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THE DEGREE OF PRIMITIVE SEQUENCES AND ERDŐS CONJECTURE

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Summary. A sequence A of strictly positive integers is said to be primitive if none of its term divides another. Z. Zhang proved a result, conjectured by Erdős and Zhang in 1993, on the primitive sequences whose the number of the prime factors of its terms counted with multiplicity is at most 4. In this paper, we extend this result to the primitive sequences whose the number of the prime factors of its terms counted with multiplicity is at most 5.

1.INTRODUCTION

A sequence A of strictly positive integers is said to be primitive if none of its elements divide another. From the sequence of prime numbers $P = (p_n)_{n\geq 1}$ we can construct an infinite collection of primitive sequences. According to the prime number theorem, the n-th prime number p_n is asymptotically equal to $nlog\ n$; this ensures the convergence of the series

$$f(P) = \sum_{p\in P} \frac{1}{p\log p}.$$

A computation for $f(P)$ was obtained in [1] by Cohen as:

$$f(P) = 1.63661632335126086856965800392186367118159707613129\ldots .$$

Throughout this paper, we let $\Omega(a)$ denote the number of prime factors of a counted with multiplicity. For a primitive sequence A the number $max\{\Omega(a)\colon a \in A\}$ is called the degree of A. It is noted $deg(A)$. By convention $deg(\{1\}) = deg(\emptyset) = 0$. For any primitive sequence A we pose $f(A) = \sum_{a\in A}\frac{1}{a\,log\,a}$. We agree that $f(A) = 0$ if $deg(A) = 0$. For any primitive sequence A and any integer $m \geq 1$, we put:

$$A_m = \{a \in A,\ \text{the prime factors of } a \text{ are} \geq p_m\},$$

$$A'_m = \{a \in A_m, p_m \mid a\},$$

$$A''_m = \left\{\frac{a}{p_m} : a \in A'_m\right\}.$$

Then we have $A'_i \cap A'_j = \emptyset$ for i≠j and $A = \bigcup_{m\geq 1} A'_m$ is disjoint. In the case when A is finit, we have $deg(A''_m) < deg(A)$. In [2], Erdős proved that the series $f(A)$ converges for any primitive sequence A and in [3], Erdős asked if it is true that $f(A) \leq f(P)$ for any primitive sequence A. In [4], Erdős and Zhang showed that $f(A) \leq 1.84$ for any primitive sequence A, and in [5], Clark improved this result $f(A) \leq e^{\gamma}$ (where γ is the Euler constant)

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in the special case when A is a primitive set of composite numbers. Several years later in [6], Lichtman and Pomerance proved that $f(A) < e^{\gamma} \cong 1.781$. Moreover, in [2], Erdős conjectured that $f(A) \leq f(P)$ for any primitive sequence A, then in [7,8], Zhang proved this conjecture for any primitive sequence A of degree ≤ 4 and for some special cases of primitive sequences. In [9], the auteurs simplified the proof of Zhang over the primitive sequences of degree ≤ 4. In this note, we prove this result:

Theorem. For any primitive sequence A where $deg(A) \leq 5$, we have:

$$\sum_{a \in A,\ a \leq n} \frac{1}{a \log a} \leq \sum_{p \in P,\ p \leq n} \frac{1}{p \log p}\ for\ n > 1.$$

The proof of this result is based on the upper bound of $f(A_i')$ where $i \geq 1$. We introduce the following constants, $K_0=0$, $K_1=0.1578$, $K_2=0.4687$, $K_3=1.1971$, $K_4=2{,}77258$, $\alpha=1.11012$ and $\beta=0.0642$. We define the sequences $((\chi_i(m))_{m \geq 1}$ as follows: $\chi_i(m) = 1\ for\ m \geq 2\ , j \in \{1,2,3,4\}$ and $\chi_4(1) = 1, \chi_3(1) = 1.096,\ \chi_2(1) = 1.03,\ \chi_1(1) = 1.012\ ,\ \chi_0(1) = 1$.

2. MAIN RESULTS

We need the following lemmas.

Lemma 2.1 Let $n > 1$ be an integer, put $F(n) = \log n + \log\log n - 1$ then we have

$$p_n \geq nF(n), for\ n \geq 2\ ([10]) \tag{1}$$

$$p_n \geq n\left(F(n) + \frac{loglogn + 2.25}{logn}\right), for\ n \geq 2\ ([10]\ ) \tag{2}$$

$$p_n \leq n(F(n) + \beta), for\ n \geq 7022 \tag{3}$$

$$p_n > n(\log(nF(n)) - \alpha)\ , for\ n \geq 2. \tag{4}$$

Proof. Inequality (3) stems from inequality $p_n \leq n(log\ n + loglog\ n - 0.9385)$([11]). According to (2) we have:

$$\frac{\mathrm{p_n}}{\mathrm{n}} - \log(\mathrm{nF(n)}) \geq -1 - \log\left(1 + \frac{(\mathrm{loglogn} - 1}{\mathrm{logn}}\right) + \frac{\mathrm{loglogn} + 2.25}{\mathrm{logn}}\ \ \mathrm{for}\ \ \mathrm{n} \geq 3.$$

Knowing that the function

$$x \longmapsto f(x) = -1 - \ log\ (1 + (loglogx - 1)/logx) + (loglogx + 2.25)/logx$$

is increasing on $[41\times10^3,+\infty)$, then $\frac{p_n}{n} - \log(nF(n)) \geq f(41 \times 10^3) \geq -\alpha$. A computer calculation shows that, for $2 \leq n \leq 41 \times 10^3$ we have :

$$\frac{p_n}{n} - log\ (nF(n)) \geq -\alpha.$$

This completes the proof of (4).

Lemma 2.2 For $m \geq 1$ and $j \in \{1, 2, 3,4\}$, we have:

$$\sum_{i \geq m} \frac{\chi_j(m)}{p_i(K_j + \log p_i)} < \frac{\chi_{j-1}(m)}{K_{j-1} + \log p_m}.$$

Proof. For $j \in \{1, 2, 3,4\}$, we put $N = 7022, C = 0.00654$,

$$U_1 = 0.02348,\ U_2 = 0.17929,\ U_3 = 0.54349,\ U_4 = 1.30221;$$
$$V_1 = 0, \qquad V_2 = 0, \qquad V_3 = 0 \quad and \quad V_4 = -0.05804.$$

It is clear that for $m \geq N$ and $j \in \{1, 2, 3,4\}$ we have:

$$\begin{aligned} &C \geq -\log\big(F(m)\big) + \log(1 + \frac{1}{m}) + \log(F(m+1) + \beta) \\ &C \leq U_j - K_{j-1}, \\ &V_j = \alpha - K_j + 2U_j - 1. \end{aligned} \tag{5}$$

We put

$$h_j(m) = \sum_{i \geq 1} \frac{\chi_j(m)}{p_i(K_j + \log p_i)}.$$

By (1) and (4) we have, for $m \geq N$ and $j \in \{1, 2, 3,4\}$,

$$p_i(K_j + \log p_i) > i\big(\log\big(iF(i)\big) - \alpha\big)\big(K_j + \log\big(iF(i)\big)\big),$$

Since $x \to \log(xF(x))$ increases for $x \geq 3$, it follows that

$$h_j(m+1) < \int_m^\infty \frac{dt}{t\big(\log\big(tF(t)\big) - \alpha\big)\big(\log\big(tF(t)\big) + K_j\big)},$$

use the change of variable $x = \log t$, we obtain:

$$h_j(m+1) < \int_{\log m}^\infty \frac{dt}{(\mathrm{L(x)} - \alpha)\big(\mathrm{L(x)} + K_j\big)}, where\ \mathrm{L(x)} = \log(\mathrm{e^x F(e^x)}).$$

Since, for $x > \log N$,

$$\frac{1}{L'(x)} < \Big(1 - \frac{1}{L(x) - 1}\Big),$$

then

$$h_j(m+1) < \int_{\log m}^\infty \frac{\Big(1 - \frac{1}{L(x)-1}\Big) L'(x) dx}{(\mathrm{L(x)} - \alpha)\big(\mathrm{L(x)} + K_j\big)},$$

by setting $y = L(x)$ and $y_m = L(\log m)$ we get:

$$h_j(m+1) < \int_{y_m}^\infty \frac{(\mathrm{y} - 2)dy}{(\mathrm{y} - 1)(\mathrm{y} - \alpha)\big(\mathrm{y} + K_j\big)}.$$

For $m \geq N$ and $j \in \{1, 2, 3, 4\}$ we put:

$$g_j(m) = \frac{\chi_{j-1}(m)}{K_{j-1} + \log p_m},$$

then according to (3) and (5) we have:

$$g_j(m+1) \geq \frac{1}{K_{j-1} + \log\big((m+1)(F(m+1)+\beta)\big)}$$

$$> \frac{1}{\log\big(mF(m)\big) + U_j} = \int_{y_m}^{\infty} \frac{dy}{(y+U_j)^2}.$$

We have for $m \geq N$ and $j \in \{1,2,3,4\}$,

$$(\mathrm{y}-2)(y+U_j)^2 - (\mathrm{y}-1)(\mathrm{y}-\alpha)\big(\mathrm{y}+K_j\big) \leq 0.$$

So, for $m \geq N$ and $j \in \{1,2,3,4\}$, we have $h_j(m+1) < g_j(m+1)$ i.e.

$$h_j(m) < g_j(m) \; for \; m > N.$$

For $1 \leq \mathrm{m} \leq \mathrm{N}$ and by definition of $\chi_j(i)$, we have for $\mathrm{j} \in \{1,2,3,4\}$ a computer calculation shows that:

$$\sum_{i \geq m} \frac{\chi_j\,(i)}{p_i\,(K_j + logp_i)} = \sum_{i \geq m}^{N} \frac{\chi_j\,(i)}{p_i(K_j + \log p_i)} + h_j(N+1)$$

$$< \sum_{i \geq m}^{N} \frac{\chi_j(i)}{p_i(K_j + \log p_i)} + \frac{1}{\log\big(NF(N)\big) + U_j}$$

$$< g_j(m).$$

This completes the proof.

Lemma 2.3. Let $m \geq 1$ be fixed and let $B = B_m$ be primitive with $\deg(B) \leq 4$. For $1 \leq t \leq 5 - \deg(B)$, we have:

$$\sum_{b \in B} \frac{1}{b(t \log p_m + \log b)} < \frac{\chi_{t-1}\,(m)}{K_{t-1} + \log p_m}, \tag{6}$$

$$\sum_{b \in B} \frac{1}{b(t \log p_m + \log b)} < \frac{1}{\log p_m}. \tag{7}$$

Proof. For $m \geq 1$ and $1 \leq t \leq 5 - \deg(B)$ put

$$g_t(B) = \sum_{b \in B} \frac{1}{b(t \log p_m + \log b)} \;\; where \; (g_t(\emptyset) = 0).$$

By induction on $\deg(B)$. If $\deg(B) = 1$ and $1 \leq t \leq 4$ we have $t \log p_m \geq t \log 2 > K_t$, so according to Lemma 2.2, we get:

$$g_t(B) = \sum_{b\in B} \frac{1}{b(t\log p_m + \log b)} < \sum_{i\geq m} \frac{1}{p_i(t\log p_1 + \log p_i)}$$
$$\leq \sum_{i\geq m} \frac{\chi_t\ (i)}{p_i(K_t + \log p_i)} < \frac{\chi_{t-1}\ (m)}{K_{t-1} + \log p_m}.$$

We assume that inequality (6) is true for $1 \leq \deg(B) \leq s < 5$ and $1 \leq t \leq 5 - \deg(B)$ and we show that it remains true for $\deg(B) = s - 1$. We have $B = \bigcup_{i\geq m} B_i{}'$ is dis joint, so we have:

$$g_t(B) = \sum_{i\geq m} g_t(B_i{}').$$

Let i$\geq m$. If $\deg(B_i{}') \leq 1$ we have:

$$g_t(B_i{}') < \frac{1}{p_i(t\log p_1 + \log p_i)} < \frac{\chi_t\ (i)}{p_i(K_t + \log p_i)}. \tag{8}$$

If $\deg(B_i{}') > 1$ we have:

$$g_t(B_i{}') = \sum_{b\in B_i{}''} \frac{1}{p_i b((t+1)\log p_i + \log b)}$$
$$= \frac{1}{p_i} g_{t+1}(B_i{}''),$$

since $\deg(B_i{}'')\ < s$ and $t + 1 \leq 5 - \deg(B_i{}'')$ so we have:

$$g_{t+1}(B_i{}'') < \frac{\chi_t\ (i)}{K_t + \log p_i},$$

thus

$$g_t(B_i{}') < \frac{\chi_t\ (i)}{p_i(K_t + \log p_i)}. \tag{9}$$

So from (8), (9), and Lemma 2.2, we get:

$$g_t(B) < \frac{\chi_{t-1}\ (m)}{K_{t-1} + \log p_m}.$$

For $t = 1$ we get the inequality (7), which ends the proof.

Proof of theorem Let n be fixed and let $A = \{a\colon a \in \boldsymbol{A}, a \leq n\}$ be subsequence of $\boldsymbol{A}$ where $\deg A \leq 5$. Put $\pi(n) = m$, the number of primes $\leq n$; then $A = \bigcup_{1\leq i\leq m} A_i{}'$ is disjoint and $f(A) = \sum_{1\leq i\leq m} f(A_i')$. Let $1 \leq i \leq m$. If $\deg A_i{}' \leq 1$ then $f(A_i') \leq \frac{1}{p_i \log p_i}$ and if $\deg A_i' > 1$ then

$$f(A_i') = \frac{1}{p_i} \sum_{b\in A_i''} \frac{1}{b(\log p_i + \log b)}$$

and $\deg A_i'' \leq \deg A_i' - 1 \leq 4$, so according to (7), we obtain:

$$\sum_{b \in A_i''} \frac{1}{b(\log p_i + \log b)} < \frac{1}{\log p_i}$$

therefore

$$f(A_i') \leq \frac{1}{p_i \log p_i}$$

Thus

$$f(A) = \sum_{1 \leq i \leq m} f(A_i') \leq \sum_{1 \leq i \leq m} \frac{1}{p_i \log p_i}.$$

This completes the proof.

3. CONCLUSION

Using a new value of the constants $\chi_i(m)$ will prove the theorem for greater degrees. Why not establish a recursive relationship on the degree of any sequence A, will prove this conjecture.

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