ON THE COMPLEXITY OF K-STEP AND K-HOP DOMINATING SETS IN GRAPHS

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Summary. Given a positive integer $k \geq 2$, two vertices in a graph are said to k-step dominate each other if they are at distance k apart. A set S of vertices in a graph G is a k-step dominating set of G if every vertex is k-step dominated by some vertex of S. The k-step domination number of G is the minimum cardinality of a k-step dominating set of G. A subset S of vertices of G is a k-hop dominating set if every vertex outside S is k-step dominated by some vertex of S. The k-hop domination number of G is the minimum cardinality of a k-hop dominating set of G. In this paper, we show that for any integer $k \geq 2$, the decision problems for the k-step dominating set and k-hop dominating set problems are NP-complete for planar bipartite graphs and planar chordal graphs.

1 INTRODUCTION

For notation and graph theory terminology not given here, we refer to [6]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n(G) = |V(G)|$ and the *size* of G is $m(G) = |E(G)|$. A graph is *non-empty* if it contains at least one edge. The *open neighborhood* of a vertex v is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. The *degree* of v, denoted by deg(v), is $|N_G(v)|$. The *open neighborhood* of a subset $S \subseteq V$, is $N_G(S) = \bigcup_{v \in S} N_G(v)$, and the *closed neighborhood* of S is the set $N_G[S] = N_G(S) \cup S$. A subset S of vertices of a graph G is a *dominating set* of G if every vertex in $V(G) \setminus S$ has a neighbor in S. The *domination number* of G is the minimum cardinality of a dominating set of G. The *distance* between two vertices u and v in G, denoted $d(u,v)$ is the minimum length of a (u,v)-path in G. A *chordal graph* is a graph that does not contain an induced cycle of length greater than 3. A *planar graph* is a graph which can be drawn in the plane without any edges crossing.

For an integer $k \geq 1$, two vertices in a graph G are said to *k-step dominate* each other if they are at distance exactly k apart in G. A set S of vertices in G is a *k-step dominating set* of G if every vertex in $V(G)$ is k-step dominated by some vertex of S. The *k-step domination number*, $\gamma_{kstep}(G)$, of G, is the minimum cardinality of a k-step dominating set of

2010 Mathematics Subject Classification: 05C69.
Keywords and phrases: k-Step dominating set, k-Step domination number, k-Hop dominating set, k-Hop domination number, Step dominating set, NP-complete.

G. The concept of 2-step domination in graphs was introduced by Chartrand, Harary, Hossain, and Schultz [3] and further studied, for example in [4,8,11].

Recently, Ayyaswamy and Natarajan [1] introduced a parameter similar to the 2-step domination number, namely the hop domination number of a graph. A subset S of vertices of G is a *hop dominating set* if every vertex outside S is 2-step dominated by some vertex of S. The *hop domination number*, $\gamma_h(G)$, of G is the minimum cardinality of a hop dominating set of G. The concept of hop domination was further studied, for example, in [2,10].

Henning et al. [7] studied the complexity issue of the 2-step domination as well as the hop domination in a graph, and showed that the decision problems for the 2-step dominating set and hop dominating set problems are NP-complete for planar bipartite graphs and planar chordal graphs. In this paper, we generalize these results for any integer $k \geq 2$. For an integer $k \geq 2$, a subset S of vertices of G is called a *k-hop dominating set* if every vertex outside S is k-step dominated by some vertex of S. The *k-hop domination number,* $\gamma_{kh}(G)$, of G is the minimum cardinality of a k-hop dominating set of G. We show that for any integer $k \geq 2$, the decision problems for the k-step dominating set and k-hop dominating set problems are NP-complete for planar bipartite graphs and planar chordal graphs.

2 MAIN RESULTS

We will state the corresponding decision problems in the standard Instance Question form [5] and indicate the polynomial-time reduction used to prove that it is NP-complete. Let $k \geq 2$ be a positive integer. Consider the following decision problems:

k -Step Dominating Set Problem (kSDP).

Instance: A non-empty graph G, and a positive integer t.
Question: Does G have a k-step dominating set of size at most t?

k -Hop Dominating Set Problem (HDP).

Instance: A non-empty graph G, and a positive integer t.
Question: Does G have a k-hop dominating set of size at most t?

We use a transformation of the Vertex Cover Problem which was one of Karp's 21 NP-complete problems [9]. A *vertex cover* of a graph is a set of vertices such that each edge of the graph is incident with at least one vertex of the set. The Vertex Cover Problem is the following decision problem.

Vertex Cover Problem (VCP).

Instance: A non-empty graph G, and a positive integer k.
Question: Does G have a vertex cover of size at most k?

We first consider the k-step dominating set problem.

Theorem 1 The $kSDP$ is NP-complete for planar bipartite graphs.

Proof. Clearly, the $kSDP$ is in NP, since it is easy to verify a "yes" instance of the $kSDP$ in polynomial time. We show how to transform the vertex cover problem to the $kSDP$ so that one of them has a solution if and only if the other has a solution.

Let G be a connected planar graph of order $n = n_G$ and size $m = m_G \geq 2$. Let H be the graph obtained from G as follows. For each edge $e = uv \in E(G)$, we subdivide the edge e, $2k-1$ times and let $x_1^e, x_2^e, \ldots, x_{2k-1}^e$ be the vertices that resulted from subdividing the edge e, $2k-1$ times, and add a path $v_1^e v_2^e \ldots v_{2k}^e$, and join v_1^e to both u and v. The resulting graph H has order $n_H = n_G + (2k-1)m_G + 2km_G = n_G + (4k-1)m_G$ and size $m_H = (4k+1)m_G$. Clearly the transformation can be performed in polynomial time. We note that H is connected and planar, since G is connected and planar. Further, by the construction, H doesn`t contain odd contour, so H is bipartite. Thus, H is a connected planar bipartite graph. We show that G has a vertex cover of size at most t if and only if H has a k-step dominating set of size at most $t + km_G$. Assume that G has a vertex cover S_G, of size at most t. We now consider the set

$$S_H = S_G \bigcup_{e \in E(G)} \{ v_1^e, v_2^e, \ldots, v_k^e \}.$$

Since $m_G \geq 2$, we find that $S_G \neq \emptyset$. For every edge $e = uv \in E(G)$ and $1 \leq i \leq k-1$, the vertex v_i^e, k-step dominates the vertices v_{k+i}^e, x_{k-i}^e an x_{k+i}^e, and the vertex v_k^e, k-step dominates the vertices v_{2k}^e, u and v in H. Further, since G is connected and $m_G \geq 2$, for every two adjacent edges e and f in G, for $1 \leq i \leq k-1$, the vertex v_i^e is k-step dominated by the vertex v_{k-i}^f. Since S_G is a vertex cover in G, v_k^e and x_k^e are k-step dominated by the set S_G in H. Therefore, the set S_H is a k-step dominating set of size at most $t + km_G$ in H.

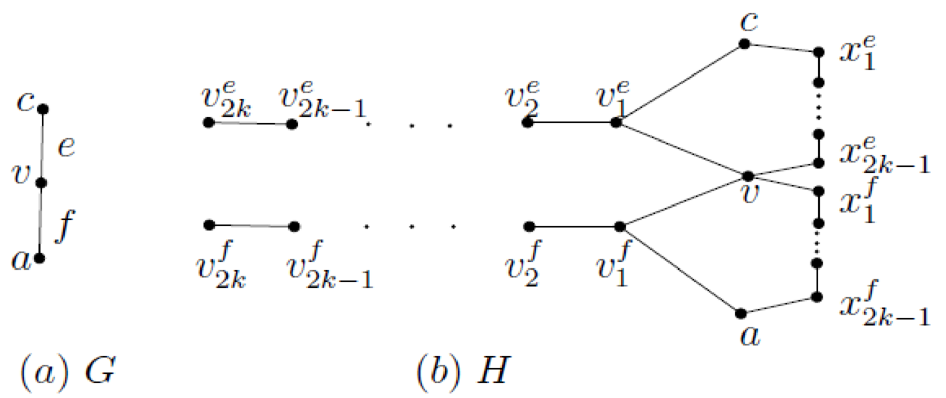

Figure 1: The graphs G and H in the proof of Theorem 1.

Suppose next that H has a k-step dominating set D_H of size at most $t + km_G$. Let $e = uv \in E(G)$. In order to k-step dominate v_i^e for $k+1 \le i \le 2k$ in H, the vertices v_i^e, $1 \le i \le k$, belongs to the set D_H. In order to k-step dominate the vertex x_k^e in H, we note that u or v or both u and v belong to D_H. Thus, $D_G = D_H \cap V(G)$ is a vertex cover of G. Further, since for $1 \le i \le k$, v_i^e, belong to D_H for every $e = uv \in E(G)$, we note that $|D_G| \le |D_H| - km_G = t$. Thus, G has a vertex cover of size at most t.

Theorem 2 The $kSDP$ is NP-complete for planar chordal graphs.

Proof . As noted in the proof of Theorem 1, the $kSDP$ is in NP. Now let us show how to transform the vertex cover problem to the $kSDP$ so that one of them has a solution if and only if the other has a solution.

Let G be a connected planar chordal graph of order n_G and size $m_G \ge 2$. Let H be the graph obtained from G as follows. For each edge $e = uv \in E(G)$ we add a path $P_{2k}: v_1^e v_2^e \ldots v_{2k}^e$, and join v_1^e to both u and v, and add a path $P_k : a_1^e a_2^e \ldots a_k^e$, and join a_1^e to both u and v. The resulting graph H has order $n_H = n_G + 3km_G$ and size $m_H = (3k+3)m_G$. The transformation can clearly be performed in polynomial time. We note that since G is a connected planar chordal graph, so too is H. We show that G has a vertex cover of size at most t if and only if H has a k-step dominating set of size at most $t + km_G$. Assume that G has a vertex cover S_G of size at most t. We now consider the set

$$S_H = S_G \bigcup_{e \in E(G)} \{ v_1^e, v_2^e, \ldots, v_k^e \}.$$

Since $m_G \ge 2$, we find that $S_G \ne \emptyset$. For every edge $e = uv \in E(G)$ and $1 \le i \le k-1$, the vertex v_i^e, k-step dominates the vertices v_{k+i}^e and a_{k-i}^e, and also the vertex v_k^e, k-step dominates the vertices v_{2k}^e, u and v in H.

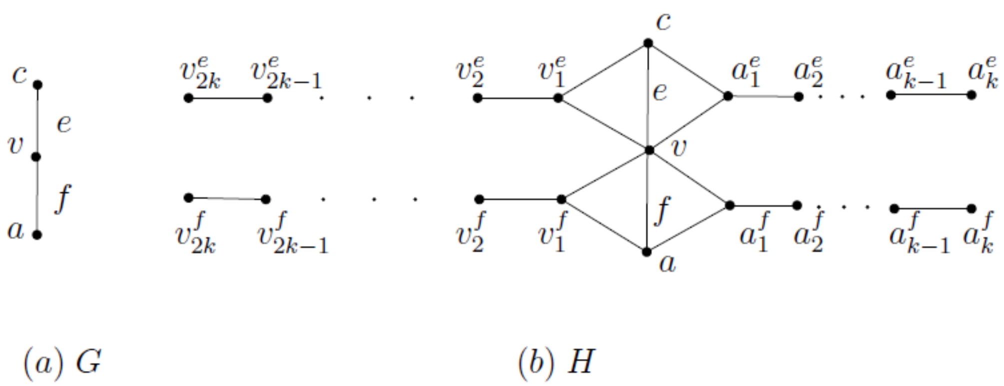

Figure 2: The graphs G and H in the proof of Theorem 2

Since G is connected and $m_G \geq 2$, for every two adjacent edges e and f in G, for $1 \leq i \leq k-1$, the vertex v_i^e is k-step dominated by the vertex $v_{k-i}{}^f$. Since S_G is a vertex cover in G, $v_k{}^e$ and $a_k{}^e$ are k-step dominated by the set S_G in H. Therefore, the set S_H is a k-step dominating set of size at most $t+\ km_G$ in H. Let $e\ =\ uv \in E(G)$. In order to k-step dominate v_i^e, $k+1 \leq i \leq 2k$ in H, the vertices v_i^e, $1 \leq i \leq k$, belongs to the set D_H. In order to k-step dominate the vertex $a_k{}^e$ in H, we note that u or v or both u and v belong to D_H. Thus, $D_G\ =\ D_H \cap V\ (G)$ is a vertex cover of G. Further, since for $1 \leq i \leq k$, v_i^e, belong to D_H for every $e\ =\ uv \in E(G)$, we note that $|\ D_G| \leq |\ D_H|\ -\ km_G\ =\ t$. Thus, G has a vertex cover of size at most t.

We next consider the k-hop dominating set problem.

Theorem 3 The $kHDP$ is NP-complete for planar bipartite graphs.

Proof . Let G be a graph of order n_G and size m_G, and let H be the connected planar bipartite graph constructed in the proof of Theorem 1. We show that G has a vertex cover of size at most t if and only if H has a khop dominating set of size at most $t+km_G$. If G has a vertex cover S_G of size at most t, then this is immediate since the k-step dominating set S_H constructed in the proof of Theorem 1 is also a khop dominating set in H of size $|\ S_H| \leq t+km_G$. Suppose next that H has a khop dominating set D_H of size at most $t\ +\ km_G$. If $\left|D_H \cap \left\{v_1{}^e, v_2{}^e,\ \ldots\ , v_{2k}{}^e\right\}\right| \leq k-1$ for some edge $e \in E(G)$, then at least one vertex in $\left\{v_{k+1}{}^e, v_{k+2}{}^e,\ \ldots\ ,\ v_{2k}{}^e\right\}$ is not khop dominated by D_H, a contradiction. Therefore, $\left|D_H \cap \left\{v_1{}^e, v_2{}^e,\ \ldots\ ,\ v_{2k}{}^e\right\}\right| \geq k$ for every edge $e \in E(G)$. Let $e\ =\ uv$ be an arbitrary edge of G. If $x_k{}^e \notin D_H$, then in order to khop dominate the vertex $x_k{}^e$ in H, we note that u or v or both u and v belong to D_H. We now consider the set D_G obtained from $D_H \cap V(G)$ as follows. For each vertex $x_k{}^e$ associated with an edge $e \in E(G)$, if $x_k{}^e \in D_H$, then we add u or v to the set D_G. The resulting set D_G is a vertex cover of G of size at most $|\ D_H|\ -\ km_G \leq t$. Thus G has a vertex cover of size at most t.

Theorem 4 The $kHDP$ is NP-complete for planar chordal graphs.

Proof . Let G be a graph of order n_G and size m_G, and let H be the connected planar chordal graph constructed in the proof of Theorem 2. We show that G has a vertex cover of size at most t if and only if H has a khop dominating set of size at most $t+\ km_G$. If G has a vertex cover S_G of size at most t, then this is immediate since the k-step dominating set S_H constructed in the proof of Theorem 2 is also a khop dominating set in H of size

$|S_H| \le t + km_G$. Suppose next that H has a khop dominating set D_H of size at most $t + km_G$. If $\left|D_H \cap \{v_1^e, v_2^e, \ldots, v_{2k}^e\}\right| \le k-1$ for some edge $e \in E(G)$, then at least one vertex of $\{v_{k+1}^e, v_{k+2}^e, \ldots, v_{2k}^e\}$ is not khop dominated by D_H, a contradiction. Therefore, $\left|D_H \cap \{v_1^e, v_2^e, \ldots, v_{2k}^e\}\right| \ge k$ for every edge $e \in E(G)$. Let $e = uv$ be an arbitrary edge of G. If $a \notin D_H$, then in order to khop dominate the vertex a_k^e in H, we note that u or v or both u and v belong to D_H. We now consider the set D_G obtained from $D_H \cap V(G)$ as follows.

For each vertex a_k^e associated with an edge $e \in E(G)$, if $a_k^e \in D_H$, then we add u or v to the set D_G. The resulting set D_G is a vertex cover of G of size at most $|D_H| - km_G \le t$. Thus, G has a vertex cover of size at most t.

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Received September 10, 2017