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MODELING FLOW OF INVESTOR'S INCOME IN VIEW OF OPPORTUNITY COST

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Summary. To justify the investment decision in favor of one of the two projects of different scale, modeled the dynamics of the increment of income in the alternative investment projects. According to the results computing experiment are determined the internal rates of return, is investigated dynamics of incremental income, impact of alternative costs risk on increase income of projects.

1 INTRODUCTION

Growing investment activity is of paramount importance for the development of industrial and agricultural production, and consequently in solving critical social problems of the state. In this regard, the problem of evaluating investment projects is a topical. Qualitative evaluation of the effectiveness of investment in view of the features of the economic situation and the alternative opportunities is an essential condition making an investment decision. The urgency of this direction is determined by its practical importance for the development of investment activity in the sphere of production.[1, 2]

To justify the investment decisions using the criteria described well in the economic literature.[1-5] The most common of these are the internal rate of return and net present value of the project. However, the problem of evaluating the effectiveness still remains difficult primarily because of the need to work with the expected results of projects, because of their different scales, the need to consider alternative investment options and the time structure of discount rates. Variants of investing in view of the latent (alternative) costs, by which to understand rate of return on alternative investments, the risk of which is equivalent to the risk of the project. Latent costs associated with the loss of profit, and one of the important factors (but rarely used), which characterizes the stability of the market positions of the investment project. One of the reasons that this indicator is rarely used in the analysis of effectiveness of investments, are the difficulties associated with its quantitative expression. In some papers

[3, 5 - 7], opportunity costs are used in the evaluation of effectiveness of projects, but in their calculations the authors used the empirical estimates.

In this paper, the dynamics of the increment of income (with an alternative investment subject to the market conditions) is determined by the methods of mathematical modeling.[8-10] This indicator is used for validating the decision to invest in one of the two projects.

2 PROBLEM DEFINITION

Consider the following example. The company's management must evaluate the effectiveness of investment projects A and B with different distributions of cash flows over time and take one of them. Project A is a short-duration, it lasts for T^A years; project B is a long-term, it lasts for T^B years and is likely to continue. The cost of the projects s_t are equal in magnitude and time structure, $s_t^A = s_t^B$. The incomes have different time structure. The total incomes of the projects are about the same $\sum_t c_t^B \approx \sum_t c_t^A$.

Because of the different scales of projects A and B, compare their effectiveness raises some difficulties. In the market of capitals the complexity of the problem increases, since there is the possibility of diversion of funds to an alternative project. For comparison of different assets is necessary to determine the present value of expected cash flows, discounting them at a certain rate of *i*, which characterizes the profitability of alternative projects. The impact of this rate, also called the opportunity cost of capital[3-7], on the decision to invest is very large, as it characterizes the market conditions and income growth of investor, who provided the financial flows of the project, in any market situation. Assume that the opportunity costs *i* can range from 0 to 1. To study the dynamics of income of the investor in alternative investment opportunities will present the projects as the movement of financial flows. With information about the growth of income at different opportunities, an investor can more efficiently manage the available funds.

Fig. 1 shows a diagram of the interaction of financial flows of the project, depending on the set of opportunity costs *i*. The flow of investment *S(I)*, expended on the development of the company through some time interval, causes a positive flow of *C(i)*, which serves as a source of compensation for all costs associated with the project. The increment of the investor's income, provided by the specified cash flows of the project at various investment options, is marked by the circle; financial flows *C(i)* and *S(i)* are marked as arrows.

Changing income Δy is regarded as the balance f the increment of income and cost of the project when changing the opportunity cost Δi. The increase in *y(i)* is due to the increase in expected profit *C(i)*, the decrease - due to increased investment *S(i)* when changing the opportunity cost. The function y(i) describes the dynamics of income for the project in view of the time structure of the project's financial flows. Therefore the first step in modeling increment of income projects A and B is the analysis of the structure of financial flows, which have different time and

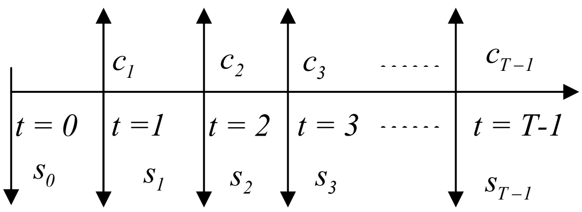

Fig. 2. The time diagram of financial flows of the projects A: $T = T^A$ and B: $T = T^B$.

quantitative structure (by the data). Fig. 2. shows the time diagram of financial flows of the projects. In what follows, $T = T^A$ and $T = T^B$ for projects A and B, respectively.

Investment flows, marked on the diagram as s_t characterize the outflow of capital, so we will consider them negative ($t = 0,...,T\text{-}1$ - the time period of investment). All financial flows to respond to investment (income in periods) will consider positive, referred to the end of period *t*. At the time diagram (Fig. 2) incomes in periods are indicated as c_t.

To evaluate the effectiveness of the project at the time $t = 0$ is used discounting cash flows (determined their present value).

$$\begin{aligned} S(i) &= \sum_{t=0}^{T-1} s_t (1+i)^{-t} \\ C(i) &= \sum_{t=0}^{T-1} c_t (1+i)^{-t} \end{aligned} . \qquad (1)$$

3 MATHEMATICAL MODEL

Mathematical description of the problem shown in the diagram of financial flows projects A and B interaction (Fig. 1) is the Cauchy problem (differential equation with initial condition).

$$\begin{aligned} &\frac{\partial y}{\partial i} = C(i) - S(i), \qquad i \in (0,\ldots 1), \\ &y(i_0) = y_0, \end{aligned} \qquad (2)$$

where *S(i)* and *C(i)* – current value of cash flows of projects A and B (1), i - opportunity cost.

The differential problem (2) subject to (1) has the form

$$\begin{aligned} &\frac{\partial y}{\partial i} = \sum_{t=0}^{T-1} c_t (1+i)^{-t} - \sum_{t=0}^{T-1} s_t (1+i)^{-t}, \quad i \in (0,\ldots 1), \\ &y(i_0) = y_0 \end{aligned} \qquad (3)$$

Introduce the notation:

$$f(i,y) = \sum_{t=0}^{T-1} c_t (1+i)^{-t} - \sum_{t=0}^{T-1} s_t (1+i)^{-t} = \sum_{t=0}^{T-1} (c_t - s_t) \cdot (1+i)^{-t} \qquad (4)$$

Subject to (4) the differential problem (3) has the form

$$\begin{aligned} &\frac{\partial y}{\partial i} = f(i,y), \quad i \in (0,\ldots 1), \\ &y(i_0) = y_0 \end{aligned} \qquad (5)$$

To solve the problem (5) is necessary to determine the initial condition – the value of the function $y(i_0)$. By the data opportunity costs vary in the range $0 \le i \le 1$, so $i_0 = 0$. Suppose that at a discount rate equal to zero, the increment of income will also be zero (zero rate gives zero growth of financial flows). In view of our arguments differential problem (4.5) takes the form

$$\frac{\partial y}{\partial i} = f(i,y), \quad i \in (0,\ldots 1),$$
$$f(i,y) = \sum_{t=0}^{T\text{-}1} (c_t - s_t)\cdot(1+i)^{-t} \tag{6}$$
$$y(0) = 0$$

The function *f(i,y)* is the net present value for each of the projects A and B.

4 RESULTS OF MATHEMATICAL MODELING

3.1 Determination of increment of income at the project

To study the dynamics of the increment of income, depending on various opportunity costs generated by the market, it is necessary to find a solution the differential problem (6). The problem (6) has an analytical solution. As a result of integration we obtain

$$y(i) = i\cdot(c_0 - s_0) + \ell n(1+i)\cdot(c_1 - s_1) - (1+i)\cdot\sum_{t=2}^{T-1}\frac{(c_t - s_t)}{(t-1)\cdot(1+i)^t} + C,\ \ i \in (0,\ldots,1) \tag{7}$$

Constant C (7) is defined by the initial condition of (6).

$$C = \sum_{t=2}^{T-1}(c_t - s_t)\cdot(t-1)^{-1} \tag{8}$$

Substituting the constant (8) in expression (7), after some transformations we obtain

$$y(i) = i\cdot(c_0 - s_0) + \ell n(1+i)\cdot(c_1 - s_1) + \sum_{t=2}^{T-1}(c_t - s_t)\cdot\frac{\left(1-(1+i)^{1-t}\right)}{(t-1)},\ \ i \in (0,\ldots,1). \tag{9}$$

Integral function *y(i)* describes the dynamics of the increment of income at the project depending on changes market conditions *i*. Thus *y(i)* shows an increase of investor's income and the amount of income on which the investor shall be refused if he invests in the alternative project at various discount rate *i*.

3.2 Determination of the internal rate of return

Singular point of the integral function *y(i)* on interval $0 \le i \le 1$ is the value of opportunity cost *i*, with a maximum value of this function. This value is determined from the extremum conditions.

$$\left[f(i,y) = \sum_{t=0}^{T-1}(c_t - s_t)(1+i)^{-t} = 0 \right]_{k=A,B} \quad . \tag{10}$$

Equation (10) has the meaning of equality to zero net present value of the project at a discount rate *i*. The solutions of (10) for the project A, $i = x_A$ and for the project B, $i = x_B$ is a effectiveness relative measure of implementation of the relevant investment project – the internal rate of return. Internal rate of return balances the discounted positive and negative flows of the project, distributed in time, and characterizes the calculated interest rate, which provides the maximum increment of investor's income.

3.3 Impact of opportunity cost on the increment of income at the project

The function *y(i)* (9) represents the sum of weighted profits $(c_t - s_t)$ for the periods t. Denote the weights coefficients of profit in periods as $p_t(i)$.

$$p_0(i) = i, \tag{11}$$

$$p_1(i) = ln(1+i), \tag{12}$$

$$p_t(i) = \frac{1}{t-1}\left(1 - \frac{1}{(1+i)^{t-1}}\right), \quad t \geq 2 \tag{13}$$

then the function increment of income can be represented as

$$\left[y(i) = \sum_{t=0}^{T-1} (c_t - s_t) \cdot p_t(i) \right]_{k=A,B} . \tag{14}$$

Each summand of the function (14) shows the contribution of profit in period *t* to increment income, and the weighting factors (11-13) are functions of the opportunity cost *i*. Because of this, *y(i)* is the sensitivity of growth of income of the project to a change in opportunity cost *i*, and the time structure of the financial flows c_t and s_t allowance in the model (6) also makes possible to trace growth of project's income on the period *t* of profit earning.

Because the weighting coefficients (11-13) depend only on the opportunity costs *i*, they can serve as indicators of the sensitivity of income in each period to changing the opportunity costs *i*, generated by the market. Changing the weights $p_t(i)$ with increase in opportunity costs *i* and periods of profit earning *t* is shown in Fig. 3 a, b.

The most sensitive to changes in *i* is the income in the initial period (11), the weight of $p_0(i)$ depends linearly on the opportunity cost *i* and varies $0 \leq p_0(i) \leq 1$ in the range of opportunity costs $i \in (0, \ldots, 1)$ (Fig.3 a).

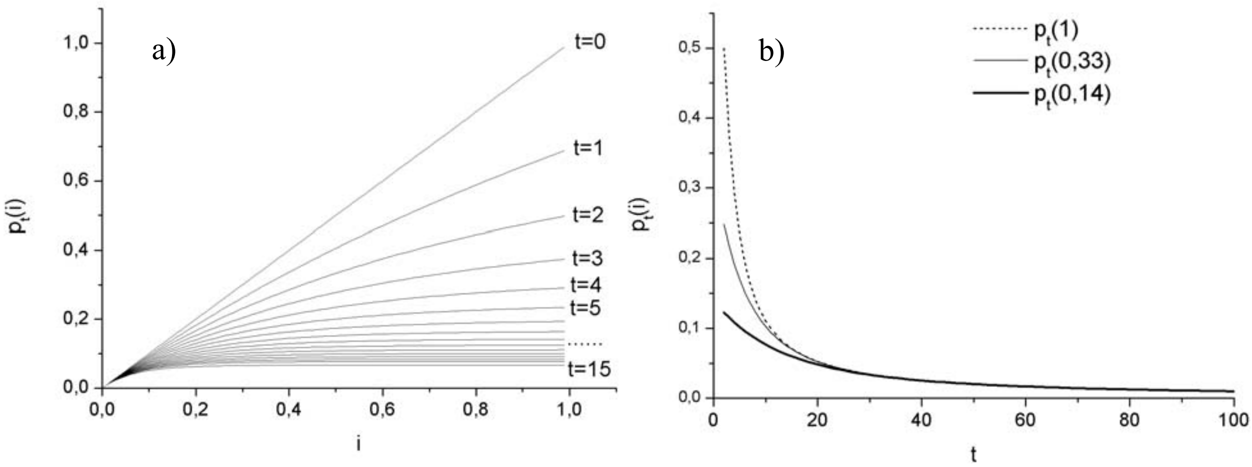

Fig. 3. Weight of profit $p_t(i)$ dependence a) on opportunity cost $0 \leq i \leq 1$; b) on number of the period *t* at *i = 0.14, 0.33, 1*.

The contribution of income in the initial period to increase of project's income increasing is directly proportional to the discount rate.

Increasing numbers of period t leads to a decrease in the sensitivity of the weighting coefficients $p_t(i)$ a change in opportunity cost i. Weight coefficient the period $t = 1$, $p_1(i)$ (12) varies from 0 ($i = 0$) to 0,693 ($i = 1$) and is a logarithmic dependence of the opportunity cost on i.

Even less change weighting coefficients (13) of each of following periods of profit. For $t \geq 2$ $p_t(i)$ represent the power dependence on the opportunity cost i with a negative exponent. For periods $t > 10$ $p_t(i)$ practically does not react to changes in opportunity costs (Fig. 3a). Weighting income coefficients $p_t(i)$ reach their maximum values at $i = 1$.

Fig. 3b shows the time dependence of the weighting coefficients $p_t(i)$ $(t \geq 2)$ for values $i = x_A = 0,33$, $i = x_B = 0,14$ (the solutions of equation (10) for projects A and B), which are the internal rate of return these projects, and for $i = 1$. Changing opportunity cost affects the weighting coefficients of periods $2 \leq t \leq 20$. For $t \geq 20$ the values of opportunity cost i have no effect on weighting coefficients, which are slowly and monotonically asymptotically approaching zero (Fig. 3b).

Analysis of the behavior of weights $p_t(i)$ the function of the increment of income (14) shows that it's influenced by the risk of market fluctuations (risk opportunity cost). The risk of fluctuations in the opportunity cost has a significant impact on the profit share in the increment of income in the periods close to the initial $t = 0$. At the same time that risk has no effect on the value of the contributions profit distant periods.

3.4. Analysis of the effectiveness of projects A and B. Consider the use of increment of income of the project as a criterion of its effectiveness. Table 1 presents data on costs and income for the periods of the projects A and B. The project A has a duration $T^A = 4$ years, and the project B – $T^B = 10$ years. In this problem, we assume that all costs of the company associated with the implementation of the project, already included in the values c_t for each project.

Project	Financial flow (m. u.)	period t (years)					
		0	1	2	3	…….	9
A	Investments s_t	9 000					
	Incomes c_t		6 000	5 000	4 000		
B	Investments s_t	9 000					
	Incjmes c_t		1 800	1 800	1 800	…….	1 800

Table 1. Investments and incomes of projects A and B.

Fig. 4 shows the function increment of income (14) calculated for projects A and B. Integral functions (solid lines on the graph) are characterized the position of each project on the market. The economic meaning of the function $y(i)$ - the share of free financial flows for each of the projects that are at risk of losing the investor, referring to alternative possibilities. The value of loss depends on the opportunity cost i, determined by the market. The increment of income up to a maximum at values of opportunity cost $i=x_A$ and $i=x_B$, calculated by solving the equation (10) for each project. The values x_A и x_B are also the internal rate of return projects. Nonlinear equations (10) were solved by Newton's method.[8, 11] Internal rate of

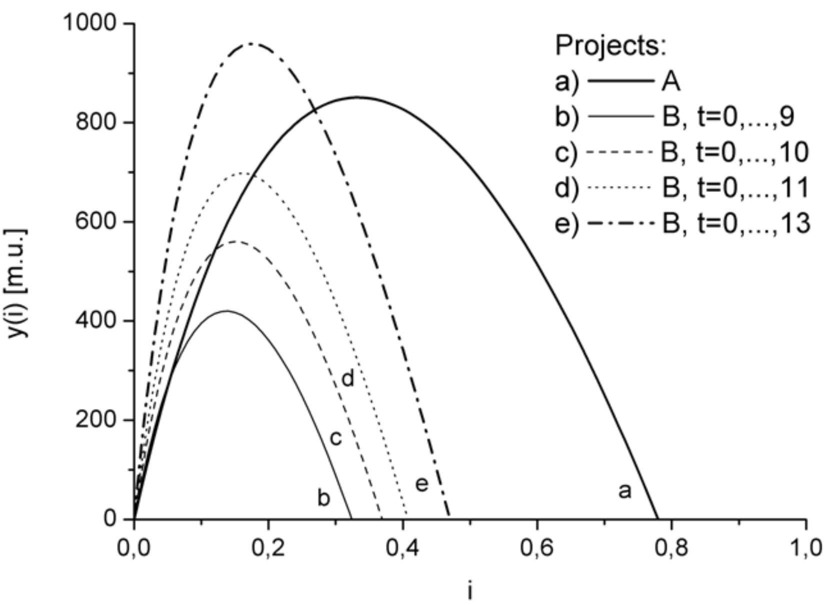

Fig. 4. The functions increment of income *y(i)* for the project A and project B with an increased number of periods *t*.

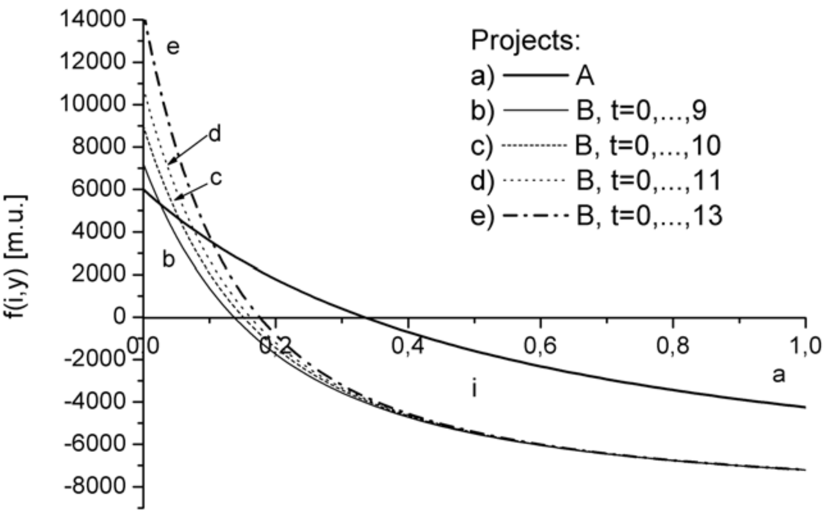

Fig. 5. Net present value for the project A and project B with an increased number of periods *t*.

return (IRR) for project A $x_A = 0,33$, for a project B – $x_B = 0,14$. As you can see, IRR of project A is significantly greater than IRR of project B. Net present value (NPV) of project A is greater than NPV of project B in almost the whole range of opportunity costs $0,02 \le i \le 1$ (Fig. 5, graphs a, b). Project B has a greater NPV in a very short range of alternative cost $0 \le i \le 0,025$. The investment decision under this, the traditional criteria of efficiency would be in favor for the project A.

But analysis of the behavior of the function *y(i)* shows that in the range $0 \le i \le 0,053$ increment of income at the project B exceeds the increment of income at project A,

$y_B(i) > y_A(i)$ (Fig. 4, graphs a, b). In the case of the project B continuation for at least one period (t = 0,…,10), there is a significant excess of increment of income $y_B(i) > y_A(i)$ in the range 0 ≤ i ≤0,1 (Fig. 4, graphs a, c). IRR will be increased only up to $x_B = 0,15$, which is still less than the x_A. Project's B NPV exceeds project's A NPV in a short range of $0 \le i \le 0,054$ (Fig. 5, graphs a, c).

A further increase in duration of the project B gives the following results:

- $t = 0,\ldots 11$- significant, at 66%, increase in increment of income $y(i)$ project B, which is $y_B(i) = 698,3$ m.u. (Fig. 4, graph d). The slight, 2%, increase in IRR to $x_B = 0,16$ (compared to base variant). The range of rates, in which can be observed excess of NPV, expanding slightly $0 \le i \le 0,075$ (Fig. 5. graphs a, d).
- $t = 0,\ldots 13$ – A project IRR is still higher than the IRR of the project B ($x_A = 0,33$, $x_B = 0,17$), but the maximum increment of income for project B is significantly higher than for the project A, $y(x_B) > y(x_A)$ (Fig. 4, plot e), inclining investment decision to the project B. If the investor refuses to project B in favor of the project A in the range of opportunity costs $0 \le i \le 0,3$, he will lose revenue, determined by the difference $|y_B(i) - y_A(i)|$. If the opportunity cost $i = x_B$, then the investor's financial losses will amount to 269,4 m.u.

Results of modeling show that stable over time, a small income for the period of the project B is more preferable to short-term project A, having a greater IRR. This conclusion was not obvious when assessing the effectiveness of the projects only criteria IRR and NPV.

4 CONCLUSIONS

There is a mathematical model of increment of income on the project with an alternative investment. The model includes the time structure of the financial flows associated with the project, and the dynamics of the opportunity cost of capital determined by the market. Obtained function that approximates the increase of income of the project. Analyzed the sensitivity of the integral increment of income to changes opportunity cost to weight coefficients of income each of the periods of the project. Found that the share of profits in the increment of income is significantly affected by the risk of opportunity cost, if the period of the profit is close to the initial. The risk of this type has no effect on the profit share in the increment of income received in the distant period.

According to the results of mathematical modeling analyzed the stability of the two projects, identified the maximum loss of profits by the alternative investment, calculated internal rate of return projects.

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