have

$$\begin{aligned} g_j(m+1) &\geq \frac{1}{k_{j-1}+\log\big((m+1)(F(m+1)+\beta)\big)} \\ &> \frac{1}{\log\big(mF(m)\big)+u_j} = \int_{y_m}^{\infty} \frac{dy}{(y+u_j)^2}. \end{aligned}$$

We have for $m \geq N$ and $j \in \{1, 2, 3\}$,

$$(\mathrm{y}-2)(y+u_j)^2 - (\mathrm{y}-1)(\mathrm{y}-\alpha)\left(\mathrm{y}+k_j\right) \leq 0.$$

So, for $m \geq N$ and $j \in \{1, 2, 3\}$, we have $h_j(m+1) < g_j(m+1)$ i.e.

$$h_j(m+1) < g_j(m+1) \; for \; m \geq N.$$

A computer calculation gives for $1 \leq m \leq N$ and $j \in \{1, 2, 3\}$,

$$\begin{aligned} h_j(m) &= \sum_{i \geq \max(m, j-1)}^{N} \frac{1}{p_i(k_j+\log p_i)} + h_j(N+1) \\ &< \sum_{i \geq \max(m, j-1)}^{N} \frac{1}{p_i(k_j+\log p_i)} + \frac{1}{\log\big(NF(N)\big)+u_j} < g_j(m). \end{aligned}$$

This completes the proof.

**Lemma 2.3** Let $m \geq 1$ be fixed and let $B = B_m$ be primitive with $\deg(B) \leq 3$. For $1 \leq t \leq 4 - \deg(B)$, we have

$$\sum_{b\in B}\frac{1}{b(t\log p_m+\log b)}<\frac{1}{k_{t-1}+\log p_m}\ where\ p_1{}^{4-t}\notin B_1, \tag{5}$$
$$\sum_{b\in B}\frac{1}{b(t\log p_m+\log b)}<\frac{1}{k_0+\log p_m}\ where\ p_1{}^{3}\notin B_1. \tag{6}$$

**Proof.** For $m\geq 1$ and $1\leq t\leq 4-\deg(B)$ put

$$g_t(B)=\sum_{b\in B}\frac{1}{b(t\log p_m+\log b)}\ \ where\ (g_t(\emptyset)=0).$$

By induction on $\deg(B)$. If $\deg(B)=1$ and $1\leq t\leq 3$ we have $t\log p_m\geq t\log 2>k_t$ and $p_1\neq B_1$ when $t=3$, then by lemma 2 we get

$$g_t(B)=\sum_{b\in B}\frac{1}{b(t\log p_m+\log b)}\ <\sum_{i\geq\max(m,t-1)}\frac{1}{p_i(k_t+\log p_i)}<\frac{1}{k_{t-1}+\log p_m}.$$

If $\deg(B)=s>1$ and $1\leq t\leq 4-s$, we know that $B=\bigcup_{i\geq m}B_i{}'$ is disjoint, so,

$$g_t(B)=\sum_{i\geq m}g_t(B_i{}')\ where\ \ p_1{}^{4-t}\notin B_1'.$$

We have two cases: if $\deg(B_i{}')\leq 1$ then

$$g_t(B_i{}')<\frac{1}{p_i(k_t+\log p_i)}, \tag{7}$$

if $\deg(B_i{}')>1$ then

$$\begin{aligned}g_t(B_i{}')&=\sum_{b\in B_i{}''}\frac{1}{p_ib((t+1)\log p_i+\log b)}\\&=\frac{1}{p_i}g_{t+1}(B_i{}'')\ where\ p_i{}^{3-t}\notin B_1'',\end{aligned}$$

since $\deg(B_i{}'')\ <s$ and $t+1\leq 4-\deg(B_i{}'')$ we have

$$g_{t+1}(B_i{}'')\ <\frac{1}{k_t+\log p_i}\ where\ p_1{}^{4-(t+1)}\notin B_1'',$$

thus

$$g_t(B_i{}') < \frac{1}{p_i(k_t + \log p_i)}. \tag{8}$$

So, from (7), (8) and lemma 2 we obtain

$$g_t(B) < \frac{1}{k_{t-1} + \log p_m} \ where \ p_1{}^{4-t} \notin B_1.$$

For $t = 1$ we get the inequality (6), which ends the proof.

**Proof of theorem 2.4** Let $n$ be fixed and let $A = \{a : a \in A, a \leq n\}$ be subsequence of $A$ where $\deg A \leq 4$. Put $\pi(n) = m$, the number of primes $\leq n$; then $A = \bigcup_{1\leq i\leq m} A_i{}'$ is disjoint and $f(A) = \sum_{1\leq i\leq m} f(A_i')$. Let $1 \leq i \leq m$, we distinguish the two following cases:

case 1: we suppose that $p_1{}^4 \notin A$, i.e. , $p_1{}^3 \notin A_1''$. If $\deg A_i{}' \leq 1$ then $f(A_i') \leq \frac{1}{p_i \log p_i}$ and if $\deg A_i' > 1$ then

$$f(A_i') = \frac{1}{p_i} \sum_{b\in A_i''} \frac{1}{b(\log p_i + \log b)}$$

where $p_1{}^3 \notin A_1''$ and $\deg A_i'' \leq \deg A_i' - 1 \leq 3$,
so, according to (6), we get

$$\sum_{b\in A_i{}''} \frac{1}{b(\log p_i + \log b)} < \frac{1}{k_0 + \log p_i} < \frac{1}{\log p_i} \ where \ p_1{}^3 \notin A_1'',$$

therefore

$$f(A_i') \leq \frac{1}{p_i \log p_i} \ for \ 1 \leq i \leq m. \tag{9}$$

Case 2: if $p_1{}^4 \in A$, since $A$ is a primitive sequence then $p_1 \notin A_1'$, so, $\deg(A_1' - \{p_1{}^4\}) \neq 1$, i.e. ,

$$f(A_1' - \{p_1{}^4\}) < \frac{1}{p_1(k_0 + \log p_1)},$$

Thus

$$\begin{aligned} f(A_1') &= f(\{p_1{}^4\}) + f(A_1' - \{p_1{}^4\}) \\ &= \frac{1}{p_1{}^4 \log p_1{}^4} + \frac{1}{p_1(k_0 + \log p_1)} < \frac{1}{p_1 \log p_1}. \end{aligned}$$

And from (9) we have $f(A_1') \leq \frac{1}{p_i \log p_i}$ for $2 \leq i \leq m$,
Then

$$f(A_1') \leq \frac{1}{p_i \log p_i} \text{ for } 1 \leq i \leq m, \tag{10}$$

thus, by (9) and (10) we get

$$f(A) = \sum_{1 \leq i \leq m} \frac{1}{p_i \log p_i}.$$

This completes the proof.

## 3. CONCLUSION

Using a new estimate of n-th prime with appropriate division of primitive sequence lead us to simplify the complexity. It would of interest to apply the obtained result to study the Erdős conjecture for primitive sequences of higher degree.

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