

## WEAK INSERTION OF A CONTRA-CONTINUOUS FUNCTION BETWEEN TWO COMPARABLE CONTRA-PRECONTINUOUS (CONTRA-SEMI-CONTINUOUS) FUNCTIONS

MAJID MIRMIRAN

Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran  
e-mail: mirmir@sci.ui.ac.ir, web page: <http://www.ui.ac.ir>

**Summary.** A sufficient condition in terms of lower cut sets are given for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.

### 1 INTRODUCTION

The concept of preopen set in a topological space was introduced by H.H. Corson and E. Michael in 1964 [4], under the name locally dense. Subset  $A$  of a topological space  $(X, \tau)$  is called preopen or locally dense or nearly open if  $A \subseteq \text{Int}(\text{Cl}(A))$ . Set  $A$  is called preclosed if its complement is preopen or equivalently if  $\text{Cl}(\text{Int}(A)) \subseteq A$ . The term ,preopen, was used for the first time by A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb [20].

The concept of semi-open set in a topological space was introduced by

N. Levine in 1963 [17]. Subset  $A$  of a topological space  $(X, \tau)$  is called semi-open [10] if  $A \subseteq \text{Cl}(\text{Int}(A))$ . Set  $A$  is called semi-closed if its complement is semi-open or equivalently if  $\text{Int}(\text{Cl}(A)) \subseteq A$ .

A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them  $V$ -sets. Complements of  $V$ -sets, i.e., sets that are intersection of open sets are called  $\Lambda$ -sets [19].

Recall that a real-valued function  $f$  defined on a topological space  $X$  is called  $A$ -continuous [23] if the preimage of every open subset of  $\mathbb{R}$  belongs to  $A$ , where  $A$  is a collection of subsets of  $X$ . Most of the definitions of function used throughout this paper are consequences of the definition of  $A$ -continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [6] introduced a new class of mappings called contra-continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 12, 13, 22].

Hence, a real-valued function  $f$  defined on a topological space  $X$  is called contra-continuous (resp. contra-semi-continuous, contra-precontinuous) if the preimage of every open subset of  $\mathbb{R}$  is closed (resp. semi-closed, pre-closed) in  $X$ [6].

Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between

**2010 Mathematics Subject Classification:** Primary 54C08, 54C10, 54C50; Secondary 26A15, 54C30.

**Key words and Phrases:** Insertion, Strong binary relation, Semi-open set, Preopen set, Contra-continuous function, Lower cut set.

two comparable real-valued functions on such topological spaces that  $\Lambda$ -sets or kernel of sets are open [19].

If  $g$  and  $f$  are real-valued functions defined on a space  $X$ , we write  $g \leq f$  in case  $g(x) \leq f(x)$  for all  $x$  in  $X$ .

The following definitions are modifications of conditions considered in [16].

A property  $P$  defined relative to a real-valued function on a topological space is  $cc$ -property provided that any constant function has property  $P$  and provided that the sum of a function with property  $P$  and any contra-continuous function also has property  $P$ .

If  $P_1$  and  $P_2$  are  $cc$ -properties, the following terminology is used: A space  $X$  has the weak  $cc$ -insertion property for  $(P_1, P_2)$  if and only if for any functions  $g$  and  $f$  on  $X$  such that  $g \leq f$ ,  $g$  has property  $P_1$  and  $f$  has property  $P_2$ , then there exists a contra-continuous function  $h$  such that  $g \leq h \leq f$ .

In this paper, for a topological space whose  $\Lambda$ -sets or kernel of sets are open, is given a sufficient condition for the weak  $cc$ -insertion property. Several insertion theorems are obtained as corollaries of these results.

## 2 THE MAIN RESULT

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.

**Definition 2.1.** Let  $A$  be a subset of a topological space  $(X, \tau)$ . We define the subsets  $A^\wedge$  and  $A^\vee$  as follows:

$$A^\wedge = \bigcap \{O : O \supseteq A, O \in \tau\} \text{ and } A^\vee = \bigcup \{F : F \subseteq A, F^c \in \tau\}.$$

In [7, 18, 21],  $A^\wedge$  is called the kernel of  $A$ .

The family of all preopen, preclosed, semi-open and semi-closed will be denoted by  $pO(X, \tau)$ ,  $pC(X, \tau)$ ,  $sO(X, \tau)$  and  $sC(X, \tau)$ , respectively.

We define the subsets  $p(A^\wedge)$ ,  $p(A^\vee)$ ,  $s(A^\wedge)$  and  $s(A^\vee)$  as follows:

$$p(A^\vee) = \bigcup \{F : F \subseteq A, F \in pC(X, \tau)\},$$

$$p(A^\wedge) = \bigcap \{O : O \supseteq A, O \in pO(X, \tau)\},$$

$$s(A^\wedge) = \bigcap \{O : O \supseteq A, O \in sO(X, \tau)\} \text{ and}$$

$$s(A^\vee) = \bigcup \{F : F \subseteq A, F \in sC(X, \tau)\}.$$

$p(A^\wedge)$  (resp.  $s(A^\wedge)$ ) is called the prekernel (resp. semi - kernel) of  $A$ .

The following two definitions are modifications of conditions considered in [14, 15].

**Definition 2.2.** If  $\rho$  is a binary relation in set  $S$  then  $\rho^-$  is defined as follows:

$x\rho^-y$  if and only if  $y\rho v$  implies  $x\rho v$  and  $u\rho x$  implies  $u\rho y$  for any  $u$  and  $v$  in  $S$ .

**Definition 2.3.** A binary relation  $\rho$  in the power set  $P(X)$  of a topological space  $X$  is called a strong binary relation in  $P(X)$  in case  $\rho$  satisfies each of the following conditions:

- 1) If  $A_i \rho B_j$  for any  $i \in \{1, \dots, m\}$  and for any  $j \in \{1, \dots, n\}$ , then there exists a set  $C$  in  $P(X)$

such that  $A_i \rho C$  and  $C \rho B_j$  for any  $i \in \{1, \dots, m\}$  and any  $j \in \{1, \dots, n\}$ .

- 2) If  $A \subseteq B$ , then  $A \overset{\wedge}{\rho} B$ .
- 3) If  $A \rho B$ , then  $A \overset{\wedge}{\subseteq} B$  and  $A \overset{\vee}{\subseteq} B$ .

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

**Definition 2.4.** If  $f$  is a real-valued function defined on a space  $X$  and if  $\{x \in X : f(x) < l\} \subseteq A(f, l) \subseteq \{x \in X : f(x) \leq l\}$  for a real number  $l$ , then  $A(f, l)$  is called a lower indefinite cut set in the domain of  $f$  at the level  $l$ .

We now give the following main result:

**Theorem 2.1.** Let  $g$  and  $f$  be real-valued functions on a topological space  $X$ , in which kernel sets are open, with  $g \leq f$ . If there exists a strong binary relation  $\rho$  on the power set of  $X$  and if there exist lower indefinite cut sets  $A(f, t)$  and  $A(g, t)$  in the domain of  $f$  and  $g$  at the level  $t$  for each rational number  $t$  such that if  $t_1 < t_2$ , then  $A(f, t_1) \rho A(g, t_2)$ , then there exists a contra-continuous function  $h$  defined on  $X$  such that  $g \leq h \leq f$ .

**Proof.** Let  $g$  and  $f$  be real-valued functions defined on  $X$  such that  $g \leq f$ . By hypothesis there exists a strong binary relation  $\rho$  on the power set of  $X$  and there exist lower indefinite cut sets  $A(f, t)$  and  $A(g, t)$  in the domain of  $f$  and  $g$  at the level  $t$  for each rational number  $t$  such that if  $t_1 < t_2$ , then  $A(f, t_1) \rho A(g, t_2)$ .

Define functions  $F$  and  $G$  mapping the rational numbers  $Q$  into the power set of  $X$  by  $F(t) = A(f, t)$  and  $G(t) = A(g, t)$ . If  $t_1$  and  $t_2$  are any elements of  $Q$  with  $t_1 < t_2$ , then  $F(t_1) \rho F(t_2)$ ,  $G(t_1) \rho G(t_2)$ , and  $F(t_1) \rho G(t_2)$ . By Lemmas 1 and 2 of [15] it follows that there exists a function  $H$  mapping  $Q$  into the power set of  $X$  such that if  $t_1$  and  $t_2$  are any rational numbers with  $t_1 < t_2$ , then  $F(t_1) \rho H(t_2)$ ,  $H(t_1) \rho H(t_2)$  and  $H(t_1) \rho G(t_2)$ .

For any  $x$  in  $X$ , let  $h(x) = \inf\{t \in Q : x \in H(t)\}$ .

We first verify that  $g \leq h \leq f$ : If  $x$  is in  $H(t)$  then  $x$  is in  $G(t)$  for any  $t > h(x)$ ; since  $x$  is in  $G(t) = A(g, t)$  implies that  $g(x) \leq t$ , it follows that  $g(x) \leq h(x)$ . Hence  $g \leq h$ . If  $x$  is not in  $H(t)$ , then  $x$  is not in  $F(t)$  for any  $t < h(x)$ ; since  $x$  is not in  $F(t) = A(f, t)$  implies that  $f(x) > t$ , it follows that  $f(x) \geq h(x)$ . Hence  $h \leq f$ .

Also, for any rational numbers  $t_1$  and  $t_2$  with  $t_1 < t_2$ , we have  $h^{-1}(t_1, t_2) = H(t_2) \setminus H(t_1)$ . Hence  $h^{-1}(t_1, t_2)$  is closed in  $X$ , i.e.,  $h$  is a contra-continuous function on  $X$ .

The above proof used the technique of Theorem 1 in [14].

### 3 APPLICATIONS

The abbreviations *cpc* and *csc* are used for contra-precontinuous and contra-semi-continuous, respectively.

Before stating the consequences of Theorems 2.1, we suppose that  $X$  is a topological space whose kernel sets are open.

**Corollary 3.1.** If for each pair of disjoint preopen (resp. semi-open) sets  $G_1, G_2$  of  $X$ , there

exist closed sets  $F_1$  and  $F_2$  of  $X$  such that  $G_1 \subseteq F_1$ ,  $G_2 \subseteq F_2$  and  $F_1 \cap F_2 = \emptyset$ , then  $X$  has the weak cc–insertion property for (cpc, cpc) (resp. (csc, csc)).

**Proof.** Let  $g$  and  $f$  be real-valued functions defined on  $X$ , such that  $f$  and  $g$  are cpc (resp. csc), and  $g \leq f$ . If a binary relation  $\rho$  is defined by  $A\rho B$  in case  $p(A^{\wedge}) \subseteq p(B^{\vee})$  (resp.  $s(A^{\wedge}) \subseteq s(B^{\vee})$ ), then by hypothesis  $\rho$  is a strong binary relation in the power set of  $X$ . If  $t_1$  and  $t_2$  are any elements of  $Q$  with  $t_1 < t_2$ , then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since  $\{x \in X : f(x) \leq t_1\}$  is preopen (resp. semi–open) set and since  $\{x \in X : g(x) < t_2\}$  is preclosed (resp. semi–closed) set, it follows that  $p(A(f, t_1)^{\wedge}) \subseteq p(A(g, t_2)^{\vee})$  (resp.  $s(A(f, t_1)^{\wedge}) \subseteq s(A(g, t_2)^{\vee})$ ). Hence  $t_1 < t_2$  implies that  $A(f, t_1) \rho A(g, t_2)$ . The proof follows from Theorem 2.1. •

**Corollary 3.2.** If for each pair of disjoint preopen (resp. semi–open) sets  $G_1, G_2$ , there exist closed sets  $F_1$  and  $F_2$  such that  $G_1 \subseteq F_1$ ,  $G_2 \subseteq F_2$  and  $F_1 \cap F_2 = \emptyset$ , then every contra-precontinuous (resp. contra-semi–continuous) function is contra-continuous.

**Proof.** Let  $f$  be a real-valued contra-precontinuous (resp. contra-semi–continuous) function defined on  $X$ . Set  $g = f$ , then by Corollary 3.1, there exists a contra-continuous function  $h$  such that  $g = h = f$ . •

**Corollary 3.3.** If for each pair of disjoint subsets  $G_1, G_2$  of  $X$ , such that  $G_1$  is preopen and  $G_2$  is semi–open, there exist closed subsets  $F_1$  and  $F_2$  of  $X$  such that  $G_1 \subseteq F_1$ ,  $G_2 \subseteq F_2$  and  $F_1 \cap F_2 = \emptyset$ , then  $X$  have the weak cc–insertion property for (cpc, csc) and (csc, cpc).

**Proof.** Let  $g$  and  $f$  be real-valued functions defined on  $X$ , such that  $g$  is cpc (resp. csc) and  $f$  is csc (resp. cpc), with  $g \leq f$ . If a binary relation  $\rho$  is defined by  $A\rho B$  in case  $s(A^{\wedge}) \subseteq p(B^{\vee})$  (resp.  $p(A^{\wedge}) \subseteq s(B^{\vee})$ ), then by hypothesis  $\rho$  is a strong binary relation in the power set of  $X$ . If  $t_1$  and  $t_2$  are any elements of  $Q$  with  $t_1 < t_2$ , then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since  $\{x \in X : f(x) \leq t_1\}$  is semi–open (resp. preopen) set and since  $\{x \in X : g(x) < t_2\}$  is preclosed (resp. semi–closed) set, it follows that  $s(A(f, t_1)^{\wedge}) \subseteq p(A(g, t_2)^{\vee})$  (resp.  $p(A(f, t_1)^{\wedge}) \subseteq s(A(g, t_2)^{\vee})$ ). Hence  $t_1 < t_2$  implies that  $A(f, t_1) \rho A(g, t_2)$ . The proof follows from Theorem 2.1. •

**Acknowledgement:** This research was partially supported by Centre of Excellence for Mathematics (University of Isfahan).

## REFERENCES

- [1] A. Al-Omari and M.S. Md Noorani, “Some properties of contra-bcontinuous and almost contra-b-continuous functions”, *European J. Pure. Appl. Math.*, **2(2)**, 213-230 (2009).
- [2] F. Brooks, “Indefinite cut sets for real functions”, *Amer. Math. Monthly*, **78**, 1007-1010 (1971).
- [3] M. Caldas and S. Jafari, “Some properties of contra- $\beta$ -continuous functions”, *Mem. Fac. Sci. Kochi. Univ.*, **22**, 19-28 (2001).

- [4] H.H. Corson and E. Michael, "Metrizability of certain countable unions", *Illinois J. Math.*, **8**, 351-360 (1964).
- [5] J. Dontchev, "The characterization of some peculiar topological space via  $\alpha$ - and  $\beta$ -sets", *Acta Math. Hungar.*, **69(1-2)**, 67-71 (1995).
- [6] J. Dontchev, "Contra-continuous functions and strongly S-closed space", *Intrnat. J. Math. Math. Sci.*, **19(2)**, 303-310 (1996).
- [7] J. Dontchev, and H. Maki, "On sg-closed sets and semi- $\lambda$ -closed sets", *Questions Answers Gen. Topology*, **15(2)**, 259-266 (1997).
- [8] E. Ekici, "On contra-continuity", *Annales Univ. Sci. Bodapest*, **47**, 127-137 (2004).
- [9] E. Ekici, "New forms of contra-continuity", *Carpathian J. Math.*, **24(1)**, 37-45 (2008).
- [10] A.I. El-Magbrabi, Some properties of contra-continuous mappings, *Int. J. General Topol.*, **3(1-2)**, 55-64 (2010).
- [11] M. Ganster and I. Reilly, "A decomposition of continuity", *Acta Math. Hungar.*, **56(3-4)**, 299-301 (1990).
- [12] S. Jafari and T. Noiri, "Contra-continuous function between topological spaces", *Iranian Int. J. Sci.*, **2**, 153-167 (2001).
- [13] S. Jafari and T. Noiri, "On contra-precontinuous functions", *Bull. Malaysian Math. Sc. Soc.*, **25**, 115-128 (2002).
- [14] M. Katětov, "On real-valued functions in topological spaces", *Fund. Math.*, **38**, 85-91 (1951).
- [15] M. Katětov, "Correction to On real-valued functions in topological spaces", *Fund. Math.*, **40**, 203-205 (1953).
- [16] E. Lane, "Insertion of a continuous function", *Pacific J. Math.*, **66**, 181-190 (1976).
- [17] N. Levine, "Semi-open sets and semi-continuity in topological space", *Amer. Math. Monthly*, **70**, 36-41 (1963).
- [18] S. N. Maheshwari and R. Prasad, "On ROs-spaces", *Portugal. Math.*, **34**, 213-217 (1975).
- [19] H. Maki, "Generalized  $\Lambda$ -sets and the associated closure operator", *The special Issue in commemoration of Prof. Kazuada IKEDA's Retirement*, 139-146 (1986).
- [20] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, "On pre-continuous and weak pre-continuous mappings", *Proc. Math. Phys. Soc. Egypt*, **53**, 47-53 (1982).
- [21] M. Mrsevic, "On pairwise R and pairwise R1 bitopological spaces", *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, **30**, 141-145 (1986).
- [22] A.A. Nasef, "Some properties of contra-continuous functions", *Chaos Solitons Fractals*, **24**, 471-477 (2005).
- [23] M. Przemski, "A decomposition of continuity and  $\alpha$ -continuity", *Acta Math. Hungar.*, **61(1-2)**, 93-98 (1993).

Received March 1, 2018