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WEAKLY CHAIN SEPARATED SETS IN A TOPOLOGICAL SPACE

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Summary. In this paper we introduce the notion of pair of weakly chain separated sets in a topological space. If two sets are chain separated in the topological space, then they are weakly chain separated in the same space. We give an example of weakly chain separated sets in a topological space that are not chain separated in the space. Then we study the properties of these sets. Also we mention the criteria for two kind of topological spaces by using the notion of chain. The topological space is totally separated if and only if any two different singletons (unit subsets) are weakly chain separated in the space, and it is the discrete if and only if any pair of different nonempty subsets are chain separated. Moreover we give a criterion for chain connected set in a topological space by using the notion of weakly chain separateness. This criterion seems to be better than the criterion of chain connectedness by using the notion of pair of chain separated sets. Then we prove the properties of chain connected, and as a consequence of connected sets in a topological space by using the notion of weakly chain separateness.

1 INTRODUCTION

The definition of connectedness, that is considered as standard, was first given in the beginning of 20th century by Riesz and Hausdorff.

In 1883, Cantor gave the definition of connectedness in $\mathbb{R}^n$ by using the notion of chain. Later the definition is generalized to all topological spaces: The topological space X is connected if for every $x, y \in X$ and every open covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y ([3, 4]). For more details about connectedness see [5-8].

In papers [1] and [2], rather than as a space, is generalized the notion of connectedness as a set in a topological space that is called a chain connected set.

We will write some definitions and statements from article [1].

By a covering we understand a covering consisting of open sets.

Suppose X is a set, $\mathcal{U}$ is a family of subsets of X, and $x, y \in X$. **A chain in $\mathcal{U}$ that connects x and y** (from x to y, from y to x) is a finite sequence $U_1, U_2, ..., U_n$ in $\mathcal{U}$, such that $U_i \cap U_{i+1} \neq \emptyset$, $i = 1,2,...,n-1$, and $x \in U_1$, $y \in U_n$. In this paper a set is a topological subspace, and a chain is a chain that consists of open sets.

Let X be a topological space and $C \subseteq X$.

Definition 1.1. The set C is **chain connected** in X, if for every covering $\mathcal{U}$ of X in X and every $x, y \in C$, there exists a chain in $\mathcal{U}$ that connects x and y.

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So, the space X is connected if and only if X is chain connected in X.

Let X be a topological space, and let A and B be nonempty subsets of X.

Definition 1.2. The sets A and B are **chain separated** in X, if there exists a covering $\mathcal{U}$ of X in X such that for every point $x \in A$ and every $y \in B$, there is no chain in $\mathcal{U}$ that connects x and y.

Theorem 1.1. The sets A and B are separated if and only if A and B are chain separated in $A \cup B$.■

Theorem 1.2. The set C is chain connected in X, if and only if C cannot be represented as a union of two chain separated sets A and B in X.■

Let X be a topological space and $x, y \in X$.

Definition 1.3. The element x is **chain related** to y in X, and we denote it by $x \underset{X}{\sim} y$, if for every covering $\mathcal{U}$ of X in X there exists a chain in $\mathcal{U}$ that connects x and y.

The chain relation in a topological space X is an equivalence relation, and it depends on the set X and the topology τ of X. The chain relation splits the space into classes.

We denote $x \underset{\mathcal{U},X}{\sim} y$, if for the covering $\mathcal{U}$ of X in X there exists a chain in $\mathcal{U}$ that connects x and y.

We denote by $A_X(x,\mathcal{U})$ the set that consists of all elements $y \in X$ such that $x \underset{\mathcal{U},X}{\sim} y$. The set $A_X(x,\mathcal{U})$ is nonempty, open, and closed.

Let X be a topological space, $x \in X$ and $\mathcal{U}$ be a covering of X. The infinite star of $x \in X$ and $\mathcal{U}$ in X is denoted by $st^{\infty}(x,\mathcal{U})$ [1, 9].

2 PAIR OF WEAKLY CHAIN SEPARATED SETS

Let X be a topological space, and let $A, B \subseteq X$.

Definition 2.1. The nonempty sets A and B are **weakly chain separated in** X, if for every point $x \in A$ and every $y \in B$, there exists a covering $\mathcal{U} = \mathcal{U}(x,y)$ of X in X such that there is no chain in $\mathcal{U}$ that connects x and y.

The notion is similar to the notion of pair of chain separated sets in a topological space. Therefore, the analogue theorems to chain separated sets, presented in [1], are valid for weakly chain separated sets.

From the definition, it follows that:

Proposition 2.1. If A and B are weakly chain separated in X, then any pair of nonempty sets C and D, where $C \subseteq A$ and $D \subseteq B$, are weakly chain separated in X.■

The following theorem will show us that two sets, which are weakly chain separated in a space, are also weakly chain separated in every subspace.

Let X be a topological space, let $Y \subseteq X$, and let A and B be nonempty subsets of Y.

Theorem 2.1. If A and B are weakly chain separated in X, then A and B are weakly chain separated in Y.

Proof. Let the sets A and B be weakly chain separated in X and let $x \in A$ and $y \in B$. It follows that there exists a covering $\mathcal{U}$ of X in X such that there is no chain in $\mathcal{U}$ that connects x and y. Then

$$\mathcal{U}_y = \mathcal{U} \cap Y = \{ U \cap Y | U \in \mathcal{U} \}$$

is covering of Y in Y such that there is no chain in $\mathcal{U}_Y$ that connects x and y.■

Remark 2.1. The most important case of the previous theorem is when $Y = A \cup B$.■

The definition of pair of chain separated sets, the definition of pair weakly chain separated sets, and the properties of quantifiers, leads to the following statement.

Theorem 2.2. If the sets A and B are chain separated in X, then A and B are weakly chain separated in X.■

The next example shows that the converse statement does not hold in general.

Example 2.1. Let $A = \{0\}$, $B = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$, and $X = A \cup B$. The sets A and B are weakly chain separated in X, but A and B are not chain separated in X i.e. A and B are not separated.

Proof. Let $b \in B$. Then $b = \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$, and for the covering:

$$\mathcal{U} = \left\{ \left[0, \frac{1}{n_0} \right) \cap X, \left\{ \frac{1}{n_0} \right\}, \left\{ \frac{1}{n_0 + 1} \right\}, \dots \right\},$$

there is no chain in $\mathcal{U}$ that connects 0 and b. It follows that A and B are weakly chain separated in X.

On the other hand, since the topology on X is relative to $\mathbb{R}$, every element of arbitrary covering of X that contains the point 0, also contains a point from the set B. It follows that A and B are not chain separated in X.■

The definition of pair of chain separated sets, the definition of pair weakly chain separated sets, and the criterion of chain separated sets in their union by using the notion of separated sets (see introduction), leads to the following two statements.

Corollary 2.1. If A and B are separated then A and B are weakly chain separated in $A \cup B$.■

Theorem 2.3. Singletons A and B are weakly chain separated in X if and only if they are chain separated in X.■

The definition of pair of weakly chain separated sets and the definition of chain relation, lead to the following statement which is a criterion for weakly chain separated sets by using the chain relation. The chain relation has a short notation and therefore we will use this criterion in the proofs.

Proposition 2.2. Two sets A and B are weakly chain separated in X if and only if for every $x \in A$ and $y \in B$, $x \underset{X}{\not\sim} y$.■

The last proposition in case of chain separateness (remark 4.4 in [1]) is valid in one direction. From the last proposition it follows the next statement:

Corollary 2.2. Let $x, y \in X$. Then $x \underset{X}{\sim} y$, if and only if X cannot be represented as a union of two weakly chain separated sets A and B that contain x and y, respectively.■

It is easily seen that the next statement is valid:

Theorem 2.4. If the function $f : X \to \{0,1\}$, such that $f(A) = \{0\}$ and $f(B) = \{1\}$, is continuous then the sets A and B are chain separated (weakly chain separated) in X.■

According to [1], two nonempty sets A and B are functionally separated in X if there exists a continuous function $f : X \to \{0,1\}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. So, the next statement is valid.

Corollary 2.3. If A and B are functionally separated in X then A and B are chain separated (weakly chain separated) in X.

From the last corollary it follows that if A and B are functionally separated in $A \cup B$ then A and B are weakly chain separated in $A \cup B$. This statement in the case of chain separateness (corollary 4.3 in [1]) is valid in both direction.

Corollary 2.4. If $f : X \to [0,1]$ is a continuous function such that $f(A) = \{0\}$ and $f(B) = \{1\}$, then the sets A and B are chain separated (weakly chain separated) in $f^{-1}(0) \cup f^{-1}(1)$.■

From the last corollary it follows that if $f: X \to [0,1]$ is a continuous function such that $f(A) = \{0\}$ and $f(B) = \{1\}$, then the sets A and B are chain separated (weakly chain separated) in $A \cup B$. The reverse claim does not have to be valid.

Example 2.2. Let $X = [-1,1]$, $A = [-1,0)$ and $B = (0,1]$. Then sets A and B are weakly chain separated in $A \cup B$, but there is no continuous function $f: X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

At the end we mention the criteria for two kind of topological spaces by using the notion of chain.

Let X be a topological space.

Theorem 2.5. The space X is the discrete if and only if any two disjoint nonempty subsets of X are chain separated in X.

Proof. Let the space X be the discrete space and let $x, y \in X$, $x \neq y$. Then for the covering $\mathcal{U} = \{\{x\}, X \setminus \{x\}\}$ it follows that $x \underset{\mathcal{U},X}{\nsim} y$ i.e. $\{x\}$ and $\{y\}$ are chain separated in X.

Let any two disjoint subsets of X be chain separated in X, and let $x \in X$. Then $\{x\}$ is open set, since otherwise $\{x\}$ and $X \setminus \{x\}$ will not be chain separated.■

So, X is the discrete if and only if for every $A, B \subseteq X$ such that $A, B \neq \varnothing$ and $A \cap B = \varnothing$, there exist a covering $\mathcal{U}$ of X, such that for every $x \in A$ and every $y \in B$ there is no chain in $\mathcal{U}$ that connects x and y. From the last theorem it follows that we can prove the properties of the discrete space by using the notion of chain.

Theorem 2.6. The space X is totally separated if and only if any two disjoint singletons subsets are weakly chain separated in X.

Proof. ($\Rightarrow$) Let X be totally separated and let $x, y \in X$, $x \neq y$. Since X is totally separated, then there is an open and closed subset $U \subseteq X$ such that $x \in U$ and $y \in X \setminus U$. Then for the covering $\mathcal{U} = \{U, X \setminus U\}$ it follows that $x \underset{\mathcal{U},X}{\nsim} y$, hence $x \underset{X}{\nsim} y$.

($\Leftarrow$) Let any two different singletons be weakly chain separated and let $x, y \in X$, $x \neq y$. Since $x \underset{X}{\nsim} y$, then there exists a covering $\mathcal{U}$ such that there is no chain in $\mathcal{U}$ that connects x and y. Then the sets $A = A_X(x, \mathcal{U})$ and $X \setminus A$ are open and closed in X such that $x \in A$ and $y \in X \setminus A$. It follows that X is totally separated.■

At the end of the section we generalise the notion of totally separated space by using the notion of chain, to a set in a topological space.

Let X be a topolgical space and let A be a subspace of X.

Definition 2.2. The set A is totally weakly chain separated in X if any two disjoint singletons subsets of A are weakly chain separated in X.

So, the set A is totally weakly chain separated in X if for every $x, y \in A$ there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y.

3 STRONGLY CHAIN CONNECTED SETS

Let X be a topological space, and $C \subseteq X$.

Definition 3.1. A set C is **strongly chain connected** in X if C cannot be represented as a union of two weakly chain separated sets A and B in X.

From the definition it follows that a space X is strongly chain connected in X if it cannot be represented as a union of two weakly chain separated sets A and B in X.

Since the property of weak chain connectedness is weaker then the property of chain connectedness, we expect the property of strong chain connectedness to be stronger then the property of chain connectedness. Actually, the next theorem shows that they are equivalent.

Theorem 3.1. The set C is strongly chain connected in X if and only if C is chain connected in X.

Proof. ($\Rightarrow$) If the set C is not chain connected in X, then there exists a pair of chain separated sets A and B in X, such that $C = A \cup B$. But then A and B are weakly chain separated in X i.e. C is not strongly chain connected in X.

($\Leftarrow$) If C is not strongly chain connected in X i.e. C can be represented as a union of two weakly chain separated sets A and B in X it follows that there exist $x, y \in C$ and a covering $\mathcal{U}$ of X in X such that there is no chain in $\mathcal{U}$ that connects x and y. Therefore C is not chain connected in X.■

Remark 3.1. The most important case of the theorem is when $C = X$.■

From the last definition it follows that if the set C is strongly chain connected in X then for every covering $\mathcal{U}$ of X in X and every $x, y \in C$, there is no chain in $\mathcal{U}$ that connects x and y.

4 PROPERTIES OF STRONGLY CHAIN CONNECTED SET

In this section we reformulate the statements from [1], by changing the notion of chain connected set with strongly chain connected set and we rewrite some of the proofs to be more readable. In the proofs we use the new chain sonnected criterion by using weakly chain separated sets or we use the short notation of chain connected relation. We also formulate and prove some analogoues statements by changing the notion of chain separated (case of chain separateness) with weakly chain separated sets (case of weakly chain separateness).

The next proposition follows from the definition of strongly chain connected set in a topological space.

Proposition 4.1. If the set C is strongly chain connected in X, then each subset of C is strongly chain connected in X.■

Let X be a topological space and $C \subseteq Y \subseteq X$.

Theorem 4.1. If the set C is strongly chain connected in Y, then C is strongly chain connected in X.

Proof. If the set C is not strongly chain connected in X then C can be represented as a union of two weakly chain separated sets A and B in X. From theorem 2.3. it follows that A and B are weakly chain separated sets in Y i.e. C is not strongly chain connected in Y.■

Remark 4.1. The most important case of the previous theorem is when $C = Y$.■

Example 4.1. Let $X = [-1,1]$ and $Y = [-1,0) \cup (0,1]$. Then Y is strongly chain connected in X, but it is not strongly chain connected in Y. Moreover Y is not connected.■

From the theorem 3.1 it follows that:

Corollary 4.1. The set C is strongly chain connected in X if and only if for every $x, y \in C$, $x \underset{X}{\sim} y$.■

Remark 4.2. The most important case of the previous corollary is when $C = X$.■

Therefore, C is not strongly chain connected in X if and only if there exist $x, y \in C$ such that $x \underset{X}{\not\sim} y$.

Theorem 4.2. A space X is connected if and only if X is strongly chain connected in X.

Proof. If X is empty or a singleton, then X is connected and strongly chain connected in X. Let X be composed of at least two elements.

($\Rightarrow$) If X is not strongly chain connected in X i.e., from theorem 3.1, it is not chain connected in X then it follows that there exists a covering $\mathcal{U}$ of X and there exist elements $x, y \in X$, such that $x \underset{\mathcal{U},X}{\not\sim} y$. Hence A and $X \setminus A$ where $A = A_X(x, \mathcal{U})$ are nonempty, open and closed sets whose union is X. It follows that X is not connected.

($\Leftarrow$) If X is not connected then X can be represented as a union of two open and closed sets A and B. Hence $\{A, B\}$ is covering of X such that for every $x \in A$ and $y \in B$, $x \underset{X}{\not\sim} y$, i.e. A and B are chain separated sets in X. Therefore A and B are weakly chain separated sets in X i.e. X is not strongly chain connected in X.■

The last theorem allows us to prove the properties of connected spaces by using the notion of weakly chain connectedness.

Notice that, as a consequence, it follows that if C is connected, then C is strongly chain connected in every super space X. In addition, a topological space C is strongly chain connected in C, if and only if it cannot be represented as a union of two separated sets.

Let X be a topological space and $C \subseteq Y \subseteq X$.

Theorem 4.3. Let $X = A \cup B$, where A and B are weakly chain separated sets in X, and C is a strongly chain connected set in X. Then $C \subseteq A$ or $C \subseteq B$.

Proof. Let C be a strongly chain connected set in X, i.e. for every $x, y \in C$, $x \underset{X}{\sim} y$.

If there exist $x \in A \cap C$ and $y \in B \cap C$, since A and B are weakly chain separated sets in X, it follows that $x \underset{X}{\not\sim} y$, which contradicts the assumption. So, $C \subseteq A$ or $C \subseteq B$.■

A direct consequence of the last theorem are the next two corollaries:

Corollary 4.2. Let $X = A \cup B$, where A and B are chain separated sets in X, and C is a strongly chain connected set in X. Then $C \subseteq A$ or $C \subseteq B$.■

Corollary 4.3. Let $X = A \cup B$, where A and B are separated. If C is a connected set, then $C \subseteq A$ or $C \subseteq B$.■

The next theorem and its remark show that chain relation and strongly chain connected set is invariant with respect to continuous function.

Theorem 4.4. Let $x, y \in X$. If $x \underset{X}{\sim} y$ and $f : X \to Y$ is a continuous function, then $f(x) \underset{f(X)}{\sim} f(y)$.

Proof. The function $f : X \to Y$ is continuous if and only if $f : X \to f(X)$ is continuous.

Let $f(x), f(y) \in f(X)$ and $\mathcal{V}$ be a covering of $f(X)$.

Then $\mathcal{U} = f^{-1}(\mathcal{V})$ is a covering of X and since $x \underset{X}{\sim} y$, there exists a chain

$$f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_n)$$

in $\mathcal{U}$ that connects x and y. Since $f^{-1}(V_i) \cap f^{-1}(V_{i+1}) \neq \varnothing$, $i = 1,2,\ldots,n-1$, it follows that

$$\varnothing \neq f\left(f^{-1}(V_i)\right) \cap f\left(f^{-1}(V_{i+1})\right) = V_i \cap V_{i+1},\ i = 1,2,\ldots,n-1,$$

i.e. $V_1, V_2, \ldots, V_n$ is a chain in $\mathcal{V}$ that connects $f(x)$ and $f(y)$. Therefore $f(x) \underset{f(X)}{\sim} f(y)$.■

Corollary 4.4. If C is strongly chain connected in X and $f : X \to Y$ is a continuous function, then $f(C)$ is strongly chain connected in $f(X)$.■

The next well known result is a consequence of the last corollary:

Remark 4.3. Let $C \subseteq X$. If C is a connected set and $f : X \to Y$ is a continuous function, then $f(C)$ is connected.■

Theorem 4.5. If $C \subseteq D \subseteq \overline{C} \subseteq X$. The set C is strongly chain connected in X, if and only if D is strongly chain connected in X.

Proof. Clearly, if D is strongly chain connected in X then proposition 4.1 implies that C is strongly chain connected in X.

Let C be strongly chain connected in X, let $x, y \in D$, and let $\mathcal{U}$ be a covering of X. Since $D \subseteq \overline{C}$, members U and V of $\mathcal{U}$ that contain x and y, also contain some elements $x_1 \in C$ and $y_1 \in C$, respectively. Then $x \underset{X}{\sim} x_1$, $x_1 \underset{X}{\sim} y_1$, and $y_1 \underset{X}{\sim} y$, i.e. $x \underset{X}{\sim} y$. It follows that D is strongly chain connected in X.■

Remark 4.4. The most important case of the theorem is when $D = \overline{C}$.■

As a consequence, an analogous statement for connected sets holds, but only in one direction.

Corollary 4.5. Let X be a topological space and $C \subseteq X$. If C is a connected set and $C \subseteq D \subseteq \overline{C}$, then D is connected.

Proof. If C is a connected set i.e. C is strongly chain connected in C it follows that C is strongly chain connected in D and, by the last theorem, D is strongly chain connected in D i.e. D is connected.■

Lemma 4.1. Let $C, D \subseteq X$. If C and D are strongly chain connected in X and $\overline{C} \cap \overline{D} \neq \emptyset$, then the union $\overline{C} \cup \overline{D}$ is strongly chain connected in X.

Proof. Let $x, y \in \overline{C} \cup \overline{D}$ and let $z \in \overline{C} \cap \overline{D}$. Since C and D are strongly chain connected in X, by using the previous theorem it follows that $x \underset{X}{\sim} z$ and $z \underset{X}{\sim} y$, hence $x \underset{X}{\sim} y$.■

Theorem 4.6. Let $C_i, i \in I$ be a family of strongly chain connected subsets of X. If there exists $i_0 \in I$ such that for every $i \in I$, $\overline{C}_{i_0} \cap \overline{C}_i \neq \emptyset$, then the union $\bigcup_{i \in I} \overline{C}_i$ is chain connected in X.

Proof. Let $x, y \in \bigcup_{i \in I} \overline{C}_i$, i.e. $x \in \overline{C}_x$ and $y \in \overline{C}_y$ for some $x, y \in I$. Let $x_1 \in \overline{C}_{i_0} \cap \overline{C}_x$ and $y_1 \in \overline{C}_{i_0} \cap \overline{C}_y$. Then $x \underset{X}{\sim} x_1$, $x_1 \underset{X}{\sim} y_1$, and $y_1 \underset{X}{\sim} y$. It follows that $x \underset{X}{\sim} y$.■

From the theorem it follows that if $C_i, i \in I$ is a family of strongly chain connected subsets of $\bigcup_{i \in I} C_i$, then the union $\bigcup_{i \in I} C_i$ is chain connected in $\bigcup_{i \in I} C_i$. A direct consequence of the last statement is:

Corollary 4.6. Let $C_i \subseteq X, i \in I$ be a family of connected sets. If there exists $i_0 \in I$ such that for every $i \in I$, $C_{i_0} \cap C_i \neq \emptyset$, then the union $\bigcup_{i \in I} C_i$ is connected.■

Note that we can not use the assumption $\overline{C}_{i_0} \cap \overline{C}_i \neq \emptyset$ in the last corollary since from $\overline{C}_{i_0} \cap \overline{C}_i \neq \emptyset$ in X we cannot conclude that $\overline{C}_{i_0} \cap \overline{C}_i \neq \emptyset$ in $\bigcup_{i \in I} C_i$.

It is clear that if every two points x and y of X are in a strongly chain connected set C_{xy} in X, then X is strongly chain connected. The next corollary follows.

Corollary 4.7. If for every two points x and y of X there exists a connected set C_{xy} in X containing them, then X is connected.■

Theorem 3.7 from [1] is not walid in the case of weakly chain separateness.

Theorem 4.7. Set C is strongly chain connected in X, if and only if for every $x \in C$ and every covering $\mathcal{U}$ of X, $C \subseteq st^{\infty}(x, \mathcal{U})$.■

Corollary 4.8. Space X is connected, if and only if for every $x \in X$ and every covering $\mathcal{U}$ of X, $X = st^{\infty}(x, \mathcal{U})$.■

In the next theorem we will give a strong chain connectedness criterion using continuous function.

Theorem 4.8. A space X is strongly chain connected, if and only if every continuous function $f : X \to \{0,1\}$ is constant.■

As a consequence, it follows that X is strongly chain connected, if and only if there is no continuous function $f : X \to [0,1]$, such that $f(A) = \{0\}$ and $f(B) = \{1\}$ for every nonempty pair of sets A and B such that $A \cup B = X$.

Let X be a topological space and $x \in C \subseteq X$.

Definition 4.1. The **strongly chain connected component** of the point x of C in X, denoted by $V_{CX}(x)$, is the biggest chain connected subset of C in X that contains x.■

From the last definition and the definition of chain relation follow the next three statements:

Proposition 4.2. The strongly chain connected component $V_{CX}(x)$ of the point x of C in X is the set of all points $y \in C$ such that $x \underset{X}{\sim} y$.■

Proposition 4.3. The set of all strongly chain connected subsets of C in X consist of all strongly chain connected components of C in X and their subsets.■

Proposition 4.4. For every $x \in C$, $V_{CX}(x) = C \cap V_{XX}(x)$. Each strongly chain connected component of X in X contains at most one strongly chain connected component of C in X.■

Since the chain relation is an equivalence relation, from the last proposition it follow the next two statements.

Proposition 4.5. Let $x, y \in C$. If $y \in V_{CX}(x)$, then $V_{CX}(x) = V_{CX}(y)$.■

Proposition 4.6. Let $x, y \in C$. If $V_{CX}(x) \neq V_{CX}(y)$, then $V_{CX}(x) \cap V_{CX}(y) = \emptyset$.■

As a consequence of the definition of strongly chain connected component and the last two statemenents, the next proposition is valid.

Proposition 4.7. For every $x \in C$,

$$V_{CC}(x) \subseteq V_{CX}(x) = \bigcup_{y \in V_{CX}(x)} V_{CC}(y).■$$

The proposition shows that every strongly chain connected component of C in X is a union of strongly chain connected components of C in C.

Proposition 4.8. The strongly chain connected components of X in X are closed sets, i.e. for every $x \in X$, $V_{XX}(x) = \overline{V_{XX}(x)}$.

Proof. Let $y \in \overline{V_{XX}(x)}$ and $\mathcal{U}$ be a covering of X. Then there exists a neighbourhood $U \in \mathcal{U}$ such that $y \in U$ and there exists a point $z \in U \cap V_{XX}$. Then $x \underset{X}{\sim} z$ and $z \underset{X}{\sim} y$ i.e. $x \underset{X}{\sim} y$. So, $y \in V_{XX}(x)$ i.e. $V_{XX}(x)$ is closed set.■

Proposition 4.9. Let $x \in X$ and $C(x)$ be a connected component of X. Then $C(x) \subseteq V_{XX}(x)$.■

Quasicomponent of the element x in a topological space X, denoted with $Q_X(x)$, is the intersection of all clopen (closed and open) sets in X that contain x.

Theorem 4.9. Quasicomponents and strongly chain connected components in a topological space X coincide, i.e., for every $x \in X$, $Q_X(x) = V_{XX}(x)$.

Proof. ($\Leftarrow$) If $y \notin Q_X(x)$, since $Q_X(x)$ is the quasicomponent of x, it follows that $y \notin A$ for some open and closed set A such that $Q_X(x) \subseteq A$. But then for the covering $\mathcal{U} = \{A, X / A\}$, it follows that $x \underset{\mathcal{U},X}{\not\sim} y$, and hence $x \underset{X}{\not\sim} y$ i.e. $y \notin V_{XX}(x)$.

($\Rightarrow$) If $y \notin V_{XX}(x) = \bigcap_{\mathcal{U} \in Cov(X)} A_X(x, \mathcal{U})$ where $Cov(x)$ consists of all coverings of X it follows that $y \notin A = A_X(x, \mathcal{U})$ where $\mathcal{U}$ is some covering of X. Since A is an open and closed set that contain x, it follows that $Q_X(x) \subseteq A$. Then $y \notin Q_X(x)$.■

From the last theorem it follows that quasicomponent of the point x is the biggest strongly chain connected set in X that contains x.

There exists a criterion for quasicomponenet by using the notion of chain: The quasicomponent of the point x in X consits of all $y \in X$ such that for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y [1]. Hence the quasicomponent of x in X consists of all $y \in X$ such that $x \underset{X}{\sim} y$. Therefore the propositions 4.8 and 4.9 and the theorem 4.9 are reformulations of two properties and a criterion of the quasicomponents.

The next proposition is the summary of the propositions 4.8, 4.9, and the theorem 4.9.

Proposition 4.10. For every $x \in C$,

$$Q_C(x) = V_{CC}(x) \subseteq \bigcup_{y \in V_{CX}(x)} Q_C(y) = V_{CX}(x) \subseteq V_{XX}(x) = Q_X(x). \blacksquare$$

So strongly chain connected components of a set in a topological space are a union of quasicomponents of the set i.e. for every $x \in C$,

$$V_{CX}(x) = \bigcup_{y \in V_{CX}(x)} Q_C(y),$$

and if the set agrees with the space, the strongly chain connected components match with the quasicomponents.

5 CONCLUSIONS

In this work we defined a pair of weakly chain separated sets, and strongly chain connected set in a topological space, and studied the properties of these sets. As a consequence, by using the notion of weakly chain separateness i.e. a chain, the properties of connected sets are proven.

REFERENCES

[1] Z. Misajleski, N. Shekutkovski, and A. Velkoska, “Chain Connected Sets In A Topological Space”, *Kragujevac Journal of Mathematics*, **43** (4), 575-586 (2019).
[2] N. Shekutkovski, Z. Misajleski, and E. Durmishi, “Chain Connectedness”, *AIP Conference Proceedings*, **2183**, 030015-1-030015-4 (2019).
[3] N. Shekutkovski, “On the concept of connectedness”, *Matematichki bilten,* **40** (1), 5-14 (2016).
[4] Alessandro Berarducci, Dikran Dikranjan, and Jan Pelant, “Uniform quasicomponents, thin spaces and compact separation”, *Topology and its applications*, **122**, 51-64 (2002).
[5] R.Engel'king, *Obshchaya topologiya*, Mir, Moskva (1986).
[6] J.G. Hocking and G.S. Young, *Topology*, Addison-Wesley, Reading, Massachusetts (1961).
[7] J. Munkres, *Topology*, Pearson Education (2014).
[8] K. Kuratovskii, *Topologiya*, Mir, Moskva, Tom 1, (1966), Tom 2, (1969).
[9] S.A. Naimpally and C.M. Pareek, “Of the compactness of connected sets“, *Rev. Un. Mat. Argentina*, **39**, 45-52 (1994).

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MERSENNE-LUCAS HYBRID NUMBERS

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Summary. We introduce Mersenne-Lucas hybrid numbers. We give the Binet formula, the generating function, the sum, the character, the norm and the vector representation of these numbers. We find some relations among Mersenne-Lucas hybrid numbers, Jacopsthal hybrid numbers, Jacopsthal-Lucas hybrid numbers and Mersenne hybrid numbers. Then we present some important identities such as Cassini identities for Mersenne-Lucas hybrid numbers.

1 INTRODUCTION

Number sequences continue to attract the attention of researchers for a long time. Number sequences, especially Fibonacci sequences, find application in many departments of mathematics as well as in other branches of science.

Many researchers have studied Fibonacci numbers and new number sequences created by their generalizations. [1-8].

Koshy [9] written one of the most popular books of Fibonacci and Lucas numbers, and gave numerous recurrence relations, generalizations and applications of Fibonacci and Lucas numbers.

Catarino et. al defined the Mersenne sequence and some identities of the the Mersenne sequence. Later, Saba et.al introduced Mersenne-Lucas nembers and some identity of this sequence.

Later, many researchers studied on hybrid numbers. These researchers developed hybrid numbers by relating them to other number sequences and created other number sequences. [10-18].

In this study, Mersenne-Lucas hybrid numbers will be defined by using the hybrid numbers. A generating function and a Binet formula for the Mersenne-Lucas hybrid numbers will be found. Furthermore, the sum, the character, the norm and the vector representation of these numbers will be given. Some relations among Mersenne-Lucas hybrid numbers, Jacopsthal hybrid numbers, Jacopsthal-Lucas hybrid numbers and Mersenne hybrid numbers wil be presented.

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Key words and Phrases: Mersenne hybrid numbers, Mersenne-Lucas hybrid numbers, Binet formula, Generating function, Cassini identity, Catalan identity

Later, we give Cassini identity, Catalan identitiy, Vajda identity, D'ocagne identity and Honsberger identity for Mersenne-Lucas hybrid numbers.

2 PRELIMINIARIES

Özdemir introduced the hybrid numbers [13]. The set of hybrid numbers is
$K = \{a + bi + c\varepsilon + dh : a, b, c, d \in \mathbb{R}\}$. Let be

$$z_1 = a_1 + b_1 i + c_1\varepsilon + d_1 h, \qquad z_2 = a_2 + b_2 i + c_2\varepsilon + d_2 h$$

any two hybrid numbers. Then we have the following properties.

- $z_1 = z_2 \Leftrightarrow a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2$ and $d_1 = d_2$.
- $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)\varepsilon + (d_1 + d_2)h$
- $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i + (c_1 - c_2)\varepsilon + (d_1 - d_2)h$
- $k.z_1 = ka_1 + kb_1 i + kc_1\varepsilon + kd_1 h$, where $k \in \mathbb{R}$ **.**

Some basic properties of hybrid counts are given by the following definition.

Definition 2.1. There are the following definitions where z is any hybrid number such as $z = a + bi + c\,\varepsilon + dh$ [13].

- The conjugate of z is
$$\bar{z} = a - bi - c\varepsilon - dh.$$
- The character of z is
$$C(z) = z\bar{z} = a^2 + (b-c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2.$$
- The norm for z has the form
$$\|z\| = N(z) = \sqrt{C(z)}.$$
- z is a spacelike, timelike or lightlike if
$$C(z) < 0, C(z) > 0 \text{ or } C(z) = 0.$$
respectively.
- The vector representation for z is
$$V_z = (a, b-c, c, d).$$
- The scalar section of z is
$$S(z) = a.$$
- The vector section of z is
$$V(z) = bi + c + dh.$$

Definition 2.2. The Mersenne numbers $\{M_n\}_{n=0}^{\infty}$ are defined by the following recurrence relations

$$M_{\boldsymbol{n+1}} = 2M_n + 1$$

or

$$M_{\boldsymbol{n+2}} = 3M_{n+1} - 2M_n$$

with $M_0 = 0$ and $M_1 = 1$ [19].
Definition 2.3. The Binet formula of the Mersenne numbers are defined by the following. [19]

$$M_n = 2^n - 1$$

Definition 2.4. The Mersenne-Lucas numbers $\{m_n\}_{n=0}^{\infty}$ are defined by the following recurrence

$$m_{\boldsymbol{n}} = 3m_{n-1} - 2m_{n-2}$$

with $m_0 = 2$ and $m_1 = 3$.[11]
Other definition is given by,

$$m_{\boldsymbol{n+1}} = 2m_n - 1$$

Definition 2.5. The Binet formula for Mersenne- Lucas numbers is defined by [20],

$$m_n = 2^n + 1$$

Definition 2.6. The sum of the Mersenne- Lucas numbers is given by [20],

$$\sum_{k=0}^{n} m_k = 2^{n+1} + n$$

or

$$\sum_{k=0}^{n} m_k = 2\,m_n + n - 2$$

Definition 2.7. The Jacobsthal numbers $\{J_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation,

$$J_{n+2} = J_{n+1} + 2J_n$$

with $J_0 = 0$ and $J_1 = 1$. [9]
Definition 2.8. The Jacobsthal -Lucas numbers $\{j_n\}_{n=0}^{\infty}$ are given by

$$j_{n+2} = j_{n+1} + 2j_n$$

with $j_0 = 2$ and $j_1 = 1$. [9]
Definition 2.9. The Mersenne hybrid number, $\{MH_n\}_{n=0}^{\infty}$ is defined as

$$MH_n = M_n + iM_{n+1} + \varepsilon M_{n+2} + hM_{n+3},\ n \geq 0$$

where M_n is the *n*th Mersenne number. [18]

Now let's give the preliminary information.

3 MERSENNE-LUCAS HYBRID NUMBERS

Definition 3.1. Let $n \geq 0$ be integer, Mersenne-Lucas hybrid numbers $\{mh_n\}$ for $n = 0, \dots, \infty$ is defined as,

$$mh_n = m_n + im_{n+1} + \varepsilon m_{n+2} + hm_{n+3} \tag{1}$$

where, m_n is nth Mersenne-Lucas number.

Let us give a few terms of Mersenne-Lucas hybrid numbers in Table 1.

n	mh_n
0	$2+3i+5\varepsilon+9h$
1	$3+5i+9\varepsilon+17h$
2	$5+9i+17\varepsilon+33h$
3	$9+17i+33\varepsilon+65h$

Table 1**:** Same values of Mersenne-Lucas hybrid numbers

Theorem 3.2. The Binet formula for $\{\mathrm{mh_n}\}_{\mathrm{n=0}}^{\infty}$ is defined as,

$$mh_n = 2^{\mathrm{n}}(1+2i+4\varepsilon+8h)+(1+i+\varepsilon+h),\ n \geq 0.$$

Proof. From (1) and the Binet formula for Mersenne-Lucas numbers, we have

$$\begin{aligned} mh_n &= 2^n+1+i(2^{n+1}+1)+\varepsilon(2^{n+2}+1)+h(2^{n+3}+1) \\ &= 2^n(1+2i+4\varepsilon+8h)+(1+i+\varepsilon+h) \end{aligned}$$

Thus, the proof is complete. □

Lemma 3.3. Let $n \geq 0$ be integer, the recurrance relation of Mersenne-Lucas hybrid numbers $\{mh_n\}_{n=0}^{\infty}$ is given as,

$$mh_{n+2} = 3mh_{n+1} - 2mh_n$$

Proof. From (1), we obtain

$$mh_{n+2} = m_{n+2} + im_{n+3} + \varepsilon m_{n+4} + hm_{n+5}$$

$$= (3m_{n+1}-2m_n)+i(3m_{n+2}-2m_{n+1})+\varepsilon(3m_{n+3}-2m_{n+2})+h(3m_{n+4}-2m_{n+3})$$

$$= 3(m_{n+1}+im_{n+2}+\varepsilon m_{n+3}+hm_{n+4})-2(m_n+im_{n+1}+\varepsilon m_{n+2}+hm_{n+3})$$

$$= 3mh_{n+1} - 2mh_n$$

Thus, the desired is obtained. □

Theorem 3.4. The generating function for $\{\mathrm{mh_n}\}_{\mathrm{n=0}}^{\infty}$ is given as follows,

$$G(t) = \sum_{n=0}^{\infty} mh_n t^n = \frac{2+3i+5\varepsilon+9h-t(3+4i+6\varepsilon+10h)}{(1-3t+2t^2)}$$

Proof. We have

$$G(t) = mh_0 + mh_1 t + mh_2 t^2 + \cdots + mh_n t^n + \cdots \tag{2}$$

Let us multiply Equation (2) by $-3t, 2t^2$ respectively. So, the following equations are obtained.

$$G(t) = mh_0 + mh_1 t + mh_2 t^2 + \cdots + mh_n t^n + \cdots$$

$$-3tG(t) = -3tmh_0 - 3t^2 mh_1 - 3t^3 mh_2 - \cdots - 3t^{n+1} mh_n - \cdots$$

$$2t^2 G(t) = 2t^2 mh_0 + 2t^3 mh_1 + 2t^4 mh_2 + \cdots + 2t^{n+2} mh_n + \cdots$$

If we take the necessary calculations to take advantage of the recurrence relation, we obtain the following equations

$$G(t)(1 - 3t + 2t^2) = mh_0 + mh_1 t - 3tmh_0$$

$$G(t) = \frac{2 + 3i + 5\varepsilon + 9h + t(3 + 5i + 9\varepsilon + 17h) - 3t(2 + 3i + 5\varepsilon + 9h)}{(1 - 3t + 2t^2)}$$

$$G(t) = \frac{2 + 3i + 5\varepsilon + 9h - t(3 + 4i + 6\varepsilon + 10h)}{(1 - 3t + 2t^2)}$$

In this case, the desired formula is obtained. □

Theorem 3.5. Let $S(t)$ be sum of $\{\mathrm{mh_n}\}_{\mathrm{n=0}}^{\infty}$. Then we have,

$$S(t) = \sum_{\mathrm{k=0}}^{\mathrm{n}} mh_k = 2mh_n + n - 2 + i(n - 3) + \varepsilon(n - 5) + h(n - 9)$$

Proof. We have,

$$S(t) = mh_0 + mh_1 + mh_2 + \cdots + mh_n$$

From (2.1), we have

$$\begin{aligned} S(t) &= m_0 + im_1 + \varepsilon m_2 + hm_3 + m_1 + im_2 + \varepsilon m_3 + hm_4 + \cdots + m_n \\ &\quad + im_{n+1} + \varepsilon m_{n+2} + hm_{n+3} \\ &= (m_0 + m_1 + \cdots + m_n) + i(m_1 + \cdots + m_{n+1}) \\ &\quad + \varepsilon(m_2 + \cdots + m_{n+2}) + h(m_3 + \cdots + m_{n+3}) \end{aligned}$$

From the sum of Mersenne numbers, we get the following

$$\begin{aligned} &= (2m_n + n - 2) + i(2m_{n+1} + n - 1 - 2) + \varepsilon(2m_{n+2} + n - 2 - 3) \\ &\quad + h(2m_{n+3} + n + 1 - 2 - 3 - 5) \\ &= 2(m_n + im_{n+1} + \varepsilon m_{n+2} + hm_{n+3}) + n - 2 + i(n - 3) + \varepsilon(n - 5) + h(n - 9) \\ &= 2mh_n + n - 2 + i(n - 3) + \varepsilon(n - 5) + h(n - 9) \end{aligned}$$

Thus, proof is complete. □

Theorem 3.6. The character of $\{\mathrm{mh_n}\}_{\mathrm{n=0}}^{\infty}$ is

$$C(mh_n) = -35m^2{}_n - 54m^2{}_{n+1} + 88m_n m_{n+1}, \quad n \geq 0$$

where m_n is the nth Mersenne-Lucas number.

Proof. From definition character of hybrid numbers, we get

$$\begin{aligned} C(mh_n) = mh_n\overline{mh_n} &= (m_n + im_{n+1} + \varepsilon m_{n+2} + hm_{n+3})(m_n - im_{n+1} \\ &\quad -\varepsilon m_{n+2} - hm_{n+3}) \\ &= m^2{}_n + m^2{}_{n+1} - 2m_{n+1}m_{n+2} - m^2{}_{n+3} \end{aligned}$$

Using reccurance relation of Mersenne numbers are the following,

$$m_{n+2} = 3m_{n+1} - 2m_n$$

and

$$\begin{aligned} m_{n+3} &= 3m_{n+2} - 2m_{n+1} \\ &= 3(3m_{n+1} - 2m_n) - 2m_{n+1} \\ &= 7m_{n+1} - 6m_n \end{aligned}$$

Then, we have,

$$\begin{aligned} C(mh_n) &= m^2{}_n + m^2{}_{n+1} - 2m_{n+1}(3m_{n+1} - 2m_n) - (7m_{n+1} - 6m_n)^2 \\ &= m^2{}_n + m^2{}_{n+1} - 6m^2{}_{n+1} + 4m_{n+1}m_n - 49m^2{}_{n+1} + 84m_{n+1}m_n - 36m^2{}_n \\ &= -35m^2{}_n - 54m^2{}_{n+1} + 88m_{n+1}m_n \end{aligned}$$

Thus, the proof is complete. □

Theorem 3.7. For any $n \geq 0$, Mersenne-Lucas hybrid number is spacelike.

Proof. From the definitions of Character and Binet formula of Mersenne-Lucas numbers, we obtain that,

$$\begin{aligned} C(mh_n) &= -35m^2{}_n - 54m^2{}_{n+1} + 88m_{n+1}m_n \\ &= -35(2^n + 1)^2 - 54(2^{n+1} + 1)^2 + 88(2^{n+1} + 1)(2^n + 1) \\ &= -75.\,2^{2n} - 22.\,2^n - 1 \end{aligned}$$

Since $C(mh_n) < 0$ for any $n \geq 0$, Mersenne-Lucas hybrid number is spacelike. □

Theorem 3.8. The vector representation of Mersenne-Lucas Hybrid numbers provide the following identities.

$$V_{mh_n} = 3V_{mh_{n+1}} - 2V_{mh_{n+2}}$$

Proof. By using definition the vector representation of Mersenne in Definition 1.1., we get

$$3V_{mh_{n+1}} - 2V_{mh_{n+2}}$$

$$= (3m_{n+1}, 3m_{n+2} - 3m_{n+3}, 3m_{n+4}, 3m_{n+5}) - (2m_{n+2}, 2m_{n+3} - 2m_{n+4}, 2m_{n+5}, 2m_{n+6})$$

$= (3m_{n+1} - 2m_{n+2}, 3m_{n+2} - 2m_{n+3} - 3m_{n+3} - 2m_{n+4}, 3m_{n+4} - 2m_{n+5}, 3m_{n+5}$
$-2m_{n+6})$

From the reccurence relation of Mersenne numbers,

$= (m_n, m_{n+1} - m_{n+2}, m_{n+3}, m_{n+4})$

Thus,

$$V_{mh_n} = (m_n, m_{n+1} - m_{n+2}, m_{n+3}, m_{n+4}) = 3V_{mh_{n+1}} - 2V_{mh_{n+2}}$$

is obtained. □

Definition 3.9. Let $N(mh_n)$ be norm of the Mersenne-Lucas hybrid numbers. Then we have,

$$N(mh_n) = \sqrt{C(mh_n)} = \sqrt{-35m^2{}_n - 54m^2{}_{n+1} + 88m_{n+1}m_n}\ .$$

Theorem 3.10. We have the following properties,

$$i) mh_n + mh_{n+1} = 3.\,2^n(1 + 2i + 4\varepsilon + 8h) + 2(1 + i + \varepsilon + h)$$
$$ii) mh_{n+1} = 2mh_n - (1 + i + \varepsilon + h)$$

Proof. From (2.1), we have

$i)$ $mh_n + mh_{n+1}$

$$= (m_n + im_{n+1} + \varepsilon m_{n+2} + hm_{n+3}) + (m_{n+1} + im_{n+2} + \varepsilon m_{n+3} + hm_{n+4})$$
$$= 2^n + 1 + i(2^{n+1} + 1) + \varepsilon(2^{n+2} + 1) + h(2^{n+3} + 1)$$
$$+(2^{n+1} + 1) + i(2^{n+2} + 1) + \varepsilon(2^{n+3} + 1) + h(2^{n+4} + 1)$$
$$= 3.2^n(1 + 2i + 4\varepsilon + 8h) + 2(1 + i + \varepsilon + h)$$

So, the proof is complete.

$ii)$ $mh_{n+1} = m_{n+1} + im_{n+2} + \varepsilon m_{n+3} + hm_{n+4}$

From reccurance relation of Mersenne-Lucas number, we have

$$= 2m_n - 1 + i(2m_{n+1} - 1) + \varepsilon(2m_{n+2} - 1) + h(2m_{n+3} - 1)$$
$$= 2(m_n + im_{n+1} + \varepsilon m_{n+2} + hm_{n+3}) - (1 + i + \varepsilon + h)$$
$$= 2mh_n - (1 + i + \varepsilon + h)$$

Thus, the proof is complete. □

Lemma 3.11. We have the following relations

$$i)\ \ m_n = \begin{cases} 3J_n\ \ , & if\ n\ is\ even \\ 3J_n + 2, & if\ n\ is\ odd \end{cases}$$

$$ii)\ \ m_n = \begin{cases} j_n\ \ , & if\ n\ is\ even \\ j_n + 2, & if\ n\ is\ odd \end{cases}$$

Proof. The proof is easily shown by induction over n. □

Theorem 3.12. By the following identities between Mersenne-Lucas hybrid number, Jacobshtal hyrid numbers and Jacobshtal-Lucas hyrid numbers are provided.

$$i)\ mh_n + mh_{n+1} = 3(JH_n + JH_{n+1}) + 2(1 + i + \varepsilon + h)$$
$$ii)\ mh_n + mh_{n+1} = jH_n + jH_{n+1} + 2 + 3i + 5\varepsilon + 5h$$

Proof. $i)$ Let's n is even. From Lemma 2.2., we obtain

$$\begin{aligned} mh_n &= m_n + im_{n+1} + \varepsilon m_{n+2} + hm_{n+3} \\ &= 3J_n + \mathrm{i}(3J_{n+1} + 2) + \varepsilon(3J_{n+2}) + h(3J_{n+3} + 2) \\ &= 3J_n + i3J_{n+1} + \varepsilon 3J_{n+2} + h3J_{n+3} + 2h + 2i \\ &= 3(J_n + \mathrm{i}J_{n+1} + \varepsilon J_{n+2} + \mathrm{h}J_{n+3}) + 2h + 2i \end{aligned}$$

From the definition of Jacobshtal hyrid numbers, we have

$$\begin{aligned} mh_n &= 3JH_n + 2h + 2i \\ mh_{n+1} &= m_{n+1} + im_{n+2} + \varepsilon m_{n+3} + hm_{n+4} \\ &= (3J_{n+1} + 2) + i(3J_{n+2}) + \varepsilon(3J_{n+3} + 2) + h(3J_{n+4}) \\ &= 3(J_{n+1} + iJ_{n+2} + \varepsilon J_{n+3} + hJ_{n+4}) + 2 + 2\varepsilon \end{aligned}$$

From the definition of Jacobshtal hybrid numbers, we get

$$= 3JH_{n+1} + 2 + 2\varepsilon$$

So, we obtain the following equation,

$$\begin{aligned} mh_n + mh_{n+1} &= 3JH_n + 2\mathrm{h} + 2\mathrm{i} + 3JH_{n+1} + 2 + 2\varepsilon \\ &= 3(JH_n + JH_{n+1}) + 2(1 + i + \varepsilon + h) \end{aligned}$$

Similarly, it is shown in the state of the n.

$ii)$ Let's n is even. From Lemma 2.2, we get

$$\begin{aligned} mh_n &= m_n + im_{n+1} + \varepsilon m_{n+2} + hm_{n+3} \\ &= (j_n + 2) + i(j_{n+1}) + \varepsilon(j_{n+2} + 2) + h(j_{n+3}) \\ &= j_n + ij_{n+1} + \varepsilon j_{n+2} + hj_{n+3} + 2 + i + 2\varepsilon + 3h \end{aligned}$$

From the definition of Jacobshtal-Lucas hybrid numbers, we have

$$\begin{aligned} mh_{n+1} &= m_{n+1} + im_{n+2} + \varepsilon m_{n+3} + hm_{n+4} \\ &= j_{n+1} + +i(j_{n+2} + 2) + \varepsilon(j_{n+3}) + h(j_{n+4} + 2) \\ &= j_{n+1} + ij_{n+2} + \varepsilon j_{n+3} + hj_{n+4} + 2i + 3\varepsilon + 2h \\ &= jH_{n+1} + 2i + 3\varepsilon + 2h \end{aligned}$$

So, we get

$$\begin{aligned} mh_n + mh_{n+1} &= jH_n + 2 + i + 2\varepsilon + 3h + jH_{n+1} + 2\mathrm{i} + 3\varepsilon + 2h \\ &= jH_n + jH_{n+1} + 2 + 3i + 5\varepsilon + 5h \end{aligned}$$

Similarly, it is shown in the state of the n.

Thus the proof is complete. □

Theorem 3.13. There is a relationship between Mersenne-Lucas hybrid numbers and Mersenne hyrid numbers

$$mh_n = 2MH_{n+1} - 3MH_n$$

Proof. Let's use the Binet formula for the right hand side of the equation. Then we get

$$\begin{aligned} 2MH_{n+1} - 3MH_n &= 2.2^{n+1}(1 + 2i + 4\varepsilon + 8h) - 2(1 + i + \varepsilon + h) \\ &\quad -3.2^n(1 + 2i + 4\varepsilon + 8h) + 3(1 + i + \varepsilon + h) \\ &= 2^n(1 + 2i + 4\varepsilon + 8h) + (1 + i + \varepsilon + h) = mh_n \end{aligned}$$

Thus,

$$mh_n = 2MH_{n+1} - 3MH_n$$

is obtained. □

Theorem 3.14. For $n \geq 0$, the following equations are provided

$$i)\ S(mh_n) = \begin{cases} 3J_n, & if\ n\ is\ even \\ 3J_n + 2, & if\ n\ is\ odd \end{cases}$$

$$ii)\ S(mh_n) = \begin{cases} j_n\ , & if\ n\ is\ even \\ j_n + 2, & if\ n\ is\ odd \end{cases}$$

where $S(mh_n)$ are the scalar parts of Mersenne-Lucas numbers.

Proof. i). By using the definition of the scalar parts of Mersenne in Definition 1.1., we have

$$S(mh_n) = m_n$$

From Lemma 2.2., if n is even then we get

$$S(mh_n) = m_n = 3J_n$$

If n is odd, then we have

$$S(mh_n) = m_n = 3J_n + 2$$

ii). From the definition of the scalar parts of Mersenne, we get

$$S(mh_n) = m_n$$

From Lemma 2.2., if n is even, then we get

$$S(mh_n) = m_n = \mathrm{j}_n$$

If n is odd, then we have

$$S(mh_n) = m_n = \mathrm{j}_n + 2$$

Thus, the proof is obtained. □

We will now give some important identities regarding the Mersenne-Lucas hybrid numbers.

Theorem 3.15. Cassini identity of Mersenne-Lucas hybrid numbers for $n > 0$ as follows:

$$mh_{n-1}mh_{n+1} - mh^2{}_n = 2^{n-1}(13 + 21i + 11\varepsilon + 3h)$$

Proof. For proof, let's write the left side of the equality by using the binet formula, we get

$$mh_{(n-1)}mh_{(n+1)} - mh^2{}_n$$
$$= 2^{n-1}(1 + 2i + 4\varepsilon + 8h)(1 + i + \varepsilon + h) + 2^{n+1}(1 + i + \varepsilon + h)(1 + 2i + 4\varepsilon + 8h)$$
$$-2^n(1 + 2i + 4\varepsilon + 8h)(1 + i + \varepsilon + h) - 2^n(1 + i + \varepsilon + h)(1 + 2i + 4\varepsilon + 8h)$$
$$= 2^{n-1}(13 - 3\mathrm{i} + 3\varepsilon + 11\mathrm{h}) + 2^{n+1}(13 + 9\mathrm{i} + 7\varepsilon + 7h) - 2^n(13 - 3\mathrm{i} + 3\varepsilon + 11\mathrm{h})$$
$$- 2^n(13 + 9\mathrm{i} + 7\varepsilon + 7\mathrm{h})$$
$$= -2^{n-1}(13 - 3i + 3\varepsilon + 11h) + 2^n(13 + 9i + 7\varepsilon + 7h)$$
$$= 2^{n-1}(13 + 21i + 11\varepsilon + 3h)$$

Thus the proof is obtained. □

Theorem 3.16. Catalan identity of Mersenne-Lucas hybrid numbers for $n, r \geq 0$ as follows:

$$mh_{n-r}mh_{n+r} - mh^2{}_n = 2^{n-\mathrm{r}}(1 - 2^{\mathrm{r}})(13 - 3i + 3\varepsilon + 11h$$
$$-2^{\mathrm{r}}(13 + 9i + 7\varepsilon + 7h))$$

Proof.

$$mh_{n-r}mh_{n+r} - mh^2{}_n$$
$$= 2^{n-\mathrm{r}}(1 + 2i + 4\varepsilon + 8h)(1 + i + \varepsilon + h) + 2^{n+\mathrm{r}}(1 + i + \varepsilon + h)(1 + 2i + 4\varepsilon + 8h)$$
$$-2^n(1 + 2i + 4\varepsilon + 8h)(1 + i + \varepsilon + h) - 2^n(1 + i + \varepsilon + h)(1 + 2i + 4\varepsilon + 8h)$$
$$= 2^{n-\mathrm{r}}(13 - 3i + 3\varepsilon + 11h) + 2^{n+\mathrm{r}}(13 + 9i + 7\varepsilon + 7h) - 2^n(13 - 3i + 3\varepsilon + 11\mathrm{h})$$
$$- 2^n(13 + 9i + 7\varepsilon + 7h)$$
$$= 2^{n-\mathrm{r}}(13 - 3i + 3\varepsilon + 11h)(1 - 2^{\mathrm{r}}) + 2^{\mathrm{n}}(13 + 9i + 7\varepsilon + 7h)(2^{\mathrm{r}} - 1)$$
$$= 2^{n-\mathrm{r}}(1 - 2^{\mathrm{r}})[13 - 3i + 3\varepsilon + 11h - 2^{\mathrm{r}}(13 + 9i + 7\varepsilon + 7h)]$$

Thus the proof is obtained. □

If we write $r = 1$, then we get the Cassini identity.

Theorem 3.17. Vajda identity of Mersenne-Lucas hybrid numbers for $n, m, r \geq 0$ as follows:

$$mh_{n+r}mh_{n+k} - mh_n mh_{n+r+k}$$
$$= 2^{2n-\mathrm{r}}(2^r - 1)[(13 - 3i + 3\varepsilon + 11h - 2^{\mathrm{k}}(13 + 9i + 7\varepsilon + 7h)]$$

Proof. For proof, let's write the left side of the equality by using the Binet formula, we get

$mh_{n+r}mh_{n+k} - mh_n mh_{n+r+k}$

$= 2^{n+\mathrm{r}}(13-3i+3\varepsilon+11h)+2^{n+k}(13+9i+7\varepsilon+7h)-2^{n}(13-3i+3\varepsilon+11h)$
$\quad -2^{n+r+k}(13+9i+7\varepsilon+7h)$

$= 2^{n}(13-3i+3\varepsilon+11h)(2^{\mathrm{r}}-1)-2^{n+k}(13+9i+7\varepsilon+7h)(2^{\mathrm{r}}-1)$

$= 2^{n}(2^{\mathrm{r}}-1)((13-3i+3\varepsilon+11h)-2^{\mathrm{k}}(13+9i+7\varepsilon+7h))$

Thus, the desired expression is obtained. □

Theorem 3.18. D'ocagne identity of Mersenne-Lucas hybrid numbers for $n, m \geq 0$ as follows:

$$mh_m mh_{n+1} - mh_n mh_{m+1} = 2^{n-m}(13+2i+11\varepsilon+3h)$$

Proof. For proof, let's write the left side of the equality by using the Binet formula,then we get

$$mh_m mh_{n+1} - mh_n mh_{m+1}$$
$= 2^{m}(1+2i+4\varepsilon+8h)+(1+i+\varepsilon+h)][2^{n+1}(1+2i+4\varepsilon+8h)+(1+i+\varepsilon+h)]$
$\quad -[2^{n}(1+2i+4\varepsilon+8h)+(1+i+\varepsilon+h)][2^{m+1}(1+2i+4\varepsilon+8h)$

$= 2^{m}(13-3i+3\varepsilon+11h)+2^{n+1}(13+9i+7\varepsilon+7h)-2^{n}(13-3i+3\varepsilon+11h)$
$\quad -2^{m+1}(13+9i+7\varepsilon+7h)$

$= (13-3i+3\varepsilon+11h)(2^{m}-2^{n})+(13+9i+7\varepsilon+7h)(2^{n+1}-2^{m+1})$

$= 2^{n-m}(13+2i+11\varepsilon+3h)$

Thus, the proof is obtained. □

Theorem 3.19. Honsberger identity of Mersenne-Lucas hybrid numbers for $n, m \geq 0$ as follows:

$$mh_n mh_m + mh_{n+1} mh_{m+1} = (2^{n+m}385+2^{n}39+2^{m}39+6)$$
$$+(2^{n+m}20-2^{n}9+2^{m}27+4)i+(2^{n+m}40+2^{n}9+2^{m}21+4)\varepsilon$$
$$+(2^{n+m}80+2^{n}33+2^{m}21+4)h$$

Proof. For proof, let's write the left side of the equality by using the Binet formula, then we get

$$mh_n mh_m + mh_{n+1} mh_{m+1}$$
$= 2^{n}(1+2i+4\varepsilon+8h)+(1+i+\varepsilon+h)][2^{m}(1+2i+4\varepsilon+8h)$

$+(1+i+\varepsilon+h)]+[2^{n+1}(1+2i+4\varepsilon+8h)+(1+i+\varepsilon+h)]$

$[2^{m+1}(1+2i+4\varepsilon+8h)+(1+i+\varepsilon+h)]$

$= (2^{n+m}5(77+4i+8\varepsilon+16h))+2^{n}3(13-3i+3\varepsilon+11h)$

$+2^{m}3(13+9i+7\varepsilon+7h)+(6+4i+4\varepsilon+4h)$

$= (2^{n+m}385+2^{n}39+2^{m}39+6)+(2^{n+m}20-2^{n}9+2^{m}27+4)i$

$+(2^{n+m}40 + 2^n 9 + 2^m 21 + 4)\varepsilon + (2^{n+m}80 + 2^n 33 + 2^m 21 + 4)h$

Thus, the proof is obtained.

4 CONCLUSIONS

We presented Mersenne-Lucas hybrid numbers. We have given the Binet formula, the generating function, the character and the norm for Mersenne-Lucas hybrid numbers. Also, we have given relations among these numbers. Then we have obtained Cassini identity, Catalan identity, Vajda identity and D'ocagne identity for Mersenne-Lucas hybrid numbers.

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REFERENCES

[1] S. Çelik, İ. Durukan, E. Özkan, ‘‘New recurrences on Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, and Jacobsthal-Lucas numbers’’, *Chaos, Solitons & Fractals*, **150**, 111173, (2021)

[2] S. Koparal, N. Ömür, ‘‘Some Congruences Involving Catalan, Pell and Fibonacci Numbers’’, *Mathematica Montisnigri*, **48**, 10-18, (2021).

[3] E. Özkan, “3-Step Fibonacci Sequences in Nilpotent Groups”, *Applied Mathematics and Computation*, **144**, 517-527, (2003).

[4] E. Özkan, “On General Fibonacci Sequences in Groups”, *Turkish Journal of Mathematics*, **27** (4), 525-537, (2003).

[5] E. Özkan, “On truncated Fibonacci sequences”, *Indian Journal of Pure and Applied Mathematics*, **38**(4), 241-251 (2007).

[6] E. Özkan, İ. Altun, A.A. Göçer, “On relationship among a new family of k-Fibonacci, k-Lucas numbers, Fibonacci and Lucas numbers”, *Chiang Mai J. Sci*., **44**(4), 1744-1750, (2017).

[7] E. Özkan, A. Aydoğdu, İ. Altun, “Some Identities for A Family of Fibonacci and Lucas Numbers”, *Journal of Mathematics and Statistical Science*, **3**(10), 295-303, (2017).

[8] N.Yilmaz, A. Aydoğdu, E. Özkan, ‘‘Some properties of ***k***-generalized Fibonacci numbers’’, *Mathematica Montisnigri*, **50**,73-79, (2021).

[9] T. Koshy, *Fibonacci and Lucas Numbers with applications*, New York: John Wiley and Sons Inc. (2001).

[10] P. Catarino, ‘‘On k−Pell hybrid numbers’’, *Journal of Discrete Mathematical Sciences and Cryptography*, **22** (1), (2019).

[11] C. Kızılates¸ ‘‘A new generalization of Fibonacci hybrid and Lucas hybrid numbers’’, *Chaos, Solitons and Fractals*, **130**, (2020).

[12] N. Kilic, ‘‘On k-Jacobsthal and k-Jacobsthal-Lucas hybrid numbers’’, *Journal of Discrete Mathematical Sciences and Cryptography*, **24**(4), 1063-1074, (2021).

[13] M. Özdemir, ‘‘Introduction to hybrid numbers’’, *Advances in Applied Clifford Algebras*, **28**, (2018).

[14] A.Szynal-Liana, ‘‘The Horadam hybrid numbers’’, *Discussiones Mathematicae–General Algebra and Applications,* **38**, (2018).

[15] A. Szynal-Liana, I. Wloch,‘‘On Pell and Pell-Lucas hybrid numbers’’, *Commentationes Mathematicae,* **58**, (2018) .

[16] A. Szynal-Liana, I. Wloch, ‘‘On Jacobsthal and Jacobsthal-Lucas hybrid numbers’’, *Annales Mathematicae Silesianae,* **33**, (2019).

[17] A. Szynal-Liana, I. Wloch, ‘‘On generalized Mersenne hybrid numbers’’, *Annales Universitatis Mariae Curie-Sklodowska Lublin-Polonia,* **74**(1), 77-84, (2020).

[18] D, Taşcı, E. Sevgi, ‘‘Some Properties between Mersenne, Jacobsthal and Jacobsthal-Lucas Hybrid Numbers’’, *Chaos, Solitons and Fractals*, **146**, 110862, (2021).

[19] P. Catarino, H. Campos, P. Vasco, ‘‘On the Mersenne sequence’’, *CM-Centro de Matemática* (2016).

[20] N. Saba, A. Boussayoud, K. V. V. Kanuri, ‘‘Mersenne Lucas numbers and complete homogeneous symmetric functions’’, *Journal of mathematics and computer science*, **24**(2), 127-139, (2021).

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ALMOST GEODESIC MAPPINGS OF TYPE π_1^* OF SPACES WITH AFFINE CONNECTION

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Summary. We consider almost geodesic mappings π_1^* of spaces with affine connections. This mappings are a special case of first type almost geodesic mappings. We have found the objects which are invariants of the mappings π_1^*. The fundamental equations of these mappings are in Cauchy form. We study π_1^* mappings of constant curvature spaces.

1 INTRODUCTION

In the theory of geodesic mappings and their generalizations many basic results were formulated as a system of differential equations in Cauchy form, see [1–14]. For almost geodesic mappings π_1 a similar result for special Ricci-Codazzi Riemannian spaces is formulated in Sinyukov monograph [1]. This result was generalized for Ricci-Codazzi spaces with affine connection and for Riemannian spaces in [15]. For π_1^* mappings of general symmetric spaces with affine connection the system of differential equations in the Cauchy form were found in works [16].

This paper is devoted to detailed study of π_1^* mappings which are characterized by the general equations in the Cauchy form. This result is significant because the equations in this form have established methods of solution.

The concept of almost geodesic mappings of type π_1^* of spaces with affine torsion-free connections was first introduced in [16]. These mappings are a special case of type π_1 almost geodesic mappings which were introduced by N. S. Sinyukov in [1].

The paper is devoted to study the general properties of π_1^* mappings. In particular, we have obtained the objects which are invariant under the mappings. Also π_1^* mappings of spaces of constant curvature and affine spaces were studied.

Let us recall the basic conceptions of the almost geodesic mappings theory presented in [1].

A curve defined in a space with an affine connection A_n is called *almost geodesic* if there exists a two-dimensional plane element parallel along the curve (relative to the affine connection) such that for any tangent vector of the curve its parallel translation along the curve belongs to the plane element.

A diffeomorphism f between spaces with affine connection A and $\overline{A}_n$ is called *almost geodesic mapping* if any geodesic curve of A is mapped under f onto an almost geodesic curve in $\overline{A}$.

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In order that a mapping of a space A_n onto a space $\overline{A}_n$ be almost geodesic it is necessary and sufficient that in a common coordinate system $x \equiv (x^1, x^2, \dots, x^n)$ which both spaces are referred to, the deformation tensor of the mapping $P_{ij}^h(x) \equiv \overline{\Gamma}_{ij}^h(x) - \Gamma_{ij}^h(x)$ must satisfy the conditions

$$A_{\alpha\beta\gamma}^h \lambda^\alpha \lambda^\beta \lambda^\gamma = a \cdot P_{\alpha\beta}^h \lambda^\alpha \lambda^\beta + b \cdot \lambda^h,$$

where $A_{ijk}^h \equiv P_{ij,k}^h + P_{ij}^\alpha P_{\alpha k}^h$, $\Gamma_{ij}^h(x)$ $\left(\overline{\Gamma}_{ij}^h(x)\right)$ are the components of the affine connection of the space A_n $\left(\overline{A}_n\right)$, $'',''$ denotes covariant derivative with respect to the connection of the space A_n, λ^h is an arbitrary vector, a and b are certain functions of variables x^h and λ^h.

Three types of almost geodesic mappings were specified, namely π_1, π_2, π_3. We have proved that for $n > 5$ other types of almost geodesic mappings except π_1, π_2, and π_3 do not exist [17].

Almost geodesic mappings of π_1 type are characterized by the following conditions for the deformation tensor:

$$A_{(ijk)}^h = \delta_{(i}^h a_{jk)} + b_{(i} P_{jk)}^h,$$

where a_{ij} is a certain symmetric tensor, b_i a certain covector, δ_i^h are the Kronecker delta, (ijk) denotes an operation called symmetrization without division with respect to the indices i, j and k.

Unlike mappings of the type π_1, the study of mappings π_2 and π_3 are devoted by a lot of papers (See e.g. [1,2]). It stems from the fact that the main equations of these mappings are much more sophisticated than equations of other ones. Hence the paper is devoted to a special case of mappings π_1, which does not degenerate into π_2, π_3 or geodesic mappings.

2 ALMOST GEODESIC MAPPINGS OF THE π_1^* TYPE

Let a mapping of A_n onto $\overline{A}_n$ satisfy the conditions [16]:

$$P_{ij,k}^h + P_{ij}^\alpha P_{\alpha k}^h = a_{ij} \delta_k^h, \tag{1}$$

where a_{ij} is a certain symmetric tensor.

These mappings are a special case of almost geodesic mappings of the π_1 type. From now on, that mappings will be denoted by π_1^*. Let us consider (1) as a system of differential equations of Cauchy type with respect to the deformation tensor P_{ij}^h and find their integrability conditions. To this end, differentiate covariantly (1) with respect to x^m in A_n, then alternate it in k and m.

Contracting the integrability conditions of the equations (1) for h and m, we get

$$(n-1) a_{ij,k} = P_{ij}^\alpha R_{\alpha k} - P_{\alpha(i}^\beta R_{j)\beta k}^\alpha - (n-1) P_{ij}^\alpha a_{\alpha k}, \tag{2}$$

where R_{ijk}^h is the Riemann tensor of the space A_n, $R_{ij} \equiv R_{ij\alpha}^\alpha$ is the Ricci tensor.

Obviously, in the space A_n the equations (1) and (2) form a closed system of PDEs of Cauchy type with respect to the functions $P_{ij}^h(x)$ and $a_{ij}(x)$. The functions must also satisfy the algebraic conditions

$$P_{ij}^h(x) = P_{ji}^h(x), \qquad a_{ij}(x) = a_{ji}(x). \tag{3}$$

Hence we have proved the theorem.

Theorem 1 *In order that a space A_n with an affine connection admits a canonical almost geodesic mapping of type π_1^* onto another space $\overline{A}_n$ with an affine connection, it is necessary and sufficient that the mixed system of differential equations of Cauchy type in covariant derivatives (1), (2), (3) has a solution with respect to the unknown functions $P_{ij}^h(x)$ and $a_{ij}(x)$.*

Let us note that Theorem 1 holds for $A_n \in C^1$ ($\Gamma_{ij}^h(x) \in C^1$), i.e. objects of affine connection Γ are differentiable. In this case, if $A_n \in C^r (r \geq 1)$ then $\overline{A}_n \in C^r$. It follows from the fact, that the solution $P_{ij}^h(x) \in C^r$ and $a_{ij}(x) \in C^{r-1}$.

The integrability conditions of the system are

$$-P_{ij}^\alpha R_{\alpha km}^h + P_{\alpha(i}^h R_{j)km}^\alpha = \frac{1}{n-1}[(P_{ij}^\alpha R_{\alpha m} - P_{\alpha(i}^\beta R_{j)m\beta}^\alpha)\delta_k^h - (P_{ij}^\alpha R_{\alpha k} - P_{\alpha(i}^\beta R_{j)k\beta}^\alpha)\delta_m^h ,$$

where $[ij]$ denotes the alternation with respect to the mentioned indices.

3 INVARIANT OBJECTS UNDER π_1^* MAPPINGS

It is known [1], that if P_{ij}^h is a deformation tensor, then the Riemann tensors R_{ijk}^h and $\overline{R}_{ijk}^h$ of the spaces A_n and $\overline{A}_n$ are related to each other by the equations

$$\overline{R}_{ijk}^h = R_{ijk}^h + P_{i[k,j]}^h + P_{i[k}^\alpha P_{j]\alpha}^h. \tag{4}$$

Using the formulas (1) and (4), we get

$$*\,\overline{W}_{ijk}^h = *\,W_{ijk}^h, \tag{5}$$

where

$$*\,W_{ijk}^h \equiv R_{ijk}^h - \frac{1}{n-1} R_{i[j}\delta_{k]}^h, \quad *\,\overline{W}_{ijk}^h \equiv \overline{R}_{ijk}^h - \frac{1}{n-1}\overline{R}_{i[j}\delta_{k]}^h. \tag{6}$$

Obviously, $*\,W_{ijk}^h$ is a tensor of type (1,3) in the space A_n, and $*\,\overline{W}_{ijk}^h$ is a tensor of the same type in the space $\overline{A}_n$. From the relations (5) it follows that the tensor is invariant under almost geodesic mappings π_1^*.

Contracting (5) for h and i, it is easy to see that it holds

$$W_{ij} = \overline{W}_{ij}, \tag{7}$$

where

$$W_{ij} \equiv R_{[ij]}, \quad \overline{W}_{ij} \equiv \overline{R}_{[ij]}. \tag{8}$$

Taking account of (7), the formulas (5) are expressible in the form

$$\overline{W}_{ijk}^h = W_{ijk}^h, \tag{9}$$

where W_{ijk}^h and $\overline{W}_{ijk}^h$ are the Weyl tensors of projective curvature of the spaces A_n and $\overline{A}_n$ respectively.

Finally we obtained the theorem.

Theorem 2 *The Weyl tensor of projective curvature W^h_{ijk}, and also the tensors $*W^h_{ijk}$ and W_{ij} defined by the formulas (6) and (8) as geometric objects of spaces with affine connections are invariant under almost geodesic mappings of type π_1^*.*

4 MAPPINGS π_1^* OF EQUIAFFINE AND PROJECTIVE EUCLIDEAN SPACES

From the Theorem 2 we obtain the next one.

Theorem 3 *If a projective Euclidean space admits an almost geodesic mappings of type π_1^* onto $\overline{A}_n$, then $\overline{A}_n$ itself is also a projective Euclidean space.*

Theorem 4 *If an equiaffine space admits an almost geodesic mappings of type π_1^* onto $\overline{A}_n$, then $\overline{A}_n$ itself is also an equiaffine space.*

Proof. Obviously, the proof of Theorem 3 and 4 follows from the facts that the Weyl tensor of projective curvature vanishes in a projective Euclidean space, and for an equiaffine space the condition $W_{ij} = 0$ holds identically, respectively.

Hence because of Theorem 2 the above mentioned tensors vanish in the space $\overline{A}_n$. This means that $\overline{A}_n$ is a projective Euclidean and equiaffine space, respectively.

Thus from Theorem 3 and 4 projective Euclidean and equiaffine spaces form closed classes with respect to mappings of type π_1^*.

It is easy to see that the Riemann tensor is preserved under mappings π_1^* if and only if the tensor a_{ij} vanishes identically. In this case the main equations of the mappings become

$$P^h_{ij,k} = -P^\alpha_{ij} P^h_{\alpha k}. \tag{10}$$

In an affine space the equations (10) are completely integrable. Consequently, a solution of the equations is determined by arbitrary initial values of $P^h_{ij}(x_0)$. If the initial values satisfy the condition $P^h_{ij}(x_0) \not\equiv \delta^h_{(i}\psi_{j)}(x_0)$, then the constructed solution determines the mapping of an affine space A_n onto another affine space $\overline{A}_n$, and the mapping is different from a geodesic one.

Hence we obtain the theorem.

Theorem 5 *There is a mapping π_1^* of affine space onto itself such that all straight lines are mapped onto plain curves, and not all the curves are straight lines.*

Moreover, since in affine spaces the integrability conditions (2) of the equations (1) are satisfied identically, the equations (1) are completely integrable.

Let us prove the theorem.

Theorem 6 *Riemannian spaces V_n of non-zero constant curvature admit non-geodesic mappings π_1^*, which are also almost geodesic mappings of type π_3. The quadratic complex of geodesics is preserved under the mappings.*

Proof. Let V_n be a Riemannian spaces with non-zero constant curvature R which admits non-geodesic mappings π_1^*. The integrability conditions are expressible in the form

$$K\left(P^h_{k(i}g_{j)l} - P^h_{l(i}g_{j)k}\right) + \delta^h_l B_{ijk} - \delta^h_k B_{ijl} = 0, \tag{11}$$

where $B_{ijk} \equiv a_{ij,k} + P^\alpha_{ij}(a_{\alpha k} + Kg_{\alpha k})$, g_{ij} is the metric tensor of the space V_n.

Let ϵ^h be a vector such that $\epsilon^\alpha \epsilon^\beta g_{\alpha\beta} = \pm 1$. Transvecting (11) with $\epsilon^j \epsilon^l$ and then symmetrizing it in i and k, we find

$$P^h_{ik} = \xi^h g_{ik} + \epsilon^h b_{ik} + \delta^h_{(i}\psi_{j)}, \tag{12}$$

where ξ^h, ψ_j are some vectors, b_{ik} is some symmetric tensor. From (12) it follows that the relation (11) becomes

$$\epsilon^h\left(b_{k(i}g_{j)l} - b_{l(i}g_{j)k}\right) + \delta^h_{[l}b_{k]ij} + \delta^h_{(i}g_{j)[l}\psi_{k]} = 0, \tag{13}$$

where b_{ijk} is some tensor.
Transvecting (13) with ϵ^l, we get

$$\delta^h_i\left(g_{j\alpha}\epsilon^\alpha\psi_k - g_{jk}\epsilon^\alpha\psi_\alpha\right) + \epsilon^h \underset{1}{b}_{ijk} + \delta^h_j \underset{2}{b}_{ik} + \delta^h_k \underset{3}{b}_{ij} = 0, \tag{14}$$

where $\underset{1}{b}_{ijk}$, $\underset{2}{b}_{ik}$, $\underset{3}{b}_{ij}$ are some tensors.

Suppose that $\psi_i \not\equiv 0$. Then $g_{j\alpha}\epsilon^\alpha\psi_k - g_{jk}\epsilon^\alpha\psi_\alpha \not\equiv 0$ and consequently there exist vectors a^j and b^k such that $a^j b^k\left(g_{j\alpha}\epsilon^\alpha\psi_k - g_{jk}\epsilon^\alpha\psi_\alpha\right) \neq 0$. Transvecting (13) with $a^j b^k$, we obtain a relation which is contrary to the assumption that $n > 3$. Hence $\psi_i = 0$. The formulas (14) can be simplified and we can show by a similar method that $b_{kij} = 0$. Then (13) becomes $b_{k(i}g_{j)l} - b_{l(i}g_{j)k} = 0$. Transvecting the latter with g^{jl}, we find that $b_{ki} = \frac{b_{\alpha\beta}g^{\alpha\beta}}{n} g_{ki}$. We have from (12) by direct calculation

$$P^h_{ij} = P^h g_{ij}, \tag{15}$$

where P^h is some vector. Hence the mapping is f-planar. Consequently, according to [1,2], such mapping is almost geodesic mapping of type π_3. And in [17] the authors have proved that the mappings $\pi_1 \cap \pi_3$ preserves the quadratic complex of geodesics [18].

Substituting (15) in (1), we have

$$P^h_{,k} + P^h P_k = \alpha\delta^h_k,$$

where α is some invariant, $P_k = P^i g_{ik}$.

Vector fields satisfying these conditions are referred to as *concircular vector fields*. One knows that concircular vector fields always exist in spaces of constant curvature.

5 EXAMPLES OF ALMOST GEODESIC MAPPINGS π_1^*

We shall give an example of an almost geodesic mapping π_1^* of a flat space A_n onto another flat space $\overline{A}_n$.

Let $x^1, x^2, \ldots, x^n$ and $\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n$ be affine coordinate systems in the spaces A_n and $\overline{A}_n$ respectively. A point mapping

$$\overline{x}^h = \tfrac{1}{2}C^h_\alpha(x^\alpha - C^\alpha)^2 + x^h_0, \tag{16}$$

where C_i^h, C^h, x_0^h are some constant, det $\| C_i^h \| \not\equiv 0$, defines the almost geodesic mapping π_1^* of the space A_n onto the space $\overline{A}_n$.

By direct calculation it is readily shown that the components of the deformation tensor P_{ij}^h in the coordinate system $x^1, x^2, \dots, x^n$ are given

$$P_{ii}^i = \frac{1}{x^i - C^i} \quad i = \overline{1, n},$$

all the other components being zero.

Obviously, the tensor satisfies the equation (10). Note that the mapping is different from mappings of types π_2 and π_3.

Straight lines which are defined in the space A_n by the equation $x^h = a^h + b^h t$ (t is a parameter along a line) are mapped into parabolas in the space $\overline{A}_n$. The parabolas are defined by the equations

$$\overline{x}^h = F^h + D^h t + E^h t^2,$$

where $F^h = \frac{1}{2} C_\alpha^h (a^\alpha - C^\alpha)^2$, $D^h = C_\alpha^h (a^\alpha - C^\alpha) b^\alpha$, $E^h = \frac{1}{2} C_\alpha^h (b^\alpha)^2$.

The exceptions are the straight lines through the point $M(C^1, C^2, \dots, C^n)$. By (16) the lines are mapped into straight lines too.

Finally we note that the formulas (16) generate a family of almost geodesic mappings π_1 of a flat space if the parameters C_i^h, C^h and x_0^h are understood as continuous values.

6 CONCLUSION

Out of the three types of almost geodesic mappings of spaces with affine connection, distinguished by N.S. Sinyukov, the least studied are almost geodesic mappings of the first type. The equations that characterize them are very complex. Therefore, the results obtained for mappings π_1^*, including for their particular cases, are very relevant and are of theoretical value from the geometrical point of view. At the same time, they can be used in the theory of relativity and theoretical mechanics.

Almost geodesic mappings are a natural generalization of geodesic mappings. The basic equations of geodesic mappings of spaces with affine connection cannot be reduced to closed systems of equations in covariant derivatives of Cauchy type, since the general solution depends on *n* arbitrary functions.

We have singled out a special case of almost geodesic mappings of the first type, denoted by π_1^*, the basic equations of which are reduced to a closed system of equations in covariant derivatives of the Cauchy type. This result is very important, since (since geodesic mappings are a special case of almost geodesic mappings) the basic equations of the first type of spaces with affine connection are not reducible to closed systems of equations in covariant derivatives of Cauchy type.

For the mappings π_1^* geometric objects of tensor nature are found that are invariant under such mappings. It turns out that the Weyl tensor is invariant not only with respect to geodesic mappings, but also with respect to more general mappings.

In the article it is proved that projective-Euclidean and equiaffine spaces form closed classes with respect to mappings π_1^*.

From geometrical point of view, an interesting is a special case of mappings π_1^*, which we have distinguished, where the Riemann tensor is invariant. In this case, the basic equations of such mappings in flat space are completely integrable. An example of mappings π_1^* of flat space onto flat space is given.

In wpresented paper, it is of interest to study the integrability conditions and their differential extensions of the obtained equations in covariant derivatives of the Cauchy type that characterize the mappings π_1^* of spaces with affine connection.

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REFERENCES

[1] N.S. Sinyukov, *On geodesic mappings of Riemannian spaces*, Moscow: Nauka, (1979).
[2] N.S. Sinyukov, "Almost geodesic mappings of affinely connected and Riemannian spaces", *Itogi Nauki Tekh., Ser. Probl. Geom.*, **13**, 30–26 (1982).
[3] M. Prvanović, "A note on holomorphically projective transformations of the Kählerian spaces", *Tensor*, **35**, 99–104 (1981).
[4] J. Mikeš et al., *Differential Geometry of Special Mappings*, Palacky Univ. Press, Olomouc, 1 ed. (2015), 2 ed. (2019).
[5] V.E. Berezovski, I. Hinterleitner and J. Mikeš, "Geodesic mappings of manifolds with affine connection onto the Ricci symmetric manifolds", *Filomat*, **32**(2), 379–385 (2018).
[6] V.E. Berezovski, I. Hinterleitner, N.I. Guseva and J. Mikeš, "Conformal mappings of Riemannian spaces onto Ricci symmetric spaces", *Math. Notes*, **103**(1-2), 304–307 (2018).
[7] V.E. Berezovski, J. Mikeš, P. Peška and L. Rýparová, "On canonical F-planar mappings of spaces with affine connection", *Filomat*, **33**(4), 1273–1278 (2019).
[8] V.E. Berezovski, J. Mikeš, L. Rýparová and A. Sabykanov, "On canonical almost geodesic mappings of typeπ_2(e)", *Mathematics*, **8**(1), N. 54 (2020).
[9] V.E. Berezovski, Y. Cherevko, J. Mikeš and L. Rýparová, "Canonical almost geodesic mappings of the first type of spaces with affine connections onto generalized m-Ricci-symmetric spaces", *Mathematics*, **9**(4), N. 437 (2021).
[10] V.V. Domashev and J. Mikeš, "Theory of holomorphically projective mappings of Kählerian spaces", *Math. Notes*, **23**, 160–163 (1978).
[11] J. Mikeš, "Special F-planar mappings of affinely connected spaces onto Riemannian spaces", *Mosc. Univ. Math. Bull.* **49**(3), 15–21 (1994).
[12] J. Mikeš, M.L. Gavril'chenko and E.I. Gladysheva, "Conformal mappings onto Einstein spaces", *Mosc. Univ. Math. Bull.* **49**(3), 10–14 (1994).
[13] J. Mikeš and V. Berezovski, "Geodesic mappings of affine-connected spaces onto Riemannian spaces", *Colloq. Math. Soc. J. Bolyai, 56. Diff. Geom. Eger Hungary*, 491–494 (1989).
[14] J. Mikeš, S. Báscó, V. Berezovski, "Geodesic mappings of weakly Berwald spaces and Berwald spaces onto Riemannian spaces", *Int. J. Pure Appl. Math.*, **45**(3), 413–418 (2008).
[15] V.E. Berezovski, J. Mikeš and A. Vanžurová, "Fundamental PDE's of the canonical almost geodesic mappings of type $\tilde{\pi}_1$", *Bull. Malays. Math. Sci. Soc.* **37**(3), 647–659 (2014).
[16] V.E. Berezovski, "On almost geodesic mappings of type π_1^* of spaces with affine connections", *Dep. in UkrNIINTI*, 8.5.1991, N. 645-91Uk, 8 p.
[17] V.E. Berezovski and J. Mikeš, "On the classification of almost geodesic mappings of affine-connected spaces", *DGA, Proc. Conf., Dubrovnik/Yugosl. 1988*, 41-48 (1989).
[18] V.M. Chernyshenko, "Affine-connected spaces with a correspondent complex of geodesics", *Collection of Works of Mech.-Math. Chair of Dnepropetrovsk Univ.*, **55**(6), 105–118 (1961).

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THE DEGREE OF PRIMITIVE SEQUENCES AND ERDŐS CONJECTURE

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Summary. A sequence A of strictly positive integers is said to be primitive if none of its term divides another. Z. Zhang proved a result, conjectured by Erdős and Zhang in 1993, on the primitive sequences whose the number of the prime factors of its terms counted with multiplicity is at most 4. In this paper, we extend this result to the primitive sequences whose the number of the prime factors of its terms counted with multiplicity is at most 5.

1.INTRODUCTION

A sequence A of strictly positive integers is said to be primitive if none of its elements divide another. From the sequence of prime numbers $P = (p_n)_{n\geq 1}$ we can construct an infinite collection of primitive sequences. According to the prime number theorem, the n-th prime number p_n is asymptotically equal to $nlog\, n$; this ensures the convergence of the series

$$f(P) = \sum_{p\in P} \frac{1}{p\log p}.$$

A computation for $f(P)$ was obtained in [1] by Cohen as:

$$f(P) = 1.63661632335126086856965800392186367118159707613129\ldots .$$

Throughout this paper, we let $\Omega(a)$ denote the number of prime factors of a counted with multiplicity. For a primitive sequence A the number $max\{\Omega(a): a \in A\}$ is called the degree of A. It is noted $deg(A)$. By convention $deg(\{1\}) = deg(\emptyset) = 0$. For any primitive sequence A we pose $f(A) = \sum_{a\in A}\frac{1}{a\, log\, a}$. We agree that $f(A) = 0$ if $deg(A) = 0$. For any primitive sequence A and any integer $m \geq 1$, we put:

$$A_m = \{a \in A, \text{ the prime factors of } a \text{ are} \geq p_m\},$$

$$A'_m = \{a \in A_m, p_m \mid a\},$$

$$A''_m = \left\{\frac{a}{p_m} : a \in A'_m\right\}.$$

Then we have $A'_i \cap A'_j = \emptyset$ for i≠j and $A = \bigcup_{m\geq 1} A'_m$ is disjoint. In the case when A is finit, we have $deg(A''_m) < deg(A)$. In [2], Erdős proved that the series $f(A)$ converges for any primitive sequence A and in [3], Erdős asked if it is true that $f(A) \leq f(P)$ for any primitive sequence A. In [4], Erdős and Zhang showed that $f(A) \leq 1.84$ for any primitive sequence A, and in [5], Clark improved this result $f(A) \leq e^{\gamma}$ (where γ is the Euler constant)

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in the special case when A is a primitive set of composite numbers. Several years later in [6], Lichtman and Pomerance proved that $f(A) < e^{\gamma} \cong 1.781$. Moreover, in [2], Erdős conjectured that $f(A) \leq f(P)$ for any primitive sequence A, then in [7,8], Zhang proved this conjecture for any primitive sequence A of degree ≤ 4 and for some special cases of primitive sequences. In [9], the auteurs simplified the proof of Zhang over the primitive sequences of degree ≤ 4. In this note, we prove this result:

Theorem. For any primitive sequence A where $deg(A) \leq 5$, we have:

$$\sum_{a\in A,\ a\leq n} \frac{1}{a\log a} \leq \sum_{p\in P,\ p\leq n} \frac{1}{p\log p}\ for\ n > 1.$$

The proof of this result is based on the upper bound of $f(A'_i)$ where i≥1. We introduce the following constants, K_0=0, K_1=0.1578, K_2=0.4687, K_3=1.1971, K_4=2,77258, α=1.11012 and β=0.0642. We define the sequences $((\chi_i(m))_{m\geq 1}$ as follows: $\chi_i(m) = 1\ for\ m \geq 2\ , j \in \{1,2,3,4\}$ and $\chi_4(1) = 1, \chi_3(1) = 1.096,\ \chi_2(1) = 1.03,\ \chi_1(1) = 1.012\ ,\ \chi_0(1) = 1$.

2. MAIN RESULTS

We need the following lemmas.

Lemma 2.1 Let $n > 1$ be an integer, put $F(n) = \log n + \log\log n - 1$ then we have

$$p_n \geq nF(n), for\ n \geq 2\ ([10]) \tag{1}$$

$$p_n \geq n\left(F(n) + \frac{loglogn + 2.25}{logn}\right), for\ n \geq 2\ ([10]\ ) \tag{2}$$

$$p_n \leq n(F(n) + \beta), for\ n \geq 7022 \tag{3}$$

$$p_n > n(\log(nF(n)) - \alpha)\ , for\ n \geq 2. \tag{4}$$

Proof. Inequality (3) stems from inequality $p_n \leq n(log\ n + loglog\ n - 0.9385)([11])$. According to (2) we have:

$$\frac{\mathrm{p_n}}{\mathrm{n}} - \log(\mathrm{nF(n)}) \geq -1 - \log\left(1 + \frac{(\mathrm{loglogn} - 1}{\mathrm{logn}}\right) + \frac{\mathrm{loglogn} + 2.25}{\mathrm{logn}}\ \text{ for }\ \mathrm{n} \geq 3.$$

Knowing that the function

$$x \longmapsto f(x) = -1 - \ log\ (1 + (loglogx - 1)/logx) + (loglogx + 2.25)/logx$$

is increasing on $[41\times10^3,+\infty)$, then $\frac{p_n}{n} - \log(nF(n)) \geq f(41 \times 10^3) \geq -\alpha$. A computer calculation shows that, for $2 \leq n \leq 41 \times 10^3$ we have :

$$\frac{p_n}{n} - log\ (nF(n)) \geq -\alpha.$$

This completes the proof of (4).

Lemma 2.2 For $m \geq 1$ and $j \in \{1, 2, 3,4\}$, we have:

$$\sum_{i \geq m} \frac{\chi_j(m)}{p_i(K_j + \log p_i)} < \frac{\chi_{j-1}(m)}{K_{j-1} + \log p_m}.$$

Proof. For $j \in \{1, 2, 3,4\}$, we put $N = 7022, C = 0.00654$,

$$U_1 = 0.02348,\ U_2 = 0.17929,\ U_3 = 0.54349,\ U_4 = 1.30221;$$

$$V_1 = 0, \qquad V_2 = 0, \qquad V_3 = 0 \quad \text{and} \quad V_4 = -0.05804.$$

It is clear that for $m \geq N$ and $j \in \{1, 2, 3,4\}$ we have:

$$\begin{aligned} &C \geq -\log\big(F(m)\big) + \log(1 + \frac{1}{m}) + \log(F(m+1) + \beta) \\ &C \leq U_j - K_{j-1}, \qquad (5) \\ &V_j = \alpha - K_j + 2U_j - 1. \end{aligned}$$

We put

$$h_j(m) = \sum_{i \geq 1} \frac{\chi_j(m)}{p_i(K_j + \log p_i)}.$$

By (1) and (4) we have, for $m \geq N$ and $j \in \{1, 2, 3,4\}$,

$$p_i(K_j + \log p_i) > i\big(\log\big(iF(i)\big) - \alpha\big)(\,K_j + \log\big(iF(i)\big)\big),$$

Since $x \to \log(xF(x))$ increases for $x \geq 3$, it follows that

$$h_j(m+1) < \int_m^\infty \frac{dt}{t\big(\log\big(tF(t)\big) - \alpha\big)(\,\log\big(tF(t)\big) + K_j\big)},$$

use the change of variable $x = \log t$, we obtain:

$$h_j(m+1) < \int_{\log m}^\infty \frac{dt}{(\mathrm{L}(\mathrm{x}) - \alpha)\big(\mathrm{L}(\mathrm{x}) + K_j\big)}, \text{where } \mathrm{L}(\mathrm{x}) = \log(\mathrm{e}^{\mathrm{x}}\mathrm{F}(\,\mathrm{e}^{\mathrm{x}})).$$

Since, for $x > \log N$,

$$\frac{1}{L'(x)} < \left(1 - \frac{1}{L(x) - 1}\right),$$

then

$$h_j(m+1) < \int_{\log m}^\infty \frac{\left(1 - \frac{1}{L(x)-1}\right) L'(x)dx}{(\mathrm{L}(\mathrm{x}) - \alpha)\big(\mathrm{L}(\mathrm{x}) + K_j\big)},$$

by setting $y = L(x)$ and $y_m = L(\log m)$ we get:

$$h_j(m+1) < \int_{y_m}^\infty \frac{(\mathrm{y} - 2)dy}{(\mathrm{y} - 1)(\mathrm{y} - \alpha)\big(\mathrm{y} + K_j\big)}.$$

For $m \geq N$ and $j \in \{1, 2, 3,4\}$ we put:

$$g_j(m) = \frac{\chi_{j-1}(m)}{K_{j-1} + \log p_m},$$

then according to (3) and (5) we have:

$$g_j(m+1) \geq \frac{1}{K_{j-1} + \log\big((m+1)(F(m+1)+\beta)\big)}$$

$$> \frac{1}{\log\big(mF(m)\big) + U_j} = \int_{y_m}^{\infty} \frac{dy}{(y+U_j)^2}.$$

We have for $m \geq N$ and $j \in \{1,2,3,4\}$,

$$(\mathrm{y}-2)(y+U_j)^2 - (\mathrm{y}-1)(\mathrm{y}-\alpha)\big(\mathrm{y}+K_j\big) \leq 0.$$

So, for $m \geq N$ and $j \in \{1,2,3,4\}$, we have $h_j(m+1) < g_j(m+1)$ i.e.

$$h_j(m) < g_j(m) \; for \; m > N.$$

For 1≤ m ≤N and by definition of $\chi_j(i)$, we have for j∈{1,2,3,4} a computer calculation shows that:

$$\sum_{i\geq m} \frac{\chi_j\,(i)}{p_i\,(K_j + logp_i)} = \sum_{i\geq m}^{N} \frac{\chi_j\,(i)}{p_i(K_j + \log p_i)} + h_j(N+1)$$

$$< \sum_{i\geq m}^{N} \frac{\chi_j(i)}{p_i(K_j + \log p_i)} + \frac{1}{\log\big(NF(N)\big) + U_j}$$

$$< g_j(m).$$

This completes the proof.

Lemma 2.3. Let $m \geq 1$ be fixed and let $B = B_m$ be primitive with $\deg(B) \leq 4$. For $1 \leq t \leq 5 - \deg(B)$, we have:

$$\sum_{b\in B} \frac{1}{b(t\log p_m + \log b)} < \frac{\chi_{t-1}\,(m)}{K_{t-1} + \log p_m}, \tag{6}$$

$$\sum_{b\in B} \frac{1}{b(t\log p_m + \log b)} < \frac{1}{\log p_m}. \tag{7}$$

Proof. For $m \geq 1$ and $1 \leq t \leq 5 - \deg(B)$ put

$$g_t(B) = \sum_{b\in B} \frac{1}{b(t\log p_m + \log b)} \;\; where \; (g_t(\emptyset) = 0).$$

By induction on $\deg(B)$. If $\deg(B) = 1$ and $1 \leq t \leq 4$ we have $t\log p_m \geq t\log 2 > K_t$, so according to Lemma 2.2, we get:

$$g_t(B) = \sum_{b \in B} \frac{1}{b(t \log p_m + \log b)} < \sum_{i \geq m} \frac{1}{p_i(t \log p_1 + \log p_i)}$$
$$\leq \sum_{i \geq m} \frac{\chi_t\ (i)}{p_i(K_t + \log p_i)} < \frac{\chi_{t-1}\ (m)}{K_{t-1} + \log p_m}.$$

We assume that inequality (6) is true for $1 \leq \deg(B) \leq s < 5$ and $1 \leq t \leq 5 - \deg(B)$ and we show that it remains true for $\deg(B) = s - 1$. We have $B = \bigcup_{i \geq m} B_i{}'$ is dis joint, so we have:

$$g_t(B) = \sum_{i \geq m} g_t(B_i{}').$$

Let i$\geq m$. If $\deg(B_i{}') \leq 1$ we have:

$$g_t(B_i{}') < \frac{1}{p_i(t \log p_1 + \log p_i)} < \frac{\chi_t\ (i)}{p_i(K_t + \log p_i)}. \tag{8}$$

If $\deg(B_i{}') > 1$ we have:

$$g_t(B_i{}') = \sum_{b \in B_i{}''} \frac{1}{p_i b((t+1) \log p_i + \log b)}$$
$$= \frac{1}{p_i} g_{t+1}(B_i{}''),$$

since $\deg(B_i{}'') < s$ and $t + 1 \leq 5 - \deg(B_i{}'')$ so we have:

$$g_{t+1}(B_i{}'') < \frac{\chi_t\ (i)}{K_t + \log p_i},$$

thus

$$g_t(B_i{}') < \frac{\chi_t\ (i)}{p_i(K_t + \log p_i)}. \tag{9}$$

So from (8), (9), and Lemma 2.2, we get:

$$g_t(B) < \frac{\chi_{t-1}\ (m)}{K_{t-1} + \log p_m}.$$

For $t = 1$ we get the inequality (7), which ends the proof.

Proof of theorem Let n be fixed and let $A = \{a : a \in \boldsymbol{A}, a \leq n\}$ be subsequence of $\boldsymbol{A}$ where $\deg A \leq 5$. Put $\pi(n) = m$, the number of primes $\leq n$; then $A = \bigcup_{1 \leq i \leq m} A_i{}'$ is disjoint and $f(A) = \sum_{1 \leq i \leq m} f(A_i')$. Let $1 \leq i \leq m$. If $\deg A_i{}' \leq 1$ then $f(A_i') \leq \frac{1}{p_i \log p_i}$ and if $\deg A_i' > 1$ then

$$f(A_i') = \frac{1}{p_i} \sum_{b \in A_i''} \frac{1}{b(\log p_i + \log b)}$$

and $\deg A_i'' \leq \deg A_i' - 1 \leq 4$, so according to (7), we obtain:

$$\sum_{b\in A_i''} \frac{1}{b(\log p_i + \log b)} < \frac{1}{\log p_i}$$

therefore

$$f(A_i') \leq \frac{1}{p_i \log p_i}$$

Thus

$$f(A) = \sum_{1\leq i\leq m} f(A_i') \leq \sum_{1\leq i\leq m} \frac{1}{p_i \log p_i}.$$

This completes the proof.

3. CONCLUSION

Using a new value of the constants $\chi_i(m)$ will prove the theorem for greater degrees. Why not establish a recursive relationship on the degree of any sequence A, will prove this conjecture.

REFERENCES

[1] H. Cohen, "High precision computation of Hardy–Littlewood constants", *Preprint*, (1991). http://www.math.u-bordeaux1.fr/cohen/hardylw.dvi.

[2] P. Erdős, "Note on sequences of integers no one of which is divisible by any other", *J.Lond. Math. Soc*, **10**, 126-128 (1935).

[3] P. Erdős, "Seminar at the University of Limoges", (1988).

[4] P. Erdős & Z. Zhang, "Upper bound of $\sum 1/a_i loga_i$, for primitive sequences", *Math.Soc*, 117, 891-895 (1993).

[5] D. A Clark, "An upper bound of $\sum 1/a_i loga_i$, for primitive sequences", *Proceedings of the American Mathematical Society*, 123, 363-365 (1995).

[6] J. D Lichtman, & C. Pomerance, "The Erdős conjecture for primitive sets", *Proceedings of the American Mathematical Society Ser*. B, 6, 1-14 (2019).

[7] Z. Zhang, "On a conjecture of Erdős on the sum $\sum 1/a_i loga_i$", *J. Number Theory*, **39**, 14-1(1991).

[8] Z. Zhang, "On a problem of Erdős concerning primitive sequences", *Mathematics of Computation*, **60**(202), 827-834 (1993).

[9] I. Laib. A. Derbal. R. Mechik. and N. Rezzoug, "NOTE ON A THEOREM OF ZEHNXIAG ZHANG", *Math. Montis.*, **50**, 44-50 (2021).

[10] P. Dusart, "The kth prime is greater than $k(\ln k + \ln\ln k - 1)$ for $k \geq 2$", *Math. Comp*. **68(**225), 411–415 (1999).

[11] G. Robin, "Estimation de la Fonction de Tchebychef θ sur le k-ième Nombre Premier et Grendes Valeurs de la Fonction $\omega(n)$ Nombre de Diviseurs premiers de n", *Acta Arith,* **52**, 367-389 (1983).

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REVERSES HARDY-TYPE INEQUALITIES VIA JENSEN INTEGRAL INEQUALITY

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Summary. The integral inequalities concerning the inverse Hardy inequalities have been studied by a large number of authors during this century, of these articles have appeared, the work of Sulaiman in 2012, followed by Banyat Sroysang who gave an extension to these inequalities in 2013. In 2020 B. Benaissa presented a generalization of inverse Hardy inequalities. In this article, we establish a new generalization of these inequalities by introducing a weight function and a second parameter. The results will be proved using the Hölder inequality and the Jensen integral inequality. Several the reverses weighted Hardy's type inequalities and the reverses Hardy's type inequalities were derived from the main results.

1 INTRODUCTION

In recent years, several researchers have obtained extensions and generalizations of Hardy's inequality in the literature, for more details see [1], Hardy type inequalities for fractional integral operators [2], Hardy type inequalities involving functions of two variables [3]- [4], Hardy-type inequalities via the Steklov operator [5], a new version of inverse Hardy inequalities on time scales that appeared in 2021 see [6]- [7] . Many researchers have obtained results of refinements and generalizations of inverse Hardy inequalities, in 2020, B. Benaissa presented the following generalizations [8, Theorem 2.2].

Let f, g be positive functions defined on [a; b] and $F(x) = \int_{a}^{x} f(t)dt$. If g is non-decreasing, then for $p \geq 1$,

$$p\int_a^b \frac{F(x)^p}{g(x)}dx \leq (b-a)^p \int_a^b \frac{f(x)^p}{g(x)}dx - \int_a^b \frac{(x-a)^p}{g(x)}f(x)^p dx \, , \qquad (1)$$

for $0<p<1$,

$$p\int_a^b \frac{F(x)^p}{g(x)}dx \geq \frac{(b-a)^p}{g(b)}\int_a^b f(x)^p dx - \frac{1}{g(b)}\int_a^b (x-a)^p f(x)^p dx \qquad (2)$$

Taking $g(x) = x^p$, we get Sulaiman result inequalities, [9, theorem 3.1] and if we putting $g(x) = x^q$; $q > 0$, we get Banyat Sroysang result inequalities, [10, theorem 2.1 and theorem 2.2]. On the other hand, convex functions play an important role in inequality theory, this class of functions has many applications in different mathematical branches (numerical

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calculation, probability theory,...), match results are obtained by the Jensen inequality and many articles relating to different versions of this inequality have been published see for example [11]. Motivated by above literature, in this work we give a generalization of Hardy's integral inequality by using a weight function μ and a second parameter q , this results will be proved by Hölder inequality and Jensen integral inequality.

All along this paper, f, g are measurable non-negatives functions on interval $(a;b)$ where $0<a<b<+\infty$ and μ is a weight function (measurable and positive) on$(a;b)$. In the set of monotone functions, non-increasing (non-decreasing) function means the function is decreasing (increasing) or constant.

2 PRELIMINARIES

In this section we state the following Lemmas which are useful in the proofs of main Theorems.

Lemma2.1. Let $0 < p \leq q < \infty$ and $f, g,$wbe non-negative measurable functions on (a,b) and suppose that $0<\int_{\boldsymbol{a}}^{\boldsymbol{b}} f^q(t)\mu(t)dt < \infty$, then

$$\int_a^b f^p(t)\mu(t)dt \leq \left(\int_a^b \mu(t)dt\right)^{\frac{q-p}{q}} \left(\int_a^b f^p(t)\mu(t)dt\right)^{\frac{p}{q}}. \qquad (3)$$

The inequality (3) holds for $-\infty < q \leq p < 0$ and inverted for $0 < q \leq p < \infty$.

Proof. Using Hölder inequality for using the parameter $\frac{q}{p} \geq 1$, we have

$$\int_a^b f^p(t)\mu(t)dt = \left(\int_a^b \mu^{\frac{q-p}{q}}(t)dt\right)\left(\int_a^b f^p(t)\mu^{\frac{p}{q}}(t)dt\right).$$

$$\leq \left(\int_a^b \mu(t)dt\right)^{\frac{q-p}{q}} \left(\int_a^b f^p(t)\mu(t)dt\right)^{\frac{p}{q}}.$$

(See the version on time scales in [6]).

The version of the Jensen integral inequality is given below:

Lemma 2.2. Let f be an integrable function defined on $(a;\ b)$ and let $\varphi : (a;\ b) \to \mathbb{R}$ be a convex function. If $\varphi\ o\ f\ \in L_{(a;\ b)}$, then

$$\phi\left(\frac{1}{b-a}\int_a^b f(t)dt\right) \leq \frac{1}{b-a}\left(\int_a^b \phi\big(f(t)\big)dt\right).$$

The above inequality is inverted if φ is a concave function. The inequality in the Lemma 2.2 can be rewritten in the following forms.

- If ϕ is a convex function, then

$$\left(\int_a^b \phi\big(f(t)\big)dt\right) \geq (b-a)\phi\left(\frac{1}{b-a}\int_a^b f(t)dt\right). \qquad (4)$$

- If ϕ is a concave function, then

$$\left(\int_a^b \phi\big(f(t)\big)dt\right) \le (b-a)\phi\left(\frac{1}{b-a}\int_a^b f(t)dt\right). \tag{5}$$

3 MAIN RESULTS

Theorem 3.1. Let f, g be integrable positives functions on [a; b], μ be a weight function on (a; b) and

$$\mathrm{F}_\mu(x) = \int_0^x f(t)\mu(t)dt\,.$$

If g is non-decreasing and μ is non-increasing, then
(i) for $1 \le p \le q$,

$$q^{\frac{p}{q}} \int_a^b \frac{\left(\mathrm{F}_\mu\right)^p(x)}{g(x)}dx$$
$$\le \mu^p(a)(b-a)^{1-\frac{p}{q}}\left[(b-a)^q\int_a^b \frac{f^q(x)}{g^{\frac{q}{p}}(x)}dx - \int_a^b (x-a)^q \frac{f^q(x)}{g^{\frac{q}{p}}(x)}dx\right]^{\frac{p}{q}}, \tag{6}$$

(ii) for $0<q \le p<1$,

$$q^{\frac{p}{q}} \int_a^b \frac{\left(\mathrm{F}_\mu\right)^p(x)}{g(x)}dx$$
$$\ge \frac{\mu^p(b)}{g(b)}(b-a)^{1-\frac{p}{q}}\left[(b-a)^q\int_a^b f^q(x)dx - \int_a^b (x-a)^q f^q(x)dx\right]^{\frac{p}{q}}. \tag{7}$$

Proof. (i) For $1 \le p \le q$, Apply the Hölder inequality for $\frac{1}{p}+\frac{1}{p'}=1$, we get

$$\begin{aligned}\int_a^b \frac{\left(\mathrm{F}_\mu\right)^p(x)}{g(x)}dx &= \int_a^b g^{-1}(x)\left(\int_0^x f(t)\mu(t)dt\right)^p dx \\ &\le \int_a^b g^{-1}(x)\left\{\left(\int_0^x f^p(t)\mu(t)dt\right)^{\frac{1}{p}}\left(\int_0^x \mu(t)dt\right)^{\frac{1}{p'}}\right\}^p dx \\ &= \int_a^b g^{-1}(x)\left(\int_0^x f^p(t)\mu(t)dt\right)\left(\int_0^x \mu(t)dt\right)^{p-1} dx\,.\end{aligned}$$

Using the inequality (3) and since μ is non-increasing function, we deduce that

$$\int_a^b \frac{\left(\mathrm{F}_\mu\right)^p(x)}{g(x)}dx \quad \le \int_a^b g^{-1}(x)\left(\int_0^x \mu(t)dt\right)^{p-\frac{p}{q}}\left(\int_0^x f^q(t)\mu(t)dt\right)^{\frac{p}{q}} dx$$

$$\leq \int_a^b g^{-1}(x)\mu^p(a)(\boldsymbol{x}-\boldsymbol{a})^{(\boldsymbol{q}-\mathbf{1})-\frac{\boldsymbol{p}}{\boldsymbol{q}}}\left(\int_{\mathbf{0}}^{\boldsymbol{x}} f^q(t)dt\right)^{\frac{\boldsymbol{p}}{\boldsymbol{q}}} dx$$

$$= \mu^p(a)\int_a^b \left(H(x)\right)^{\frac{p}{q}} dx.$$

Where

$$H(x) = \int_{\mathbf{0}}^{\boldsymbol{x}} g^{-\frac{q}{p}}(x)(\boldsymbol{x}-\boldsymbol{a})^{(\boldsymbol{q}-\mathbf{1})} f^q(t)dt.$$

Let $\phi(x) = x^{\frac{p}{q}}$ be a concave function and g be non-decreasing function, apply the inequality (5), hence

$$\begin{aligned}
\int_a^b (H(x))^{\frac{p}{q}} dx &= \int_a^b \phi(H(x))dx \\
&\leq (b-a)\phi\left(\frac{1}{b-a}\int_a^b H(x)dx\right) \\
&= (b-a)^{1-\frac{p}{q}}\left(\int_a^b \int_{\mathbf{0}}^{\boldsymbol{x}} g^{-\frac{q}{p}}(x)(\boldsymbol{x}-\boldsymbol{a})^{(\boldsymbol{q}-\mathbf{1})} f^q(t)\, dt\, dx\right)^{\frac{p}{q}} \\
&= (b-a)^{1-\frac{p}{q}}\left(\int_a^b f^q(t)\int_{\boldsymbol{t}}^{\boldsymbol{b}} g^{-\frac{q}{p}}(x)(\boldsymbol{x}-\boldsymbol{a})^{(\boldsymbol{q}-\mathbf{1})}\, dx\, dt\right)^{\frac{p}{q}} \\
&\leq (b-a)^{1-\frac{p}{q}}\left(\int_a^b f^q(t)g^{-\frac{q}{p}}(t)\int_{\boldsymbol{t}}^{\boldsymbol{b}} (\boldsymbol{x}-\boldsymbol{a})^{(\boldsymbol{q}-\mathbf{1})}\, dx\, dt\right)^{\frac{p}{q}}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\int_a^b \frac{\left(F_\mu\right)^p(x)}{g(x)}dx &\leq \mu^p(a)(b-a)^{1-\frac{p}{q}}\left(\frac{1}{q}\int_a^b f^q(t)g^{-\frac{q}{p}}(t)[(\boldsymbol{b}-\boldsymbol{a})^{\boldsymbol{q}}-(\boldsymbol{t}-\boldsymbol{a})^{\boldsymbol{q}}]dt\right)^{\frac{p}{q}} \\
&= \left(\frac{\mu(a)}{q^{\frac{1}{q}}}\right)^p (b-a)^{1-\frac{p}{q}}\left\{(\boldsymbol{b}-\boldsymbol{a})^{\boldsymbol{q}}\int_a^b \frac{f^q(t)}{g^{\frac{q}{p}}(t)}dt - \int_a^b (\boldsymbol{t}-\boldsymbol{a})^{\boldsymbol{q}}\ \frac{f^q(t)}{g^{\frac{q}{p}}(t)}dt\right\}^{\frac{p}{q}},
\end{aligned}$$

thus we get (6) .

(ii) For $0 < q \leq p < 1$, using the reverse Hölder inequality, we get

$$\int_a^b \frac{\left(F_\mu\right)^p(x)}{g(x)}dx \geq \int_a^b g^{-1}(x)\left(\int_{\mathbf{0}}^{\boldsymbol{x}} f^p(t)\mu(t)dt\right)\left(\int_{\mathbf{0}}^{\boldsymbol{x}} \mu(t)dt\right)^{\boldsymbol{p}-\mathbf{1}} dx.$$

Apply the reverse of the inequality (3) and since μ is non-increasing function, we deduce that

$$\int_a^b \frac{(\mathrm{F}_\mu)^p(x)}{g(x)}dx \geq \int_a^b g^{-1}(x)\left(\int_{\mathbf{0}}^{x}\mu(t)dt\right)^{p-\frac{p}{q}}\left(\int_{\mathbf{0}}^{x}f^q(t)\mu(t)dt\right)^{\frac{p}{q}}dx$$
$$\geq \int_a^b g^{-1}(x)\mu^p(b)(x-a)^{(q-1)-\frac{p}{q}}\left(\int_{\mathbf{0}}^{x}f^q(t)dt\right)^{\frac{p}{q}}dx$$
$$=\mu^p(b)\int_a^b (H(x))^{\frac{p}{q}}dx\,.$$

Let $\phi(x) = x^{\frac{p}{q}}$ be a convex function and g be non-decreasing function, apply the inequality (4), hence

$$\int_a^b (H(x))^{\frac{p}{q}}dx \geq (b-a)^{1-\frac{p}{q}}\left(\int_a^b\int_{\mathbf{0}}^{x} g^{-\frac{q}{p}}(x)(x-a)^{(q-1)}f^q(t)\,dt\,dx\right)^{\frac{p}{q}}$$
$$\leq (b-a)^{1-\frac{p}{q}}\left(\int_a^b f^q(t)g^{-\frac{q}{p}}(b)\int_t^b (x-a)^{(q-1)}\,dx\,dt\right)^{\frac{p}{q}}$$
$$=\frac{(b-a)^{1-\frac{p}{q}}}{g(b)}\left(\int_a^b f^q(t)\int_t^b (x-a)^{(q-1)}\,dx\,dt\right)^{\frac{p}{q}},$$

therefore

$$\int_a^b \frac{(\mathrm{F}_\mu)^p(x)}{g(x)}dx \leq \frac{\mu^p(b)(b-a)^{1-\frac{p}{q}}}{g(b)}\left(\frac{1}{q}\int_a^b f^q(t)[(b-a)^q-(t-a)^q]dt\right)^{\frac{p}{q}}$$
$$=\frac{\mu^p(b)(b-a)^{1-\frac{p}{q}}}{q^{\frac{p}{q}}g(b)}\left\{(b-a)^q\int_a^b f^q(t)dt-\int_a^b f^q(t)(t-a)^q\,dt\right\}^{\frac{p}{q}}.$$

So, the proof of Theorem 3.1. is complete.

In the same data on the functions f, g and μ, with $\mathrm{F}_\mu(x)=\int_0^x f(t)\mu(t)dt$ and by reasoning analogously to the proof of Theorem 3.1, we obtain the following remarks.

Remark 3.1. If g and μ are non-decreasing functions, then

(i) for $1 \leq p \leq q$,

$$q^{\frac{p}{q}}\int_a^b \frac{(\mathrm{F}_\mu)^p(x)}{g(x)}dx$$
$$\leq \mu^p(b)(b-a)^{1-\frac{p}{q}}\left[(b-a)^q\int_a^b \frac{f^q(x)}{g^{\frac{q}{p}}(x)}dx-\int_a^b (x-a)^q\frac{f^q(x)}{g^{\frac{q}{p}}(x)}dx\right]^{\frac{p}{q}},(8)$$

(ii) for $0<q \leq p<1$,

$$q^{\frac{p}{q}} \int_a^b \frac{\left(\mathrm{F}_\mu\right)^p(x)}{g(x)} dx$$

$$\geq \frac{\mu^p(a)}{g(b)} (b-a)^{1-\frac{p}{q}} \left[(b-a)^q \int_a^b f^q(x) dx - \int_a^b (x-a)^q f^q(x) dx \right]^{\frac{p}{q}}. \quad (9)$$

Remark 3.2. If g is non-increasing and μ is non-decreasing, then

(i) for $1 \leq p \leq q$,

$$q^{\frac{p}{q}} \int_a^b \frac{\left(\mathrm{F}_\mu\right)^p(x)}{g(x)} dx$$

$$\leq \frac{\mu^p(b)}{g(b)} (b-a)^{1-\frac{p}{q}} \left[(b-a)^q \int_a^b f^q(x) dx - \int_a^b (x-a)^q f^q(x) dx \right]^{\frac{p}{q}}, \quad (10)$$

(ii) for $0 < q \leq p < 1$,

$$q^{\frac{p}{q}} \int_a^b \frac{\left(\mathrm{F}_\mu\right)^p(x)}{g(x)} dx$$

$$\geq \mu^p(a) (b-a)^{1-\frac{p}{q}} \left[(b-a)^q \int_a^b \frac{f^q(x)}{g^{\frac{q}{p}}(x)} dx - \int_a^b (x-a)^q \frac{f^q(x)}{g^{\frac{q}{p}}(x)} dx \right]^{\frac{p}{q}}. \quad (11)$$

Remark 3.3. If g and μ are non-increasing functions, then

(i) for $1 \leq \mathrm{p} \leq \mathrm{q}$,

$$q^{\frac{p}{q}} \int_a^b \frac{\left(\mathrm{F}_\mu\right)^p(x)}{g(x)} dx$$

$$\leq \frac{\mu^p(a)}{g(b)} (b-a)^{1-\frac{p}{q}} \left[(b-a)^q \int_a^b f^q(x) dx - \int_a^b (x-a)^q f^q(x) dx \right]^{\frac{p}{q}}, \quad (12)$$

(ii) for $0 < q \leq p < 1$,

$$q^{\frac{p}{q}} \int_a^b \frac{\left(\mathrm{F}_\mu\right)^p(x)}{g(x)} dx$$

$$\geq \mu^p(b) (b-a)^{1-\frac{p}{q}} \left[(b-a)^q \int_a^b \frac{f^q(x)}{g^{\frac{q}{p}}(x)} dx - \int_a^b (x-a)^q \frac{f^q(x)}{g^{\frac{q}{p}}(x)} dx \right]^{\frac{p}{q}}. \quad (13)$$

4. APPLICATIONS

We now give some new consequences of the above results.

4.1. The reverses weighted Hardy's type inequalities

If we set $q = p$ in Theorem3.1. , we obtain the following corollary.

Corollary 4.1. Let f, g be integrable positives functions on $[a; b]$, μ be a weight function on $(a; b)$ and

$$F_\mu(x) = \int_{\mathbf{0}}^{x} f(t)\mu(t)dt\,.$$

If g is non-decreasing and μ is non-increasing then

(i) for $1 \le p$,

$$p\int_a^b \frac{\left(F_\mu\right)^p(x)}{g(x)}dx \le \mu^p(a)\left((b-a)^p\int_a^b \frac{f^p(x)}{g(x)}dx - \int_a^b (x-a)^p\frac{f^q(x)}{g(x)}dx\right), (14)$$

(ii) for $0 < p < 1$,

$$p\int_a^b \frac{\left(F_\mu\right)^p(x)}{g(x)}dx \ge \frac{\mu^p(b)}{g(b)}\left((b-a)^p\int_a^b f^p(x)dx - \int_a^b (x-a)^p f^p(x)dx\right). (15)$$

According to remarks 3.1, 3.2, 3.3 above and with the same condition $q = p$, one can deduce new results, in a similar way to the inequalities (14) and (15) via to the monotonicity of the functions g and μ.

4.2. The reverses Hardy's type inequalities

If we set $\mu \equiv 1$ in Theorem3.1. and remarks 3.1, 3.2, 3.3, we result some new inequalities with two parameters in the following corollaries.

Corollary 4.2. Let f, g be positive functions defined on $[a; b]$ and $F(x) = \int_a^x f(t)dt$.

If g is non-decreasing, then

(i) for $1 \le p \le q$,

$$q^{\frac{p}{q}}\int_a^b \frac{F^p(x)}{g(x)}dx \le (b-a)^{1-\frac{p}{q}}\left[(b-a)^q\int_a^b \frac{f^q(x)}{g^{\frac{q}{p}}(x)}dx - \int_a^b (x-a)^q\frac{f^q(x)}{g^{\frac{q}{p}}(x)}dx\right]^{\frac{p}{q}}, (16)$$

(ii) for $0 < q \le p < 1$,

$$q^{\frac{p}{q}}\int_a^b \frac{F^p(x)}{g(x)}dx \ge \frac{(b-a)^{1-\frac{p}{q}}}{g(b)}\left[(b-a)^q\int_a^b f^q(x)dx - \int_a^b (x-a)^q f^q(x)dx\right]^{\frac{p}{q}}. (17)$$

Inequalities (16) and (17) are new generalizations of inequalities (1) and (2).

Corollary 4.3. Let f, g be positive functions defined on $[a; b]$ and $F(x) = \int_a^x f(t)dt$. If g is non-increasing, then

(i) for $1 \leq p \leq q$,

$$q^{\frac{p}{q}} \int_a^b \frac{F^p(x)}{g(x)} dx \leq \frac{(b-a)^{1-\frac{p}{q}}}{g(b)} \left[(b-a)^q \int_a^b f^q(x) dx - \int_a^b (x-a)^q f^q(x) dx \right]^{\frac{p}{q}}, (18)$$

(ii) for $0 < q \leq p < 1$,

$$q^{\frac{p}{q}} \int_a^b \frac{F^p(x)}{g(x)} dx \geq (b-a)^{1-\frac{p}{q}} \left[(b-a)^q \int_a^b \frac{f^q(x)}{g^{\frac{q}{p}}(x)} dx - \int_a^b (x-a)^q \frac{f^q(x)}{g^{\frac{q}{p}}(x)} dx \right]^{\frac{p}{q}}. (19)$$

The inequalities (18) and (19) are new results with two parameters in the case where g is a non-increasing function. If we p = q put in the Corollary4.2. we obtain the following new result.

Remark4.1 Let f, g be positive functions defined on [a; b] and $F(x) = \int_a^x f(t) dt$. If g is non-increasing, then for $p \geq 1$,

$$p \int_a^b \frac{F^p(x)}{g(x)} dx \leq \frac{1}{g(b)} \left((b-a)^p \int_a^b f^p(x) dx - \int_a^b (x-a)^p f^p(x) dx \right), (20)$$

for $0 < p < 1$,

$$p \int_a^b \frac{F(x)^p}{g(x)} dx \geq (b-a)^p \int_a^b \frac{f^p(x)}{g(x)} dx - \int_a^b (x-a)^p \frac{f^p(x)}{g(x)} dx. \qquad (21)$$

5 CONCLUSIONS

By applying Hölder's inequality to two integrability parameters and Jensen's integral inequality, new generalization integral inequalities relating to the inverse-weighted Hardy inequalities have been established and proven. Some particular cases are studied according to the monotony of the functions.

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REFERENCES

[1] B. Benaissa, M.Z. Sarikaya, A. Senouci "On some new Hardy-type inequalities", *Math. Meth. Appl .Sci.*, **43**, 8488-8495 (2020). https://doi.org/10.1002/mma.6503

[2] A. Khameli, Z. Dahmani, K. Freha, and M. Z .Sarikaya, "New Riemann-Liouville generalizations for some inequalities of Hardy type", *Malaya J. Mat.*, **4**(2), 277-283 (2016).

[3] B. Benaissa, M.Z. Sarikaya, "Some Hardy-type integral inequalities involving functions of two independent variables". *Positivity*, **25**, 853-866 (2021). https://doi.org/10.1007/s11117-020-00791-5

[4] B.G Pachpatte, "On Hardy type integral inequalities for a functions of two variables", *Demonstratio Math*., XXVIII(2), (1995). https://doi.org/10.1515/dema-1995-0202

[5] B. Benaissa, A. Senouci "Some new integral inequalities through the Steklov operator", *Math. Montis.*, **49**, 49 - 56 (2020). https://doi:10.20948/mathmontis-2020-49-4

[6] B. Benaissa, "Some Inequalities on Time Scales Similar to Reverse Hardy's Inequality", *Rad Hrvat. Akad. Znan. Umjet. Mat. Znan., (2021), Forthcoming papers (with pdf preview).*

[7] B. Benaissa, “More on Minkowski and Hardy integral inequality on time scales”, *Ricerche mat* (2021). https://doi.org/10.1007/s11587-021-00595-z
[8] B. Benaissa, “More on reverses of Minkowski’s inequalities and Hardy’s integral inequalities”, *Asian-Eur. J. Math.*, **13** (3) (2020). https://doi:10.1142/S1793557120500643
[9] W.T. Sulaiman, “Reverses of Minkowski’s, Hölder’s, and Hardy’s integral inequalities”, *Int. J. Mod. Math. Sci.*, **1** (1), 14-24 (2012).
[10] Banyat Sroysang, “A Generalization of Some Integral Inequalities Similar to Hardy’s Inequality”, *Math. AEterna*, **3** (7), 593-596 (2013).
[11] S.S. Dragomir, A.K. Muhammad, A. Addisalem, ”Refinement of the Jensen integral inequality”, *Open. Math.*, **14** (1), 221-228 (2016). https://doi:10.1515/math-2016-0020

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MODELING OF THE RADIATION INDUCED ELECTROMAGNETIC FIELD IN FINELY-DISPERSE MEDIA

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Summary. Algorithms for supercomputer modeling of the radiation electromagnetic field in heterogeneous materials of a complex finely-dispersed structure are constructed. A geometric model of a heterogeneous medium is created using Stilinger-Lubachevsky algorithms for multimodal structures. The model includes a system of detectors for statistical evaluation of functionals on the space of solutions of the photon-electron cascade transport equations. Algorithms for the three-dimensional approximation of the results of modeling the radiation transport in a fine-dispersed medium to an electrodynamic difference grid are developed. The approximation methods based on the technology of neural networks. The method of numerical solution of the complete system of Maxwell's equations for calculating the electromagnetic field in a fine-dispersed medium is worked out. The results of demonstration calculations of the electromagnetic field are presented. The results of the calculations show that the spatial distribution of the radiation electromagnetic field has a sharply inhomogeneous structure caused by the presence of boundaries of materials with different radiation properties.

1. INTRODUCTION

The investigations of radiation-induced (electromagnetic radiation, laser radiation, penetrating radiation) effects in media of complex geometrical structure are actual for a lot of applications: interaction EMP with objects [1], plasma generation and relaxation [2], ionizing radiation interaction with matter [3-5] and many others. Mathematical modeling is an effective method to such investigation [6-8].

The technique of a detailed supercomputer simulation of the processes of radiation-induced electrodynamic effects using ultra-high-performance computational techniques and modern parallelization technologies (MPI, OpenMP, CUDA) is presented in this paper.

The problems of mathematical modeling of radiation-induced charge and current effects in environments of complex geometric structure are considered in [9, 10]. Algorithms of supercomputer modeling of the formation of charge and current fields in heterogeneous polydisperse materials with direct resolution of their microstructure are described. The results of demonstration calculations of the parameters of charge and current fields are presented. The spatial distribution of radiation-induced charges has a sharply inhomogeneous structure due to the presence of boundaries of materials with strongly different radiation properties. Charge separation occurs near the boundary surfaces and can lead to the generation of a strong electric field that can disrupt the functional properties of a heterogeneous material with a finely dispersed structure.

Mathematical modeling of the radiation electromagnetic field includes the following tasks:

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- construction of a geometric model of a heterogeneous fine-dispersed medium, which includes a detector system for calculating the required values (energy deposit, electric current density) when modeling the interaction of radiation with matter;
- statistical modeling of radiation transport in a fine-dispersed medium with direct resolution of its microstructure;
- 3D approximation of the results of calculations of energy deposit and currents from the detector system used in solving the radiation transport problem to a spatial difference electrodynamic grid designed for numerical solution of the electrodynamics problem;
- numerical solution of the initial boundary value problem for the complete system of Maxwell equations.

The paper presents the results of modeling radiation-induced electromagnetic fields (EMF) in a fragment of a closed-cell structure, which consists of a binder and finely dispersed dielectric inclusions.

2. GEOMETRIC MODEL OF A POLYDISPERSE MEDIUM FRAGMENT

Let us consider a fragment of a fine-dispersed medium of a closed-cellular structure consisting of a binder and eight inclusions (Fig. 1). The binder material is polybutadiene (C4H6, density $\rho = 0.95\, g / cm^3$), the inclusion material is ammonium perchlorate (NH4ClO4, $\rho = 1.95\, g / cm^3$). The size of the fragment is shown in Fig. 1.

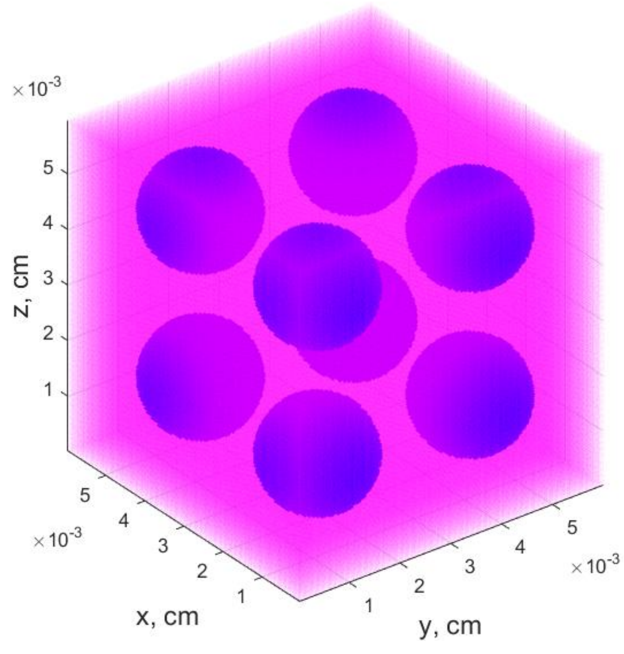

Figure 1: A fragment of a heterogeneous dispersed material

The geometric model of the medium includes a detector system for statistical evaluation of the required physical quantities. The detector (recording) system consists of a given number of spherical detectors of the same radius. The detectors should be isolated from each other (they should not intersect) and should not cross the boundaries of inclusions.

The Stillinger-Lubachevsky algorithm and its modifications are used to construct a detector system of a polydisperse medium [11-13]. An example of the constructed detector system for the fragment under consideration is shown in Fig. 2.

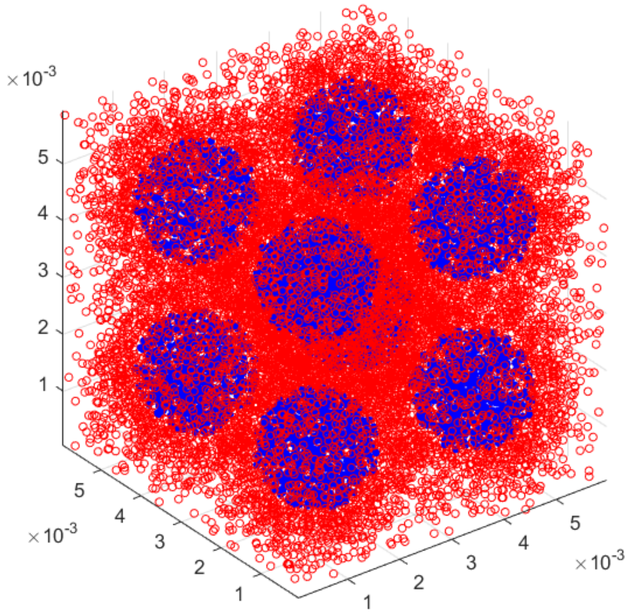

Figure 2: The detector system. Red color indicates detectors in the binder, blue – in inclusions

3. MODELING OF RADIATION TRANSPORT IN A FINE-DISPERSED MEDIUM

The processes of interaction of radiation with matter have a cascade character. The algorithms of statistical modeling of such processes are considered in detail in the works [14, 15]. These papers describe effective statistical algorithms for mathematical modeling of cascade radiation transport processes using hybrid computing technology. The algorithms are built considering the peculiarities of performing calculations on heterogeneous supercomputers using graphics accelerators as arithmetic calculators [14].

A modification of the processing scheme of the "tree" describing the cascade of particles by generations is proposed and implemented. The modification is developed on the basis of the use of stacks for temporary storage of information about the particles being born. The algorithm for filling these stacks takes into account a priori information about the relative path length of particles of various varieties.

The created method is optimized in terms of minimizing the amount of information required for processing the cascade tree. The approach to the organization of calculations in modeling the emergence and development of a photon-electron cascade is improved to achieve maximum GPU performance and increase the efficiency of simultaneous CPU/GPU loading. The algorithm for registering unlikely events is optimized in order to increase the information value of photon trajectories.

Figure 3 shows an example of the result of calculating the energy deposit in the considered dispersed fragment in the case when the object is irradiated by a photon flux with an energy of 20 keV. The direction of propagation of a flat flow is along the Z axis.

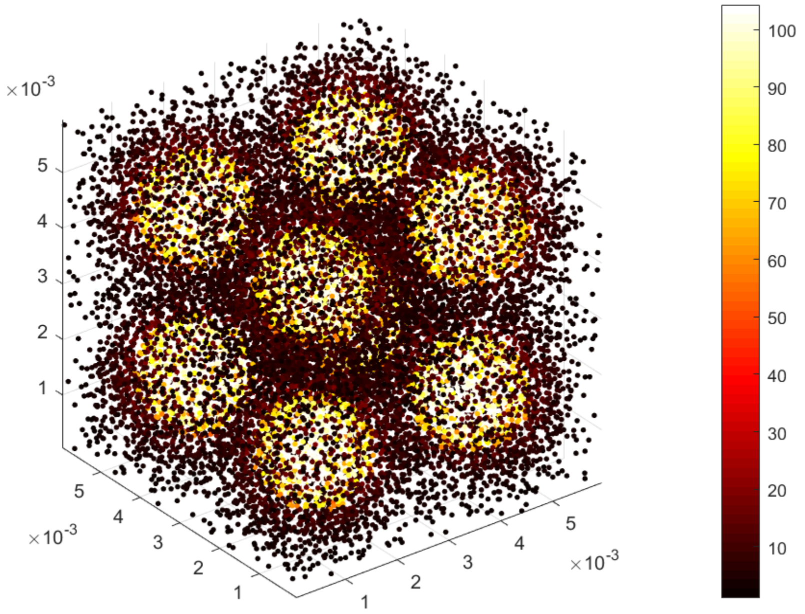

Figure 3: Results of calculation of the energy deposit density (keV / cm^3)

4. APPROXIMATION OF THE RESULTS OF MODELING THE RADIATION TRANSPORT TO AN ELECTRODYNAMIC GRID

Modeling of the radiation electromagnetic field requires the joint use of different software tools to assess the influence of various interdependent factors on the functional properties of the materials under study.

The use of mathematical models and numerical methods for computer research of processes of various physical nature (interaction of radiation with matter, secondary electrodynamic effects) makes it necessary to use various geometric approximations to describe the object. In this regard, there is a problem of integrating "according to data" the results of a numerical study of various physical processes in various mathematical models.

The problem of adequate transfer of the results of statistical modeling of the radiation energy deposit and radiative electric currents from the detector system used in modeling the interaction of radiation with matter to a rectangular Cartesian electrodynamic difference grid is solved using various approximation methods based on the technology of machine learning [16], in particular, on the technology of neural networks [17].

A multilayer perceptron [17] was used to solve approximation problems in this paper.

The network topology (3-100-30-1) and the logistic function $f(x) = 1/(1+e^{-x})$ of neuronal activation [17] were used to approximate the energy deposit density.

Figure 4 presents as an example the result of approximating the energy deposit density on an electrodynamic grid in the form of a surface in the plane z=0.0015 cm.

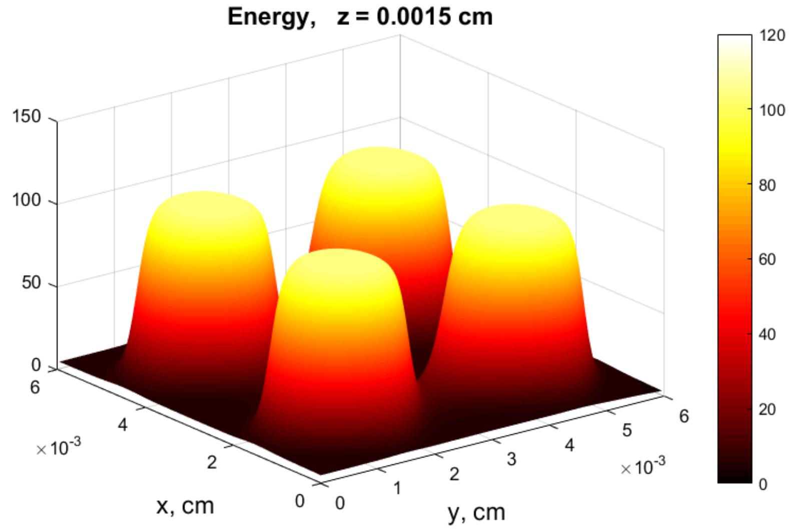

Figure 4: The energy deposit density in the plane z=0.0015 cm (keV / cm^3)

The network topology (3-60-25-8-1) for approximating the current components is used.

Different activation functions are used for approximating the current components. Tangential function ($f(x)=\tanh(x)$) is applicated for the transverse current components J_x and J_y because the transverse components J_x and J_y are odd relative to the center of the fragment and logistic one ($f(x)=1/(1+e^{-x})$) [4, 17] is used for the longitudinal component J_z. The electrodynamics grid has size 100x100x100.

As an example, the results of approximation of the transverse components of the electron flux density to the electrodynamic grid are presented in Fig. 5, 6. The results are depicted as surfaces on a slice z=0.0015 cm.

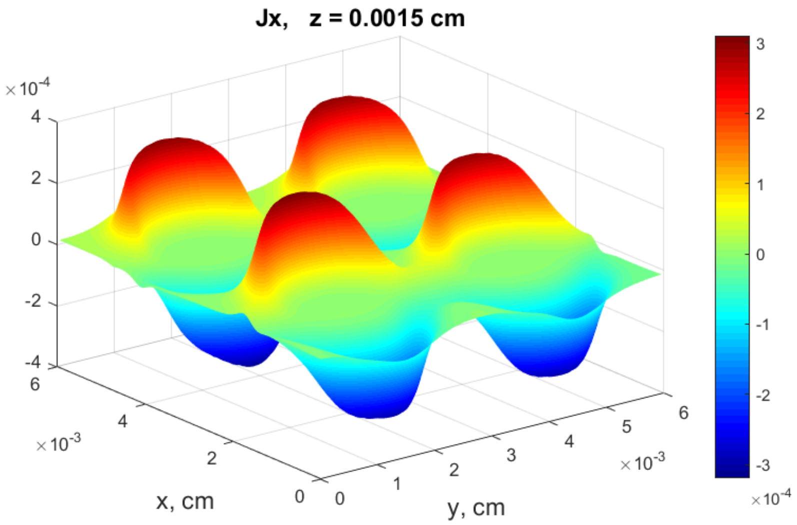

Figure 5: Function $J_x(x,y)$, **z**=0.0015cm, 1/cm^3

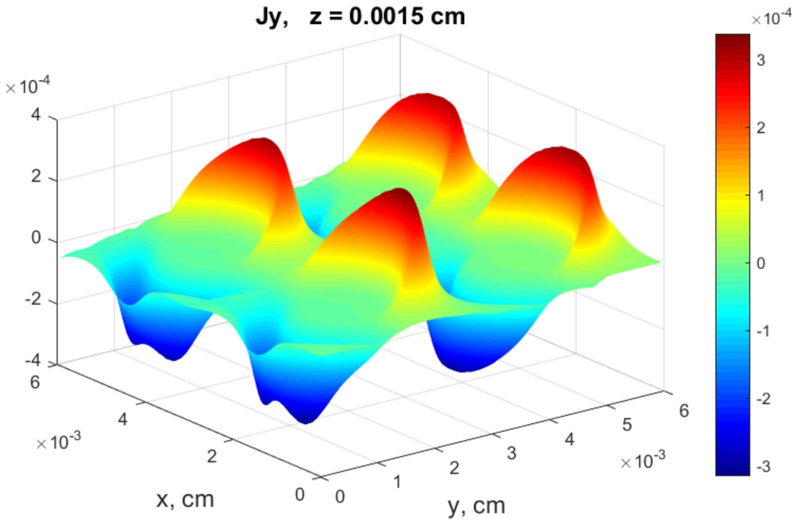

Figure 6: Function $J_y(x,y)$, **z**=0.0015cm, 1/cm^3

A constant step of gradient descent with the BFGS optimization method (a quasi-Newtonian optimization method with the Broyden–Fletcher–Goldfarb–Shanno scheme [18]) is chosen to solve the approximation problem.

5. ELECTRODYNAMIC MODEL

The EMF generation process is described by a complete system of nonstationary Maxwell equations:

$$\begin{gathered}
\operatorname{rot}\mathbf{H} = \varepsilon\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}(\sigma\mathbf{E} + \mathbf{j}), \\
\operatorname{rot}\mathbf{E} = -\mu\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \\
\frac{\partial \rho}{\partial t} + \operatorname{div}(\sigma\mathbf{E} + \mathbf{j}) = 0, \\
\mathbf{j} = \varphi(t)(\delta(z),0,0),\ \mathbf{E}|_{t=0} = \mathbf{H}|_{t=0} = \rho|_{t=0} = 0.
\end{gathered} \tag{1}$$

In (1):

$\rho = \rho(t,\mathbf{r})$ – electric charge density,

$\mathbf{j} = \mathbf{j}(t,\mathbf{r})$ – electric current density,

$\varepsilon = \varepsilon(\mathbf{r})$ – dielectric constant of material,

$\mu = \mu(\mathbf{r})$ – magnetic constant of material,

$\sigma = \sigma(\mathbf{r})$ – conductivity of materials.

Let's consider equations (1) in a bounded domain $\Omega = \{x,y,z : x\in[x_{\min},x_{\max}], y\in[y_{\min},y_{\max}], z\in[z_{\min},z_{\max}]\}$ with a boundary $\partial\Omega$. The condition $\iint_{\partial\Omega} \langle [\mathbf{E},\mathbf{H}] \cdot \mathbf{n} \rangle ds..0$, is considered fulfill at the boundary $\partial\Omega$ of the region Ω (**n** is a unit vector in the direction of the external normal to the surface $\partial\Omega$).

The equality of the tangential components of the electric and magnetic fields at the boundary is used as approximate boundary conditions.

$$
\begin{gathered}
E^y = H^z \quad E^z = -H^y \text{ at } x = x_{\max}, \quad E^y = -H^z \quad E^z = H^y \quad \text{at } x = x_{\min}, \\
E^z = H^x \quad E^x = -H^z \quad \text{at } y = y_{\max}, \quad E^z = -H^x \quad E^x = H^z \quad \text{at } y = y_{\min}, \\
E^x = H^y \quad E^y = -H^x \quad \text{at } z = z_{\max}, \quad E^x = -H^y \quad E^y = H^x \quad \text{at } z = z_{\min}, \\
t \ge 0, \mathbf{r} \in \Omega .
\end{gathered}
\tag{2}
$$

The conditions (2) ensure the coincidence of the directions of the Poynting vector and the external normal to the boundary, allowing the outflow of electromagnetic energy from the area and prohibiting the inflow. The boundary conditions (2) are called "radiation conditions".

6. NUMERICAL ALGORITHM FOR SOLVING MAXWELL'S EQUATIONS

The algorithm for the numerical solution of the initial boundary value problem for system (1) with the boundary conditions (2) is based on the difference scheme presented in [19, 20]. It has proven itself well in solving a large number of different electrodynamic problems [20].

Let's consider Maxwell's equations (1) in the Cartesian coordinate system $\mathbf{r} = (x, y, z)$:

$$\partial_y H^z - \partial_z H^y = \varepsilon \dot{E}^x + I^x, \tag{3}$$

$$\partial_z H^x - \partial_x H^z = \varepsilon \dot{E}^y + I^y, \tag{4}$$

$$\partial_x H^y - \partial_y H^x = \varepsilon \dot{E}^z + I^z, \tag{5}$$

$$\partial_z E^y - \partial_y E^z = \mu \dot{H}^x, \tag{6}$$

$$\partial_x E^z - \partial_z E^x = \mu \dot{H}^y, \tag{7}$$

$$\partial_y E^x - \partial_x E^y = \mu \dot{H}^z, \tag{8}$$

the symbols ∂_x, ∂_y, ∂_z mean partial derivatives with respect to the corresponding coordinates, the dot above the function denotes its partial derivative with respect to the variable $\xi \equiv ct$, $\{\xi : \xi \in [0, \xi_{\max}]\}$, $\mathbf{I} \equiv (4\pi / c)(\sigma\mathbf{E} + \mathbf{j})$.

The difference grid for the variable *x* is as follows:

$x_{i+1} = x_i + \Delta_i; \quad i = 0,..., N_x - 1, \quad x_0 = x_{\min}, \quad x_{N_x} = x_{\max};$

$x_{i+1/2} = (x_i + x_{i+1}) / 2; \quad i = 0,..., N_x - 1, \quad x_{-1/2} = x_0, \quad x_{N_x+1/2} = x_{N_x};$

$\delta_i = x_{i+1/2} - x_{i-1/2}; \quad i = 0,..., N_x, \quad \delta_0 = \Delta_0 / 2, \quad \delta_{N_x} = \Delta_{N_x-1} / 2.$

For variables (y, z), the difference grid is introduced in the same way.

We choose a grid so that the discontinuities of the coefficients of the equations are located on the surfaces $x = x_i$, $y = y_j$, and $z = z_k$. The coefficient values are set at grid points with fractional indexes. These points coincide with the centers of rectangular parallelepipeds formed by the intersection of the planes $x = x_i, x_{i+1}$, $y = y_j, y_{j+1}$, and $z = z_k, z_{k+1}$. All the coefficients of the system, current densities and components of the electromagnetic field are continuous inside these parallelepipeds.

The component of the electric field normal to the rupture surface ε suffers a rupture when passing through this surface. The components of the electric field tangent to this surface are continuous in this case. Therefore the grid components of the electric field E^x, E^y and E^z are defined in the middle of the corresponding edges of these rectangular parallelepipeds.

The components of the magnetic field are placed in the centers of the faces of the parallelepipeds.

Finite-difference analogs of equations (3-8) are given in [20].

A nonuniform time grid is introduced to construct a difference approximation of Maxwell's equations in time on the interval $t \in [t_{\min}; t_{\max}]$:

$$t_{n+1} = t_n + \Delta t_n; \quad n = 0,\dots,N_t - 1, \quad t_0 = t_{\min}, \quad t_{N_t} = t_{\max},$$
$$t_{n+1/2} = (t_n + t_{n+1})/2; \quad n = 0,\dots,N_t - 1,$$
$$\delta t_n = t_{n+1/2} - t_{n-1/2}; \quad n = 2,\dots,N_t - 1.$$

The components of the electric field (E^x, E^y, E^z) are given at integer moments of time t_n. The components of the magnetic field (H^x, H^y, H^z) and the electric current density (j^x, j^y, j^z) are in half-integer moments of time $t_{n+1/2}$.

The time derivatives are approximated by explicit difference equations [20]:

$$\partial_t E^x = \frac{E^x_{i+1/2,j,k,n+1} - E^x_{i+1/2,j,k,n}}{\Delta t_n}; \quad \partial_t H^x = \frac{H^x_{i,j+1/2,k+1/2,n+3/2} - H^x_{i,j+1/2,k+1/2,n+1/2}}{\delta t_{n+1}};$$
$$\partial_t E^y = \frac{E^y_{i,j+1/2,k,n+1} - E^y_{i,j+1/2,k,n}}{\Delta t}; \quad \partial_t H^y = \frac{H^y_{i+1/2,j,k+1/2,n+3/2} - H^y_{i+1/2,j,k+1/2,n+1/2}}{\delta t_{n+1}};$$
$$\partial_t E^z = \frac{E^z_{i,j,k+1/2,n+1} - E^z_{i,j,k+1/2,n}}{\Delta t_n}; \quad \partial_t H^z = \frac{H^z_{i+1/2,j+1/2,k,n+3/2} - H^z_{i+1/2,j+1/2,k,n+1/2}}{\delta t_{n+1}}.$$

The constructed numerical algorithm is implemented in the form of a software module focused on multiprocessor computing equipment [20] using MPI parallelization technology.

7. RESULTS OF MODELING OF THE RADIATION ELECTROMAGNETIC FIELD

Some results of modeling the radiation electromagnetic field in a heterogeneous fine-dispersed medium are presented in this section.

The object of irradiation is a fragment of a binder (polybutadiene) and dielectric inclusions (ammonium perchlorate) shown in Fig. 1.

The power of the gamma radiation source is described by the function

$$J = N_0 f(t),\ \ f(t) = \frac{2}{t_0}\left(1 - \frac{t}{t_0}\right),\ \ t_0 = 2\cdot 10^{-8} c;\quad \int_0^{t_0} f(\tau)\,d\tau = 1\,.$$

N_0 is selected in such a way that the amplitude of the electric field is of the order of 1 CGSE.

The radiation conductivity of the materials was determined by the formulas [12]:
$\sigma = 1.8\cdot 10^{-7} D$ for the binder and $\sigma = 1.8\cdot 10^{-6} D$ for inclusions, D is the radiation dose rate in rad/s [21].

Two-dimensional distributions of the electric field amplitude in planes orthogonal to the coordinate axes are presented in Fig. 7-10 (colormap is “jet”).

Values of fields are in CGSE.

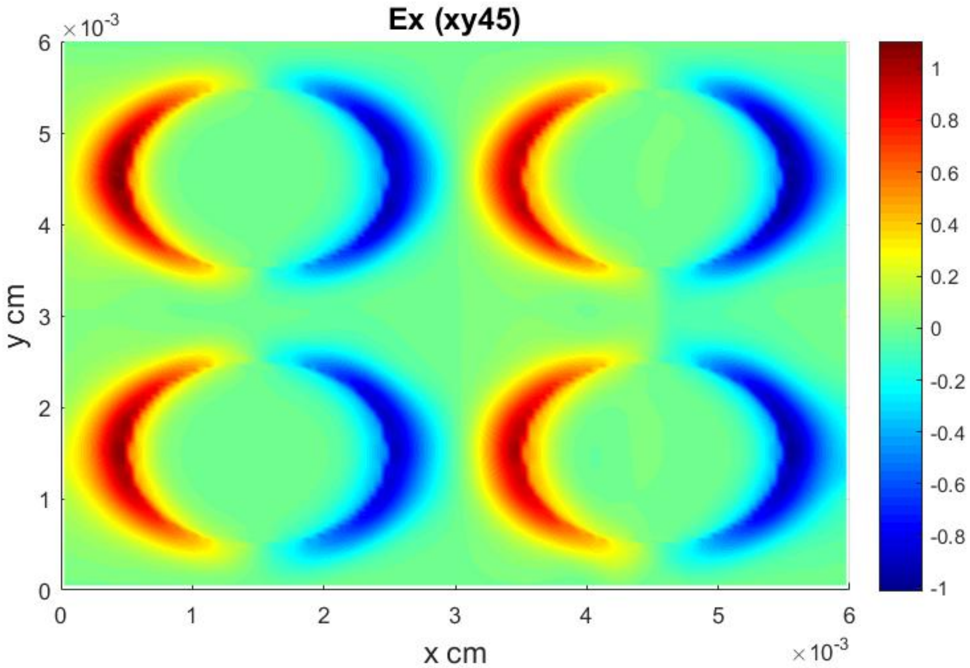

Figure 7: E_x, z=0.0045 cm

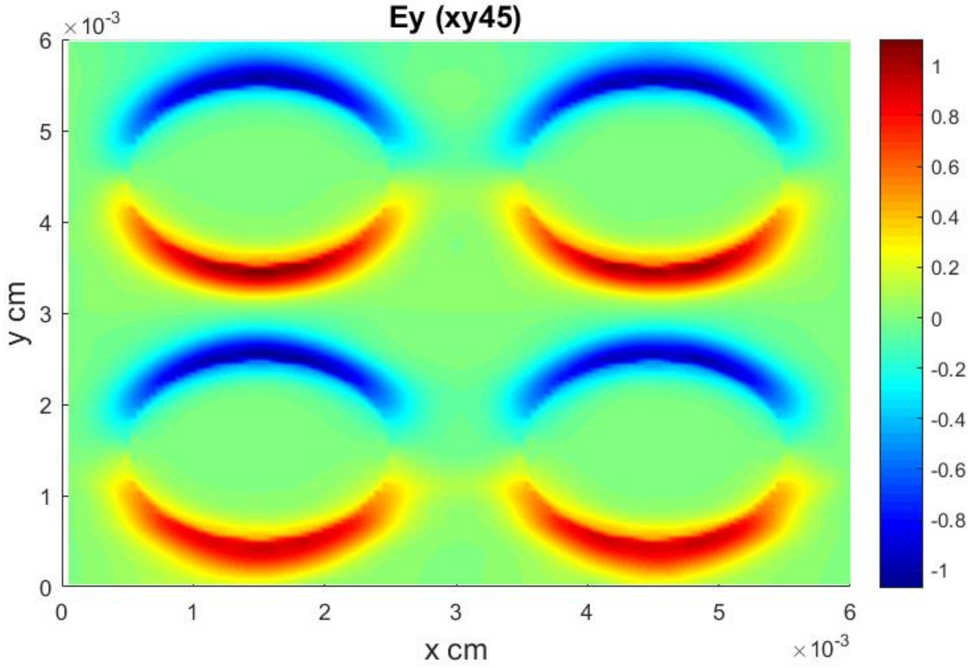

Figure 8: *Ey*, *z*=0.0045 cm

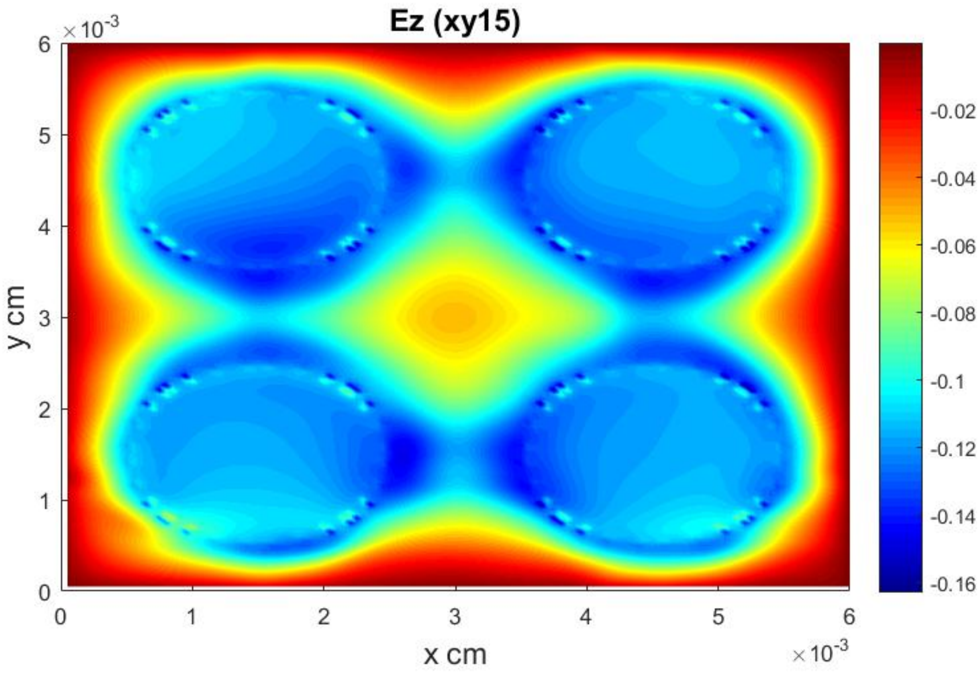

Figure 9: *Ez*, *z*=0.0015 cm

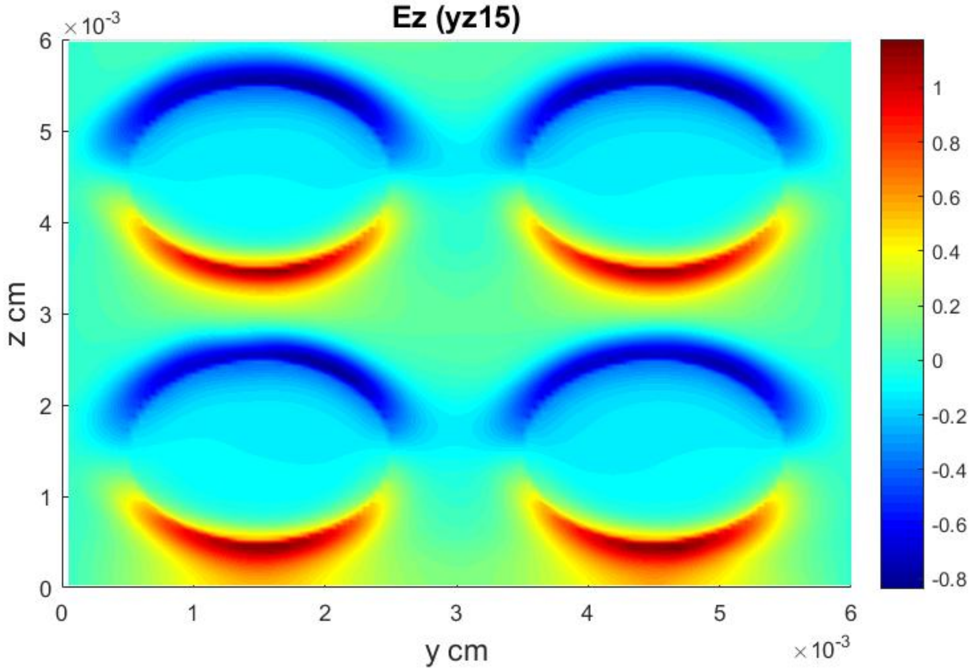

Figure 10: *Ez, x*=0.0015 cm

Two-dimensional spatial distributions of the amplitudes of the transverse (relative to the direction of the photon flux) components of the magnetic field in the plane z=0.0045 cm are presented in Fig. 11, 12.

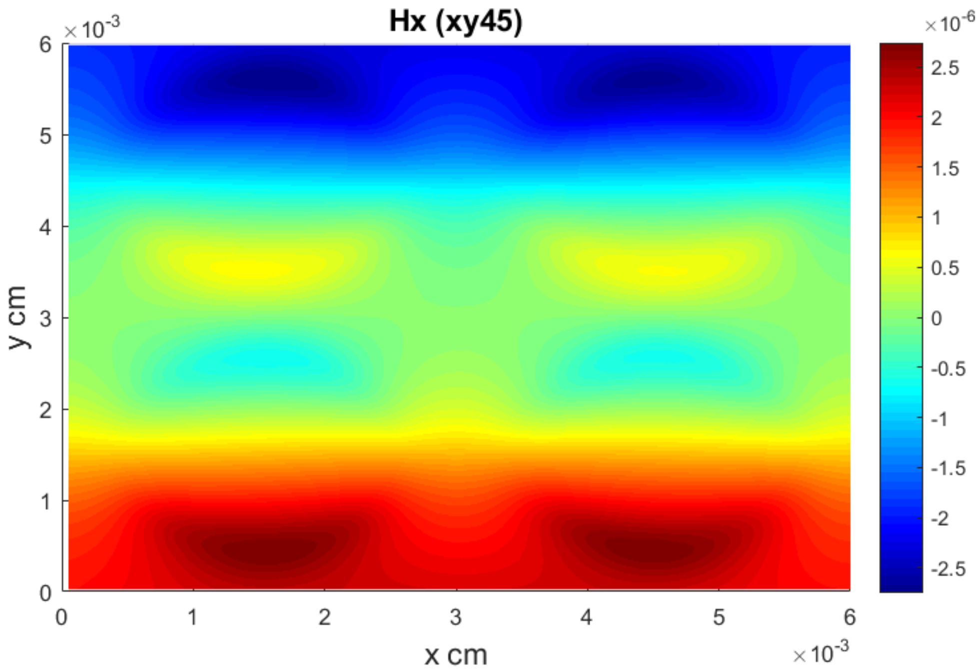

Figure 11: *Hx, z*=0.0045 cm

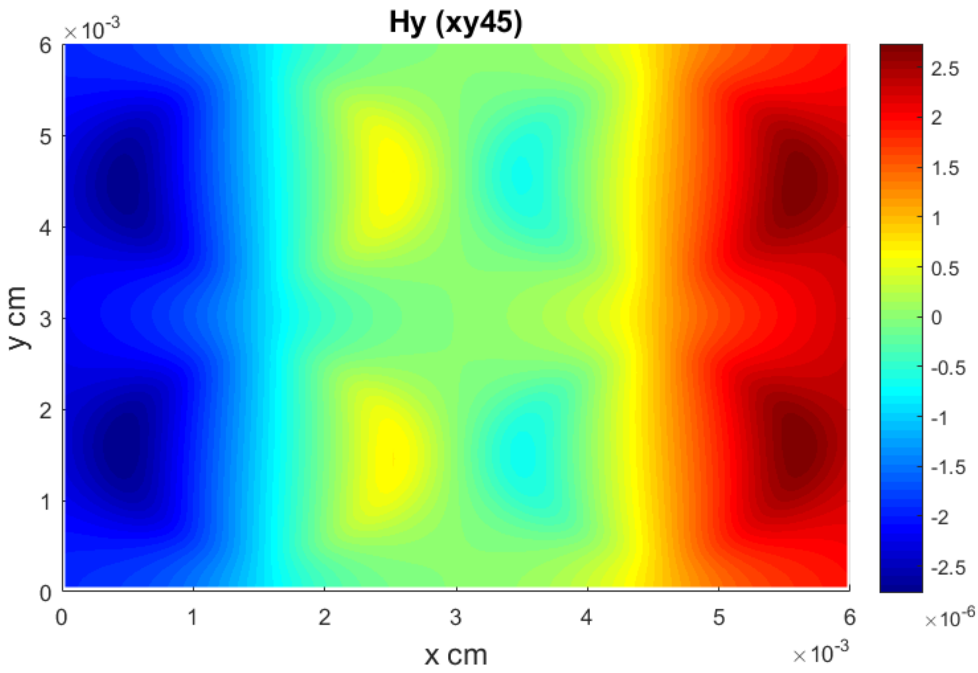

Figure 12: *Hy*, *z*=0.0045 cm

The above illustrations show that when the fragment under study is irradiated, electric fields with a sharply inhomogeneous spatial structure are generated, and inhomogeneities occur near the boundar. The generated magnetic field is negligible compared to the electric one.

8. CONCLUSIONS

Algorithms for supercomputing the radiation electromagnetic field in heterogeneous fine-dispersed materials with direct consideration of their microstructure have been developed. The modeling includes a statistical evaluation of radiation energy deposit and electric currents, an approximation of the results from the detector system to the difference electrodynamic grid, and a numerical solution of the complete system of Maxwell's equations.

A geometric model of a heterogeneous fine-dispersed medium is constructed. The model includes the detecting system for statistical estimation of functionals on the space of solutions of the photon-electron cascade transport equations.

A mathematical three-dimensional simulation of the radiation EMF in a fragment of an object of a dispersed structure was carried out. The results of modeling show that electromagnetic fields with a sharply inhomogeneous spatial structure are formed and inhomogeneities of the fields occur near the boundary surfaces of the binder and inclusions.

The generation of EMF at the boundary of two media is due to the predominance of electron emission from the inclusion (a material with a large macroscopic cross-section of photons) into the binder (a material with a greater penetrating ability of electrons) over the emission in the opposite direction.

REFERENCES

[1] R. Fisher, F. W. Smith and K. Cho, "EMP Coupling to Canonical Models Consisting of Dielectric and Metal Sections with Emphasis toward Simple Missile Models," *IEEE Transactions on Nuclear Science*, **28** (6), 4490-4494, (1981), doi: 10.1109/TNS.1981.4335752.

[2] V. I. Mazhukin, M. M. Demin and A. V. Shapranov, “High-speed Laser Ablation of Metal with Pico- and Subpicosecond Pulses”, *ASS*, **302**, 6–10 (2014).

[3] P. A. Childs, D. W. Dyke, “Analytic expressions for impact ionization rates and secondary particle energy distributions in semiconductors”, *Appl. Phys. Lett.*, **74**, 2646 (1999); https://doi.org/10.1063/1.123925.

[4] M. E. Zhukovskiy, M. B. Markov, R. V. Uskov, “Modeling of radiation-induced electric current in the materials of finely dispersed structure”, *Math. Montis.*, **47**, 65-74 (2020).

[5] A. V. Berezin, Yu. A. Volkov, M. B. Markov, I. A. Tarakanov, “The radiation-induced conductivity of silicon”. *Math. Montis.*, **33**, 69-87 (2015).

[6] F.N. Voronin, K.K. Inozemtseva, M.B. Markov, “Electromagnetic and Thermomechanical Effect Produced by an Electronic Beam on a Solid Barrier”, *Math. Models Comput. Simul.*, **10,** 407–417 (2018). Doi.org/10.1134/S2070048218040154.

[7] M.E. Zhukovskiy, R.V. Uskov, E.B. Savenkov, M. V. Alekseev, M. B. Markov, F. N. Voronin, “Model for the Radiation Transport in the Matter of Porous-Type Heterogeneous Materials”, *Math. Models Comput. Simul.*, **11,** 415–425 (2019). https://doi.org/10.1134/S2070048219030177.

[8] V. A. Egorova, F. N. Voronin, M. E. Zhukovskiy, M. B. Markov, A. I. Potapenko, R. V. Uskov, D. S. Boykov, “Model of Radiation-Induced Thermomechanical Effects in Heterogeneous Finely Dispersed Materials”, *Math. Models Comput. Simul.*, **12,** 729–739 (2020). https://doi.org/10.1134/S2070048220050063.

[9] M. E. Zhukovskiy, M. B. Markov, R. V. Uskov, “Modeling of radiation-induced electric current in the materials of finely dispersed structure”, *Math. Montis.*, **47**, 65-74 (2020).

[10] . M.V. Alekseev, V.A. Egorova, M.E. Zhukovskiy, D.N. Sadovnichiy, I.A. Tarakanov, R.V. Uskov, “On modeling of the radiation-induced charge effects in the fine dispersed materials of closed-cell structure”, *Keldysh Institute preprints*, 80, 15 pp. (2019)

[11] F.H. Stillinger, B.D. Lubachevsky, “Crystalline-Amorphous Interface Packings for Disks and Spheres”, *J. Stat. Phys*., 73(3-4), 497-514 (1993).

[12] B.D. Lubachevsky, F.H. Stillinger, “Geometric properties of random disk packings”, *J. Stat. Phys.*, **60**, 561-583 (1990).

[13] B.D. Lubachevsky, “How to Simulate Billiards and Similar Systems”, *J. Comput. Phys.*, **94**(2), 255-283 (1991).

[14] M.E. Zhukovskiy, R.V. Uskov, “Hybrid parallelization of the algorithms of radiation cascade transport modeling”, *Math. Models Comput. Simul.*, **7,** 601–610 (2015). https://doi.org/10.1134/S2070048215060101

[15] V.A. Egorova, F.N. Voronin, M.E. Zhukovskiy, *et al.*, “Model of Radiation-Induced Thermomechanical Effects in Heterogeneous Finely Dispersed Materials”, *Math. Models Comput. Simul.*, **12,** 729–739 (2020). https://doi.org/10.1134/S2070048220050063.

[16] T. Mitchell, *Machine Learning*, McGraw-Hill Science/Engineering/Math, (1997).

[17] Simon Haykin, *Neural networks. A comprehensive Foundation. Second edition*, Prentice hall, Upper Saddle River, New Jersey 07458, (1999).

[18] J. Nocedal, S.J. Wright, *Numerical Optimization*. *2nd edition*. USA: Springer, (2006). ISBN 978-0-387-30303-1.

[19] A.V. Berezin, N.S. Kellin, M.B. Markov, S.V. Parot'kin, A.V. Sysenko, “The radiative electromagnetic field in the object with the boundary of ideal conductivity”, *Matem. Mod.*, **16**(3), 3–12 (2004) http://mi.mathnet.ru/eng/mm/v16/i3/p3

[20] A.N. Andrianov, A.V. Berezin, A.S. Vorontsov, K.N. Efimkin, M.B. Markov, “The radiational electromagnetic fields modeling at the multiprocessor computing systems”, *Matem. Mod.*, **20**(3), 98–114 (2008), http://mi.mathnet.ru/eng/mm/v20/i3/p98

[21] I.P. Bezrodnykh, A.P. Tiutnev, V.T. Semenov, Radiatsionnye effecty v kosmose. Ch. 2. Vozdeistvie kosmicheskoi radiatsii na elektrotekhnicheskie materialy, M.: AO “Corporatsiia “VNIIEM” (2016).

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УСТОЙЧИВОСТЬ ТЕЧЕНИЯ И КОРРЕКТНОСТЬ ЗАДАЧИ КОШИ ДЛЯ МОДЕЛИ ДВУХСКОРОСТНОЙ СРЕДЫ С РАЗНЫМИ ДАВЛЕНИЯМИ ФАЗ

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Ключевые слова: Математическое моделирование, дисперсная смесь, двухскоростное течение, два давления.

Аннотация. В настоящее время для описания двухскоростного течения дисперсной смеси, как правило, применяют двух жидкостную модель с равным давлением фаз среды, и разными скоростями фаз. Соответствующая система уравнений без специальных, постулируемых, стабилизирующих слагаемых негиперболична. Это может приводить к сложностям в поиске решения.

В последнее время предлагается шире использовать аналогичные модели, но с различными давлениями фаз среды. Такие модели позволяют учитывать новые физические эффекты, связанные с разными давлениями фаз, и часто обеспечивают гиперболичность соответствующей системы уравнений. В данной статье анализируется влияние различности давления фаз среды на свойства системы: исследуется важность соответствующих новых эффектов, гиперболичность системы уравнений, устойчивость её стационарных решений и корректность соответствующей задачи Коши. Рассмотрены три системы. За основу первой, простейшей модельной системы взята хорошо известная негиперболическая система, которая была модернизирована. Показано, что формально задача Коши для модифицированной системы корректна, но практическая возможность использования результатов расчётов, полученных из решения этой системы, должна исследоваться в каждом конкретном случае, и зависит от расчетного шага и длительности изучаемого процесса. Методики, отработанные для решения первой простейшей системы, были использованы для других систем. В качестве второй системы рассмотрена модель течения двухфазной среды с разными давлениями фаз и двумя уравнениями импульса. Будем предполагать фазы баротропными. Постулируем уравнение, связывающее давление в фазах. Доказано, что эта система всегда гиперболическая. Исследована устойчивость её стационарных решений. Выведены соотношения, позволяющие определять в каких условиях из-за неустойчивости полученные решения недостоверные. Проведено сравнение свойств этой системы с системой двухскоростного течения дисперсной смеси с равным давлением фаз среды. В качестве третьей системы рассмотрена модель с двумя давлениями, описывающая пульсации пузырьков. Будем предполагать фазы баротропными. Определены условия, когда система негиперболическая, а задача Коши некорректна. Исследовано, для каких условий некорректность задачи Коши приводит к недостоверности решения, а при каких условий некорректность задачи Коши не приводит к недостоверности решения.

2010 Mathematics Subject Classification: 76T10, 76T15, 76T20.
Key words and Phrases: Mathematical Modeling, Dispersed Mixture, Two-speed Flow, Two Pressures.

FLOW STABILITY AND CORRECTNESS OF THE CAUCHYPROBLEM FOR A TWO-SPEED MEDIUM MODEL WITH DIFFERENT PHASE PRESSURES

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Summary. At present, to describe the two-velocity flow of a dispersed mixture, as a rule, a two-fluid model is used with equal pressure of the phases of the medium and different velocities of the phases. The corresponding system of equations without special, postulated, stabilizing terms is non-hyperbolic. This can lead to difficulties in finding a solution.

Recently, it has been proposed to use similar models more widely, but with different pressures of the phases of the medium. Such models allow one to take into account new physical effects associated with different phase pressures and often provide hyperbolicity of the corresponding system of equations. This article analyzes the influence of the difference in the pressure of the phases of the medium on the properties of the system: the importance of the corresponding new effects, the hyperbolicity of the system of equations, the stability of its stationary solutions, and the correctness of the corresponding Cauchy problem are investigated. Three systems are considered. The first, simplest model system is based on the well-known non-hyperbolic system, which has been modernized. It is shown that the Cauchy problem for the modified system is formally correct, but the practical possibility of using the calculation results obtained from the solution of this system should be investigated in each specific case, and depends on the calculated step and duration of the process under study. The techniques worked out to solve the first simplest system were used for other systems. As the second system, a model of the flow of a two-phase medium with different phase pressures and two momentum equations is considered. We will assume the phases are barotropic. Let us postulate an equation relating the pressure in the phases. It is proved that this system is always hyperbolic. The stability of its stationary solutions is investigated. Relationships are derived that make it possible to determine under what conditions, due to instability, the obtained solutions are unreliable. The properties of this system are compared with the system of two-speed flow of a dispersed mixture with equal pressure of the phases of the medium. As a third system, a two-pressure model describing bubble pulsations is considered. We will assume the phases are barotropic. Conditions are determined when the system is non-hyperbolic and the Cauchy problem is incorrect. It is investigated for what conditions the ill-posedness of the Cauchy problem leads to the unreliability of the solution, and under what conditions the ill-posedness of the Cauchy problem does not lead to the unreliability of the solution.

1 ВВЕДЕНИЕ

Широко применяемая для описания дисперсной смеси двух жидкостную модель с равным давлением фаз среды, и разными скоростями фаз была опубликована в [1] и было отмечено, что эта система уравнений негиперболична. Хорошо известны среды со сложной структурой (не дисперсные среды) в которых учитывается отличие давлений фаз [2]. До недавнего времени считалось, что в дисперсных средах различие давлений быстро приводит к сжатию или расширению фаз и давления, становятся равными, поэтому давления математических моделях предполагались равными [2]. Задача о пульсации пузырьков в пузырьковой смеси, описываемая уравнением Рэлея-Ламба, вероятно, единственное, хорошо известное исключение из этого правила [2]. Негиперболичность не является физической, а является «дефектом» математической модели, приводящей в определённых условиях к неограниченному росту объёмной концентрации фазы включений. В реальной среде значение объёмной концентрации фазы включений не может больше значения, соответствующего плотной упаковке, если не учитывать деформацию формы включений. При приближении значения объёмной концентрации к значению, соответствующему плотной упаковке, возникают силовые взаимодействия непосредственно между включениями, что не позволяет включениям приблизиться друг к другу ближе, чем на два радиуса. В экспериментах, как правило, не происходит сближения включений до состояния плотной упаковки, следовательно, в межфазной силе есть слагаемые, препятствующие сближению включений. Именно отсутствие этих слагаемых в математической модели [1] и приводит к негиперболичности системы и некорректности задачи Коши. В работе [3] была исследована некорректность соответствующей задачи Коши, и предложены пути устранения этого недостатка, формально, делающего недостоверными полученные из решения этих уравнений результаты. В настоящее время существуют два пути решения этой проблемы [4]. Первый – введение дополнительных, стабилизирующих слагаемых в уравнения импульса, подробная реализация этого подхода с обзором литературы дана в [5] и были установлены основные закономерности. В этой работе доказано, что недостаточно обеспечить гиперболичность системы, необходимо, чтобы стационарные решения были устойчивы; определены условия, когда неустойчивость решения приводит к невозможности получить достоверное решение, причем неустойчивость решения не является физической, а возникает из-за «дефекта» системы. Недостатком этого подхода является невозможность прямого получения стабилизирующего слагаемого, например, из эксперимента или путём осреднения микроуравнений. Другой подход заключается в учёте отличия давлений фаз. В последнее время именно этот подход активно развивается [6]. Этому подходу и посвящена данная работа.

Для исследования влияния различности давления фаз среды на свойства системы (гиперболичность системы уравнений, устойчивость её стационарных решений и корректность соответствующей задачи Коши) необходимо проанализировать исходную систему дифференциальных уравнений, состоящую из двух уравнений сохранения массы, двух уравнений сохранения энергии, двух уравнений сохранения энергии фаз и замыкающего уравнения связывающего давления фаз. Используя результаты полученные в [7], удаётся показать, что комплексные собственные значения, которые определяют негиперболичность уравнений, не связаны со сжимаемостью фаз, что позволяет исследовать эту проблему в простейшем случае двухскоростного течения баротропных фаз. Это существенно упрощает задачу. Из системы исключаются два уравнения энергии фаз.

2 ГИПЕРБОЛИЧНОСТЬ И УСТОЙЧИВОСТЬ МОДЕЛЬНОЙ ЗАДАЧИ

Модель двух жидкостной гидродинамики достаточно сложная для теоретического анализа. Поэтому в качестве первого шага рассмотрим простейшую тестовую задачу, для которой имеется аналитическое решение, что позволяет исследовать основные закономерности. Одна из простейших не гиперболических систем уравнений – стационарное уравнение теплопроводности в плоскости x,y (уравнение Лапласа)

$$\frac{\partial^2 \mathrm{T}}{(\partial \mathrm{x})^2}+\frac{\partial^2 \mathrm{T}}{(\partial \mathrm{y})^2}=0$$

где T–температура.

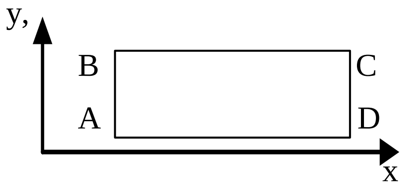

Рис. 1 Область определения температуры

Будем определять температуру в прямоугольной области, показанной на рис. 1. Задавая произвольные значения на границе ABCD, будут однозначно определены значения температуры внутри области. Формально заменяем переменную y на t. Стандартным способом перейдем от приведенного выше уравнения с заменой y на t к системе двух уравнений первого порядка, вводя переменные:

$$y_1=\frac{\partial T}{\partial x} \text{ и } y_2=\frac{\partial T}{\partial t} \tag{1}$$

$$\frac{\partial \mathrm{y}_1}{\partial \mathrm{t}}-\frac{\partial \mathrm{y}_2}{\partial \mathrm{x}}=0$$

$$\frac{\partial \mathrm{y}_2}{\partial \mathrm{t}}+\frac{\partial \mathrm{y}_1}{\partial \mathrm{x}}=0.$$

Для этой системы уравнений можно пытаться решать задачу Коши. Легко показать, что характеристики этой системы равны

$$\lambda_{1,2}=\pm i$$

Т.е. это не гиперболическая система, корни характеристического уравнения комплексно-сопряжённые, задача Коши для системы не корректна.

Разберём некорректности задачи Коши в терминах анализа устойчивости её решений. Будем искать решения в виде:

$$(\mathrm{y})=(\mathrm{y})_0+(\mathrm{A_y})e^{i(kx-\omega t)}$$

Легко показать, что условием существования таких решений будет выполнение следующего дисперсионного соотношения:

$$\mathrm{\omega}^2 + \mathrm{k}^2 = 0. \tag{2}$$

У этого уравнения есть следующие решения:

$$\mathrm{\omega}_{1,2} = \pm \mathrm{ik}$$

Видно, что малые возмущения экспоненциально растут, причем в пределе коротких волн, скорость их роста стремится к бесконечности. Т.е. любые малые возмущения за любое время бесконечно возрастают…, т.е. такие решения «физически» бессмысленны, что и дает «физический» смысл некорректности задачи Коши.

Часто встречается мнение, что такие задачи можно, в определённых условиях, численно решать, так как при численном решении максимально возможное волновое число k определяется размером расчетной сетки, а значит, при не слишком мелкой расчетной сетке и не слишком продолжительном времени решения задачи, ошибка, вызванная неточностью задания начальных условий, будет не велика. Справедливость такого мнения не очевидна, так как устойчивость численного расчёта подразумевает неизменность результатов при стремлении размера расчетной сетки к нулю, а в данной задаче это не так.

Теперь подправим систему (1), добавив еще одно уравнение:

$$\frac{\partial \mathrm{y}_1}{\partial \mathrm{t}} - \frac{\partial \mathrm{y}_2}{\partial \mathrm{x}} = 0, \tag{3}$$

$$\frac{\partial y_2}{\partial t} - m^2 \frac{\partial y_1}{\partial x} + (1 + m^2) \frac{\partial y_3}{\partial x} = 0,$$

$$\frac{\partial y_3}{\partial t} + \frac{1}{t_0}(y_3 - y_1) = 0,$$

где m–положительная константа

Третье уравнение, означает, что добавочная переменная y_3 за характерное время t_0 стремится к переменной y_1. Понятно, что в случае t_0 стремящемся к нулю, решения систем (1) и (3) должны совпадать. Тем не менее характеристики системы (3) равны

$$\lambda_1 = 0;\ \lambda_{2,3} = \pm m$$

Видно, что эта система гиперболична, и её собственные значения не зависят от t_0 и значит, формально, задача Коши для неё корректна.

Если же мы посмотрим устойчивость её решений, то аналогичное (2) дисперсионное уравнение имеет вид:

$$\mathrm{\omega}^3 + \mathrm{\omega}^2 \mathrm{i}\frac{1}{t_0} - \mathrm{\omega k}^2 m^2 + \mathrm{k}^2 \mathrm{i}\frac{1}{t_0} = 0. \tag{4}$$

Это уравнение 3 порядка можно приблизительно решать в предельных случаях. В случае t_0 стремящегося к нулю его решения имеют вид:

$$\mathrm{\omega}_1 = -\mathrm{i}\frac{1}{t_0};\ \mathrm{\omega}_{2,3} = \pm \mathrm{ik}. \tag{5}$$

Видно, что при маленьких, решения не устойчивы, и пропорциональны k - также, как и решения системы (1)

В предельном случае k стремящемся к бесконечности, решения имеют вид:

$$\omega_1 = \mathrm{i}\frac{1}{t_0}\frac{1}{m^2};\ \omega_{2,3} = \pm km \quad (6)$$

Т.е. одно решение растет, но с конечной скоростью, а другие 2 решения соответствуют не растущим волнам. Т.е. скорость роста неустойчивого решения (5), которая пропорциональна k, а при больших k, она постоянна (6) (не зависит от k).

Приближенное решение (5) справедливо до тех пор, пока первое слагаемое в уравнении (4) много меньше второго слагаемого в этом уравнении, т.е. $\omega \ll \frac{\mathrm{i}}{t_0}$. А в случае выполнения условия $\omega \gg \frac{\mathrm{i}}{t_0}$ справедливо решение (5).

Особо исследуем общий случай (не придельный). Для простоты считаем $\mathrm{m} = 1$.
Рассмотрим t_0 фиксировано, а k меняется. Введем безразмерную величину $\overline{\omega} = \omega * t_0$.
Уравнение (4) перепишем в виде В предельном случае k стремящемся к бесконечности, решения имеют вид:

$$\overline{\omega}^3 + \overline{\omega}^2\mathrm{i} - \overline{\omega}(\mathrm{k}t_0)^2 + \mathrm{i}(\mathrm{k}t_0)^2 = 0.$$

Решение выписанного выше уравнения находим по формуле Кардано [8]. На Рис. 2 показано, как меняется мнимая часть этого решения.

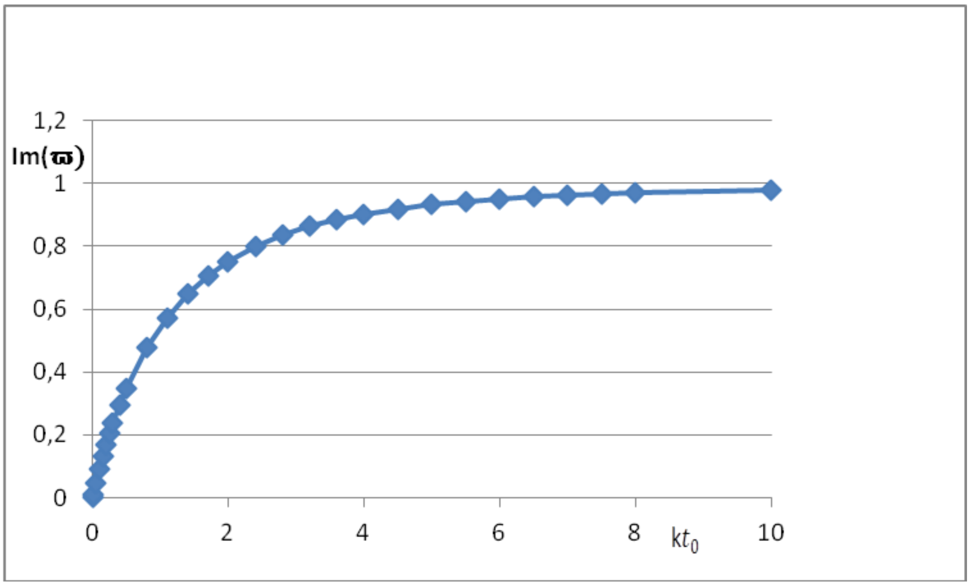

Рис. 2 Зависимость мнимой части частоты от волнового числа

Из рисунка видно, что $\mathrm{Im}\,(\overline{\omega})$ в начале линейно растет, что соответствует неустойчивости - решение (5), а затем, когда начинают преобладать слагаемые k, с $\mathrm{Im}\,(\overline{\omega})$, выходит на постоянное значение, соответствующее решению (6), что и обеспечивает формальную корректность задачи Коши.

Однако, при попытке практического решения задачи Коши, например, при t_0=0,01с, при использовании аналитических решений системы (4) погрешности задания начальных условий при $kt_0 >\ 8$ за десять секунд увеличиваются примерно в 10^{500} раз, что делает решения этой системы «физически» бессмысленными, хотя, формально, система (3) и гиперболична и задача Коши – корректна.

Достоверность (и возможность практического использования) численных решений системы (3) зависит от конкретных условий и шага расчётной сетки (который определяет максимальное значение k) и может не совпадать с приведенной выше оценкой. Например, если t_0 =0,01с, и нужно с шагом 0,1 м просчитать процесс длительностью 10 с., то максимальное значение k порядка 10. Следовательно, применимо приближение (4), и значит, погрешность начальных данных на конец расчета возрастёт примерно в 10^{50} раз, т.е. результаты таких расчётов также не имеют ни какого «физического» смысла.

Если, же, например, с шагом 10^{-3} м, нужно просчитать 10^{-2} с, то максимальное значение k порядка 10^3, и применимо приближение (5). Погрешность начальных данных в конце такого расчета возрастёт примерно в 3 раза, в этом случае, часто считается, что результаты таких расчётов имеют физический смысл.

Вывод: Таким образом, формально задача Коши для модифицированной системы (3) корректна, но практическая возможность использования результатов расчётов, полученных из решения этой системы должна исследоваться в каждом конкретном случае и зависит от расчетного шага и длительности изучаемого процесса.

3 ХАРАКТЕРИСТИКИ СИСТЕМ УРАВНЕНИЙ МОДЕЛЕЙ С ОДИНАКОВЫМИ И РАЗНЫМИ ДАВЛЕНИЯМИ ФАЗ

Выпишем и проанализируем приведенную в [6] модель течения двухфазной среды с разными давлениями фаз и двумя уравнениями импульса. Будем предполагать фазы баротропными: $\rho_i^0 = \rho_i^0(P_i)$, индекс i внизу показывает к какой фазе относится соответствующая величина: i=1 – жидкость, i=2– газ, где ρ_i^0 – истинная плотность i-ой фазы, P_i- давление i-ой фазы, тогда $c_i = \sqrt{\frac{dP_i}{d\rho_i^0}}$, где c_i - скорость звука i-ой фазы.

В этом случае тепловые параметры среды: температура, энергия – не входят в уравнения неразрывности и импульса, а входят только в уравнения энергии и, следовательно, для анализа системы можно не привлекать уравнения энергии.

Такая модель включает два уравнения неразрывности:

$$\frac{\partial}{\partial t}(\alpha_1\rho_1^0) + \frac{\partial}{\partial x}(\alpha_1\rho_1^0\vartheta_1) = 0, \qquad (7)$$

$$\frac{\partial}{\partial t}(\alpha_2\rho_2^0) + \frac{\partial}{\partial x}(\alpha_2\rho_2^0\vartheta_2) = 0,$$

два уравнения импульса

$$\frac{\partial}{\partial \mathrm{t}}(\alpha_1\rho_1^0\vartheta_1) + \frac{\partial}{\partial \mathrm{x}}[\alpha_1(\rho_1^0\vartheta_1^2 + \mathrm{P}_1)] = \mathrm{P}_{\mathrm{int}}\frac{\partial\alpha_1}{\partial x} + \mathrm{F}_{\mathrm{m}1} - \mathrm{F}_{12}, \qquad (8)$$

$$\frac{\partial}{\partial \mathrm{t}}(\alpha_2\rho_2^0\vartheta_2) + \frac{\partial}{\partial \mathrm{x}}[\alpha_2(\rho_2^0\vartheta_2^2 + \mathrm{P}_2)] = \mathrm{P}_{\mathrm{int}}\frac{\partial\alpha_2}{\partial x} + \mathrm{F}_{\mathrm{m}2} + \mathrm{F}_{12},$$

и, постулируемое в [6], связывающее давление в фазах уравнение

$$\frac{\partial \alpha_1}{\partial t} + \vartheta_{\text{int}} \frac{\partial \alpha_1}{\partial x} = \mu(P_1 - P_2) \tag{9}$$

где $\mu = \frac{A_{\text{int}}}{(\rho_1^0 c_1 + \rho_2^0 c_2)}$, α_i - объемная концентрация i-ой фазы, ϑ_i - скорость i-ой фазы, ϑ_{int} - скорость на межфазной границе [6], P_{int}- давление на межфазной границе [6], A_{int}- площадь межфазной поверхности в единице объёма смеси [6]. Для упрощения выкладок ниже будут использоваться самые простые зависимости для массовых сил F_{mi} (что не влияет на основные выводы)

$$F_{m1} = -\alpha^1 \rho_1^0 g,$$

$$F_{m2} = -\rho_2^0 \alpha^2 g,$$

где g – ускорение свободного падения, а ось x направлена вверх и самая простая, линейная зависимость для межфазной силы трения F_{12}:

$$F_{12} = f * (\vartheta_1 - \vartheta_2),$$

Оценим время выравнивания давления системы (7) - (9). Рассмотрим однородное состояние $\frac{\partial}{\partial x}$=0 для всех параметров смеси. Пусть в начальный момент t=0 давления фаз P_1 и P_2 отличаются. Обозначим параметры смеси в равновесном состоянии, в которое смесь перейдет и в котором $P_1 = P_2$ индексом «0» внизу. Выразим из уравнений (7) ρ_i^0 через α_i: случае t_0 стремящегося к нулю, его решения имеют вид:

$$\rho_1^0 = \frac{\alpha_{10}\rho_{10}^0}{\alpha_1};\ \rho_2^0 = \frac{\alpha_{20}\rho_{20}^0}{\alpha_2} \tag{10}$$

Рассматриваем малое изменение параметров α_1 : $\alpha_1 = \alpha_{10} + \alpha_1'$ ($\alpha_1' \ll \alpha_{10}$) , где штрихом обозначены отличия от равновесных значений. Подставим (10) в (9), и, используя связь давления и плотности через c_i, получим

$$\frac{d\alpha_1'}{dt} = -B * \alpha_1'$$

$$\text{где } B = \frac{\left\{\frac{c_1^2\rho_{10}^0}{\alpha_{10}} + \frac{c_2^2\rho_{20}^0}{\alpha_{20}}\right\}}{\{c_1\rho_{10}^0 + c_2\rho_{20}^0\}} A_{\text{int}}.$$

Из последнего уравнения следует, что объемная концентрация, а, значит, и все параметры выходят на равновесные значения за время 1/B, обратно пропорциональное скорости звука и межфазной поверхности.

Например, для «типичного пузырькового течения» с параметрами $\rho_{10}^0 = 1000$ кг/м^3 ; $a_{10} = 0{,}8$; $\rho_{20}^0 = 1$ кг/м^3 ; $c_1 = 1000$ м/с ; $c_2 = 300$ м/с и диаметра пузырька $d = 3$ мм , $B = 5 \times 10^5 \frac{1}{c}$, а характерное время выравнивания давлений при использовании уравнения (9) равно 2×10^{-6}с. Для расслоенного течения, в котором межфазная поверхность A_{int} много меньше, с теми же параметрами $B = 1{,}25 \times 10^3 \frac{1}{c}$, характерное время выравнивания давлений равно $0{,}8 \times 10^{-3}$с.

В случае модельной задачи маленькое время релаксации параметров приводило к быстрому развитию неустойчивости. Ниже изучено развитие неустойчивости для уравнений модели [6].

В рассматриваемом случае баротропных фаз уравнения неразрывности (7) удобно записать в виде

$$\frac{\partial\,\alpha_1}{\partial t}+\frac{\partial}{\partial x}(\alpha_1\vartheta_1)\;+\frac{\alpha_1}{\rho_1^0c_1^2}\left(\frac{\partial \mathrm{P}_1}{\partial t}\;+\;\vartheta_1\frac{\partial \mathrm{P}_1}{\partial x}\right)=0, \tag{11}$$

$$\frac{\partial\alpha_2}{\partial t}+\frac{\partial}{\partial x}(\alpha_2\vartheta_2)\;+\frac{\alpha_2}{\rho_2^0c_2^2}(\frac{\partial \mathrm{P}_2}{\partial t}\;+\;\vartheta_2\frac{\partial \mathrm{P}_2}{\partial x})=0,$$

Выпишем матрицу для определения характеристического уравнения

$$\begin{pmatrix} \vartheta_1-\lambda & \alpha_1 & 0 & \frac{\alpha_1}{\rho_1^0c_1^2}(\vartheta_1-\lambda) & 0 \\ -\vartheta_2+\lambda & 0 & \alpha_2 & 0 & \frac{\alpha_2}{\rho_2^0c_2^2}(\vartheta_2-\lambda) \\ \mathrm{P}_1-\mathrm{P}_{int} & \alpha_1\rho_1^0(\vartheta_1-\lambda) & 0 & \alpha_1 & 0 \\ \mathrm{P}_2+\mathrm{P}_{int} & 0 & \alpha_2\rho_2^0(\vartheta_2-\lambda) & 0 & \alpha_2 \\ \vartheta_{int}-\lambda & 0 & 0 & 0 & 0 \end{pmatrix}$$

Приравняв определитель выше приведенной матрицы, получим следующее характеристическое уравнение

$$\alpha_1^2\alpha_2^2(\vartheta_{int}-\lambda)\left[1-\frac{(\vartheta_1-\lambda)^2}{c_1^2}\right]\left[1-\frac{(\vartheta_2-\lambda)^2}{c_2^2}\right]=\;0 \tag{12}$$

Откуда определяются характеристики

$$\lambda_1=\vartheta_{int};\;\lambda_{2,3}=\vartheta_1\pm c_1;\;\lambda_{4,5}=\vartheta_2\pm c_2 \tag{13}$$

Характеристиками в таком виде были ранее получены в [9].

Вывод: Из (13) видно, что модель [6] с разными давлениями фаз, всегда гиперболическая, имеет пять различных действительных характеристик: одна характеристика равна скорости переноса в постулируемом в модели [6] уравнении, связывающим давление в фазах и четыре равны скоростям фаз плюс минус скорость звука в фазе.

Сравним полученные характеристики с характеристиками в аналогичной модели с одинаковым давлением фаз [1]. Принимая вышеописанные предположения, легко выписать уравнения, аналогичные уравнениям (8), но при условии равенства давления фаз ($\mathrm{P}_1=\mathrm{P}_2=\mathrm{P}$). Проведя вычисления, аналогично описанным в этом разделе выше, удаётся выписать характеристики, которые в предположении $\vartheta_i/\mathrm{c}_i\ll 1$ имеют вид

$$\lambda_{1,2}=\frac{\alpha_1\rho_2^0\vartheta_1+\;\alpha_2\rho_1^0\vartheta_2}{\alpha_1\rho_2^0+\;\alpha_2\rho_1^0}\pm\sqrt{\frac{\alpha_1\rho_2^0+\;\alpha_2\rho_1^0}{\frac{\alpha_1\rho_2^0}{c_1^2}+\frac{\alpha_2\rho_1^0}{c_2^2}}};$$

$$_{3,4} = \frac{\alpha_1\rho_2^0\vartheta_2 + \alpha_2\rho_1^0\vartheta_1 \pm i(\vartheta_1 - \vartheta_2)\sqrt{\alpha_1\alpha_2\rho_1^0\rho_2^0}}{\alpha_1\rho_2^0 + \alpha_2\rho_1^0}$$

Вывод: Система уравнений модели с 1 давлением, в случае межфазной силы, не зависящей от производных, всегда не гиперболическая, Две характеристики модели - действительные – линейная комбинация скоростей фаз плюс минус функция от скоростей звука фаз и две характеристики комплексные, причем они не зависят от скоростей звука фаз. скоростей звука фаз.

4 УСТОЙЧИВОСТЬ РЕШЕНИЯ СИСТЕМЫ УРАВНЕНИЙ МОДЕЛЕЙ С РАЗНЫМИ ДАВЛЕНИЯМИ ФАЗ

Исследуем устойчивость решения системы (11,8,9). Приравняв определитель соответствующей матрицы нулю, получим дисперсионное уравнение пятой степени относительно ω, которое ввиду его громоздкости не приводится. В общем случае это дисперсионное уравнение аналитически решить невозможно. Однако его можно решить в предельных случаях. Рассмотрим два предельных случая: 1) большие k; 2) скорости фаз много меньше скоростей звука. 1) В пределе при $\mathrm{k} \to \infty$ корни дисперсионного уравнения получаются из корней (13) характеристического уравнения при замене λ на $\frac{\omega}{\mathrm{k}}$. Как видно из (13), в нулевом приближении, все решения действительны и не позволяют оценить их устойчивость. С точностью до первого приближения по $\frac{\omega}{\mathrm{k}}$ где $|\vartheta_i|/\mathrm{c}_i \ll 1 (i = 1,2; j = 1,2)$ первый корень имеет вид

$$\frac{\omega_1}{\mathrm{k}} = \vartheta_{int} + \frac{\mathrm{i}\mu}{\mathrm{k}\alpha_1\alpha_2}[\alpha_1\rho_2^0(\vartheta_{int} - \vartheta_2)^2 + \alpha_2\rho_1^0(\vartheta_{int} - \vartheta_1)^2] - \frac{\mu\mathrm{f}}{k^2\alpha_1^2\alpha_2^2}[\vartheta_{int} - \alpha_1\vartheta_2 - \alpha_2\vartheta_1]$$

Из этой формулы видно, что это решение не устойчиво. Для «типичного пузырькового течения» при $|\vartheta_{int} - \vartheta_1| \sim 0.5\ \frac{\text{м}}{\text{сек}}$, $\mathrm{Im}(\omega_1) \sim\ 0{,}1\frac{1}{\text{сек}}$. Тогда за 10 сек. погрешность увеличивается в 3 раза, т.е. за время больше 1 мин. погрешность увеличится более чем в 1000 раз, а, значит, результатам таких расчетов доверять нельзя, т.к если исходные данные, имеют точность не выше 1% , поэтому в 1000 раз увеличенная погрешность составляет 1000%. Эта оценка получена при $\mathrm{k} \to \infty$. Оставшиеся четыре корня в тех же предположениях, что и первый корень соответствуют устойчивым решениям. Скорость затухания возмущений очень большая. Например, для «типичного пузырькового течения» $\mathrm{Im}(\omega_1) \sim$ - 2,5 E5 $\frac{1}{\text{сек}}$.

Однако, если не делать предположений о больших k, как уже говорилось, в общем случае дисперсионное уравнение 5 порядка аналитически решить невозможно, его можно решать численно. На рис 3 для «типичного пузырькового течения» с учетом присоединенной массы, показана зависимость $\mathrm{Im}(\omega_1)$ от волнового числа k.

Как видно из графика, скорость развития неустойчивости выходит на асимптоту при $\mathrm{k} \approx 100\ \frac{1}{\text{м}}$, и приведенные выше оценки уже примерно справедливы при $\mathrm{k} \approx 25\ \frac{1}{\text{м}}$. Для остальных решений выполняется условие $\frac{\omega}{\mathrm{k}} \sim C_j$. Решение $\omega_{2,3}$ почти во всём диапазоне режимных параметров устойчиво.

Можно показать, что решения $\omega_{4,5}$ всегда устойчивы. Итак, в случае $|\vartheta_i|/c_i \ll 1$ найдены все 5 решений.

Подводя итог – аналогично тестовой задаче, несмотря на гиперболичность системы (11,8,9) во многих режимах имеется не устойчивое решение ω_1, причем, скорость его роста (для «типичного пузырькового течения» $\mathrm{Im}(\omega_1) < 0{,}1\frac{1}{\text{c}}$) настолько большая (хоть и не такая огромная, как в тестовой задаче), что она может делать численные расчеты недостоверными.

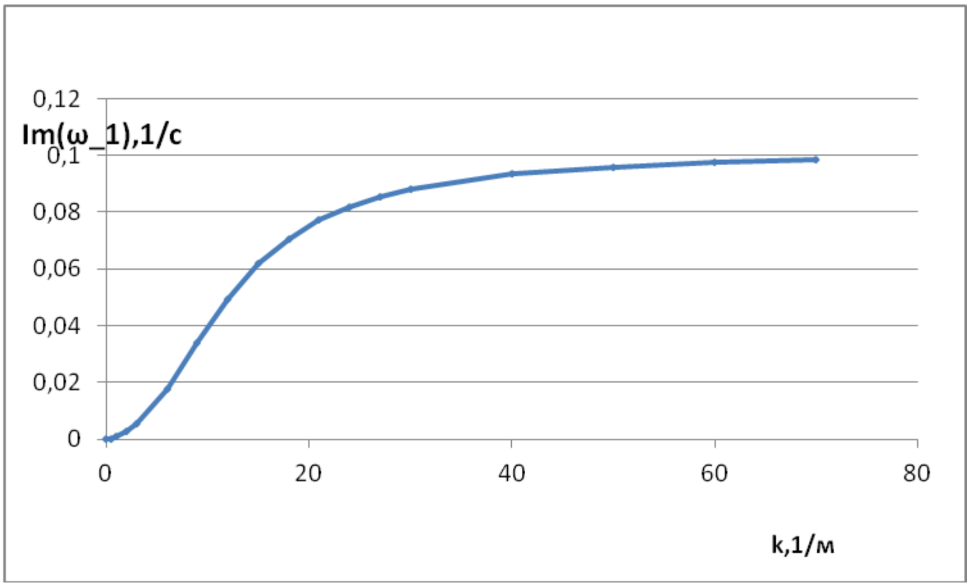

Рис. 3 Зависимость мнимой части частоты от волнового числа.

5 СИСТЕМА УРАВНЕНИЙ С РАЗНЫМИ ДАВЛЕНИЯМИ ФАЗ, ОПИСЫВАЮЩАЯ ПУЛЬСАЦИИ ПУЗЫРЬКОВ

Недостатком модели [6], является, по мнению авторов, отсутствие обоснования уравнения (9). С другой стороны, например для пузырькового режима течения, хорошо известно уравнение Релея – Ламба [2], описывающее изменение давлений фаз, которое вполне обоснованно можно использовать вместо (9) [7]. Исследуем гиперболичность математической модели этой системы. Будем предполагать фазы баротропными. Исследуем устойчивость решения системы (11,8) с уравнением Релея – Ламба и уравнением баланса массы пузырьков. Приравняв определитель соответствующей матрицы нулю получаем характеристическое уравнение 7 степени.

Три характеристики этого уравнения равны скорости пузырьков. Проведенные многочисленные численные оценки показали, что при «нефизических» условиях, когда проскальзывание много больше скорости звука: $|\vartheta_{12}| \gg c_1$ четыре оставшихся корня будут действительными, т.е. система уравнений гиперболическая. А при «нормальных» условиях, когда скорость проскальзывания меньше скорости звука, из оставшихся четырёх корней два будут действительными, а оставшиеся 2 комплексными, т.е. система уравнений – не гиперболическая, а значит задача Коши для неё – не корректна. Однако, как отмечалось выше, многие авторы допускают использование результатов численных решений таких систем, если скорость развития неустойчивости решений в связи с ограничениями на волновое число, из-за размера шага численной сетки, не велика. Для «типичного пузырькового течения» $\alpha_1 = 0{,}8$ и $\vartheta_{12} = -0{,}1 \frac{\text{м}}{\text{сек}}$ при шаге расчётной сетки 1 см $\mathrm{Im}(\omega_6) = 5 \frac{1}{\text{сек}}$ Т.е. с таким шагом можно считать только короткие процессы, длительностью - до нескольких секунд.

6 ЗАКЛЮЧЕНИЕ

Система уравнений течения двухфазной среды с разными давлениями фаз и двумя уравнениями импульса в предположении баротропности фаз и постулируемым уравнением, связывающим давления в разных фазах всегда гиперболическая. Имеет пять различных, действительных характеристик. Одна характеристика равна скорости переноса в постулируемом уравнении, связывающим давление в фазах и четыре характеристики равны скоростям фаз плюс минус скорость звука в фазе. Эта модель, несмотря на гиперболичность системы, во многих режимах имеет не устойчивое решение, причем, скорость роста возмущения (для «типичного пузырькового течения» настолько большая (хоть и не такая огромная, как для модифицированного уравнения Лапласа)), что результаты численных расчетов могут быть недостоверными.

Система уравнений с разными давлениями фаз и двумя уравнениями импульса, описывающая пульсации пузырьков при «нефизических» условиях, когда проскальзывание много больше скорости звука в жидкости гиперболическая. А при «нормальных» условиях, когда скорость проскальзывания меньше скорости звука в жидкости, система уравнений – не гиперболическая, и задача Коши для неё – не корректна. Даже если ограничить волновое число разумным шагом при численных расчетах, то большая скорость роста неустойчивости может делать использование результатов расчетов – невозможным.

REFERENCES

[1] X.A.Raxmatulin, “Osnovy` gazovoj dinamiki vzaimopronikayushhix dvizhenij sploshny`x sred”, PMM, 20 (2), 184-195 (1956).

[2] R.I. Nigmatulin, *Dynamics of Multiphase Media*, Vol.1, 2, Hemisphere, N.Y., (1990).

[3] L.A. Klebanov, A.E. Kroshilin, B.I. Nigmatulin, R.I. Nigmatulin, “O giperbolichnosti, ustojchivosti i korrektnosti zadachi Koshi dlya sistemy` differencial`ny`x uravnenij dvuxskorostnogo dvizheniya dvuxfazny`x sred”, *PMM*, **46** (1), 83-95 (1982).

[4] Daniel Lhuillier, Chih-Hao Chang, Theo G. Theofanous, “On the quest for a hyperbolic effective-field model of disperse flows”, *J. Fluid Mech*., **731**, 184-195 (2013).

[5] A.E. Kroshilin, V.E. Kroshilin, “Correct numerical simulation of a two-phase coolant”, *Thermal Engineering,* **63** (2), 98-106 (2016).

[6] R.A. Berry, J.W. Peterson, H. Zhang, R.C. Martinean, H. Zhao, L. Zou, D. Andrs, *RELAP-7 theory manual*, Idaho National Laboratory, INT/EXT-14-31366, 2014.

[7] A.F. Zhivotyagin, S.D Kalinichenko, A.E. Kroshilin, V.E. Kroshilin, “A HighlyAccurate Method for Numerical Modeling Wave Outflow of Two-Phase Media”, *Thermal Engineering*, 47(5), 412-416 (2000).

[8] Korn G., Korn T., *Spravochnik po matematike dlya nauchny`x rabotnikov i inzhenerov*, Izdatel`stvo Nauka, Glavnaya redakciya fiziko-matematicheskoj literatury`, Moskva (1977).

[9] Ling Zou, Haihua Zhao, Hongbin Zhang “A revisit to the Hicks’ hyperbolic twopressure two-phase flow model” *Proceedings of The 17th International Topical Meeting on Nuclear Reactor Thermal-Hydraulics (NURETH-17), Xi`an, China Sept. 3-8,* 1-14 (2017).

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