ДРУШТВО МАТЕМАТИЧАРА И ФИЗИЧАРА ЦРНЕ ГОРЕ
ПРИРОДНО-МАТЕМАТИЧКИ ФАКУЛТЕТ
УНИВЕРЗИТЕТА ЦРНЕ ГОРЕ

МАТЕМАТИКА ЦРНЕ ГОРЕ

КЊИГА XXVIII

MATHEMATICA MONTISNIGRI

VOLUME XXVIII

UDK 51 ISSN 0354 2238

ПОДГОРИЦА, 2013.

MATHEMATICA MONTISNIGRI
Vol XXVIII (2013) 5-121

CONTENTS

Mathematics

Mathematical modeling

Computer science applications

MATHEMATICA MONTISNIGRI
Vol XXVIII (2013) 17-24

THE GREATEST ORDER OF THE DIVISOR FUNCTION WITH INCREASING DIMENSION

GLEB V. FEDOROV*

* Mechanics and Mathematics Faculty
Moscow State University
Moscow, Russia
e-mail: glebonyat@mail.ru

Summary. We found the effect of the deformation of the upper limit of the divisor function, depending on the rate of growth of the dimension concerning the argument of the divisor function.

2010 Mathematics Subject Classification: 05A10, 05A16, 11B65, 11B75, 11D57, 11N37, 11N56.
Key words and Phrases: Divisor Function, Increasing Dimension, Multiplicative Function.

1. INTRODUCTION

Let $\tau_k(n)$ denote *multidimentional divisor function*, as usual, the number of solutions of the equation $x_1 \cdot x_2 \cdot \ldots \cdot x_k = n$ in positive integer $x_1, x_2, \ldots, x_k$ for fix integer n and k. We suppose $\tau_k(0) = 0$, $\tau_k(1) = 1$, $\tau_1(n) = 1$. For $k = 2$ value of $\tau_2(n) = \tau(n)$ is the number of distinct divisors a positive integer n. In general case of multidimensional divisor function $\tau_k(n)$ the number k we call *dimension*.

In 1907, S. Wigert [1] showed that

$$\max_{n \le x} \tau(n) = \exp\left(\ln 2 \frac{\ln x}{\ln \ln x} + \mathrm{O}\left(\frac{\ln x \cdot \ln \ln \ln x}{(\ln \ln x)^2} \right) \right),$$

hence it follows as a consequence of the existence of an upper limit

$$\limsup_{n \to \infty} \frac{\ln \tau(n) \ln \ln n}{\ln n} = \ln 2. \tag{1}$$

S. Ramanujan [2] in 1915 gave enough interesting and a simple proof of this relation. Similar techniques can be generalized for formula (1) for any fixed dimension $k \ge 2$ of the divisor function $\tau_k(n)$

$$\limsup_{n \to \infty} \frac{\ln \tau_k(n) \ln \ln n}{\ln n} = \ln k. \tag{2}$$

We define

$$f(n) = \frac{\ln \tau(n) \cdot \ln \ln n}{\ln 2 \cdot \ln n}, \quad \text{where} \quad n \ge 2.$$

J.L. Nicolas and G. Robin [3] established that the maximum of $f(n)$ is reached at number $n_0 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, and $\max f(n) = f(n_0) \approx 1.5379$. Also in the article [3] they proved a sharp inequality

$$\log_2 \tau(n) \le C_1 \frac{\ln n}{\ln \ln n},$$

where $n \ge 3$ and $C_1 \approx 1.5379$ – a constant that for $n = n_0$ equality holds.

The article [4] gives several interesting majorant estimates of the divisor function with $n \ge 56$:

$$\log_2 \tau(n) \le \frac{\ln n}{\ln \ln n} + C_2 \frac{\ln n}{(\ln \ln n)^2}, \quad \text{where} \quad C_2 \approx 1.9349;$$

$$\log_2 \tau(n) \le \frac{\ln n}{\ln \ln n} + \frac{\ln n}{(\ln \ln n)^2} + C_3 \frac{\ln n}{(\ln \ln n)^3}, \quad \text{where} \quad C_3 \approx 4.7624;$$

$$\log_2 \tau(n) \le \frac{\ln n}{\ln \ln n - C_4}, \quad \text{where} \quad C_4 \approx 1.39177.$$

In the article [5] author obtained the maximal order of the ratio of the number of divisors of "adjacent" binomial coefficients. For integer $k \ge 1$ and $n > \prod_{p \le k+1} p^{[\log_p k]} + k$ we proved the sharp inequality

$$T_k(n) = \frac{\tau\left(C_n^k\right)}{\tau\left(C_n^{k+1}\right)} \le \frac{\tau(k+1)}{2}, \tag{3}$$

and for each integer $k \ge 1$, there is an infinite number n that in (3) equality holds.

In this paper, we investigate the behavior of the upper limit of the divisor function on the set of natural numbers with increasing dimension. If $k = k(n) \to \infty$ for $n \to \infty$, then maximum value (in the sense of the upper limit) of the divisor function $\tau_k(n)$ differ from classical upper limit (2) at a sufficiently fast growth of dimension k, and the upper limit is achieved on the power sequences.

Recall that the behavior of the average value of the divisor function with increasing dimension is investigated. The paper [6] shows that the main term of the asymptotic formula for the mean value changes according to which the values of dimension k lie in each of the intervals

$$(\ln x)^{\frac{m}{m+1}} \le k \le (\ln x)^{\frac{m+1}{m+2}},$$

where m is an integer, $0 \le m \le 3$, and x – length of the interval of averaging.

Let p_m be the m-th prime number ($p_1 = 2$). We define the sequence of numbers $n_m = p_1 \cdot p_2 \cdot \ldots \cdot p_m$. Then the upper limit of the multidimensional divisor function $\tau_k(n)$ (in the sense of the upper limit with a fixed dimension) is achieved on a sequence $\{n_m\}$. In particular, $\tau_k(n_m) = k^m$, from the law of distribution of prime numbers it follows that

$$m = \pi(p_m) = \frac{p_m}{\ln p_m} + \mathrm{o}\left(\frac{p_m}{\ln p_m}\right) = \frac{\ln n_m}{\ln \ln n_m} + \mathrm{o}\left(\frac{\ln n_m}{\ln \ln n_m}\right), \tag{4}$$

$$\ln n_m = \sum_{m \le m} \ln p_m = p_m + \mathrm{o}(p_m).$$

We use the well-known expression for the multidimensional divisor function

$$\tau_k(p^\alpha) = \binom{\alpha + k - 1}{\alpha},$$

which by multiplicativity easily extended to all positive integers. Always under the symbol p we mean a prime number. If we record $n \vdots p$, we always consider some natural number $\alpha_p = \nu_p(n)$, $p^{\alpha_p} \| n$.

We use the symbols O and o in their usual sense when $n \to \infty$. It should be noted that when we use inequalities

$$F(n) \le G(n)\big(1 + \mathrm{o}(1)\big),$$

we consider that the parameter n takes quite large values, for which inequality

$$F(n) \le G(n)\big(1 + R(n)\big)$$

holds, where $R(n) \to 0$ as $n \to \infty$. In all such inequalities function $R(n)$ can be presented clearly.

2. PRELIMINARIES AND AUXILIARY RESULTS

LEMMA 1. (i) *The estimate*

$$\binom{k+\alpha-1}{\alpha} < e^{\alpha}\left(\frac{k}{\alpha}+\frac{1}{2}\right)^{\alpha}$$

holds for $\alpha \le k$.

(ii) *The estimate*

$$\binom{k+\alpha-1}{\alpha} < e^{k}\left(\frac{\alpha}{k}+\frac{1}{2}\right)^{k}$$

holds for $\alpha \ge k$.

PROOF. It suffices to prove only one of two points, as

$$\binom{k+\alpha-1}{\alpha} = \binom{\alpha+k-1}{k-1}$$

and approval of the items obtained from each other by replacing k and α .We prove the first inequality. Transformation of the binomial coefficient is

$$\binom{k+\alpha-1}{\alpha} = \frac{k(k+1)\cdot\dots\cdot(k+\alpha-1)}{\alpha!} = \frac{k^{\alpha}}{\alpha!}\cdot\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\cdot\dots\cdot\left(1+\frac{\alpha-1}{k}\right).$$

Note that for $1 \le i < \dfrac{\alpha}{2}$ following inequality holds

$$\left(1+\frac{i}{k}\right)\left(1+\frac{\alpha-i}{k}\right) \le \left(1+\frac{\alpha}{2k}\right)^{2}.$$

Consequently,

$$\binom{k+\alpha-1}{\alpha} \le \frac{k^{\alpha}}{\alpha!}\cdot\left(1+\frac{\alpha}{2k}\right)^{\alpha} < \frac{e^{\alpha}}{\alpha^{\alpha}}\cdot\left(k+\frac{\alpha}{2}\right)^{\alpha},$$

inasmuch as $\alpha! > \left(\dfrac{\alpha}{e}\right)^{\alpha}$. Lemma is proved.

For a positive integer n we write $\omega(n)$ and $\Omega(n)$ for the number of distinct prime factors of n and the total number of prime factors of n (including multiplicities), respectively. We define the functions $\Omega_k(n)$ and $\omega^k(n)$ as follows:

$$\Omega_k(n) = \sum_{\substack{p|n \\ \alpha_p \le k}} \alpha_p, \qquad \omega^k(n) = \sum_{\substack{p|n \\ \alpha_p > k}} 1.$$

LEMMA 2. *Let* $k = k(n) \to \infty$, *and* $\dfrac{\log_2 k}{\log_2 \log_2 n} \to 0$ *as* $n \to \infty$. *Then the following limit relation*

$$\limsup_{n\to\infty} \left(\frac{\Omega_k(n) \cdot \log_2 \log_2 n}{\log_2 n} \right) = 1$$

holds.

PROOF. This follows from two statements:

- the upper limit is reached on a sequence $\{(n_m)^k\}$, where $n_m = p_1 \cdots p_m$ – a product of consecutive primes, $p_1 = 2$;
- for any $n < (n_m)^k$ the inequality $\Omega_k(n) < \Omega_k\left((n_m)^k\right)$ holds.

Indeed, when $n = (n_m)^k$ we see that $\Omega_k(n) = k \cdot m$ and from (4), we have

$$m = \frac{\log_2 n_m}{\log_2 \log_2 n_m}\left(1 + o(1)\right) = \frac{\frac{1}{k}\log_2 n}{\log_2 \log_2 n - \log_2 k}\left(1 + o(1)\right).$$

Thus, when $n = n(m) = (n_m)^k$ we get

$$\lim_{m\to\infty} \left(\frac{\Omega_k(n) \cdot \log_2 \log_2 n}{\log_2 n} \right) = 1.$$

Lemma is proved.

LEMMA 3. *Let* $k = k(n) \to \infty$, *and* $\dfrac{\log_2 k}{\log_2 \log_2 n} \to 0$ *as* $n \to \infty$. *Then the following limit relation*

$$\limsup_{n\to\infty} \left(\frac{\omega^k(n) \cdot k \cdot \log_2 \log_2 n}{\log_2 n} \right) = 1$$

holds.

PROOF. This follows from two statements:

- the upper limit is reached on a sequence $\{(n_m)^k\}$, where $n_m = p_1 \cdots p_m$ – a product of consecutive primes, $p_1 = 2$;
- for any $n < (n_m)^{k+1}$ the inequality $\omega^k(n) < \omega^k\left((n_m)^k\right)$ holds.

The proof is similarly to Lemma 2.

3. PROOFS OF MAIN RESULTS

THEOREM 1. *Let* $k = k(n) \to \infty$ *and* $\dfrac{\log_2 k}{\log_2 \log_2 n} \to 0$ *as* $n \to \infty$. *Then the following equality holds*

$$\limsup_{n\to\infty} \frac{\log_2 \tau_k(n)\cdot \log_2 \log_2 n}{\log_2 k\cdot \log_2 n} = 1.$$

PROOF. We show that the upper limit is achieved on a sequence $\{n_m\}$ where $n_m = p_1 \cdots p_m$ – the product of consecutive primes, $p_1 = 2$. We have

$$\log_2 \tau_k(n_m) = \sum_{m=1}^{m} \log_2 \binom{k}{1} = m \log_2 k.$$

The expression (4) gives us for $n = n(m) = n_m$ the limiting relation

$$\lim_{m\to\infty} \left(\frac{\log_2 \tau_k(n)}{\log_2 k \cdot \frac{\log_2 n}{\log_2 \log_2 n}} \right) = 1.$$

Further it is necessary to show that for $n \to \infty$ the inequality is satisfied:

$$\log_2 \tau_k(n) \le \frac{\log_2 n \cdot \log_2 k}{\log_2 \log_2 n} \big(1 + \mathrm{o}(1)\big). \tag{5}$$

We write the divisor function in the form of two pieces:

$$\tau_k(n) = \prod_{\substack{p|n \\ \alpha_p \le k}} \binom{k+\alpha_p - 1}{\alpha_p} \cdot \prod_{\substack{p|n \\ \alpha_p > k}} \binom{k+\alpha_p - 1}{\alpha_p} = \Pi_1 \cdot \Pi_2.$$

We estimate the logarithm of the first product by using Lemma 1:

$$\log_2 \Pi_1 \le \sum_{\substack{p|n \\ \alpha_p \le k}} \log_2 \left(e^{\alpha_p} \left(\frac{k}{\alpha_p} + \frac{1}{2} \right)^{\alpha_p} \right) =$$

$$= \frac{1}{\ln 2} \sum_{\substack{p|n \\ \alpha_p \le k}} \alpha_p + \sum_{\substack{p|n \\ \alpha_p \le k}} \alpha_p \cdot \log_2 \left(\frac{k}{\alpha_p} + \frac{1}{2} \right) = \left(\log_2 \left(k + \frac{1}{2} \right) + \frac{1}{\ln 2} \right) \Omega_k(n).$$

According to Lemma 2 we find that

$$\log_2 \Pi_1 \le \frac{\log_2 n \cdot \log_2 k}{\log_2 \log_2 n} \big(1 + \mathrm{o}(1)\big). \tag{6}$$

Once again use of Lemma 1 gives us the estimation of the second piece:

$$\log_2 \Pi_2 \le \sum_{\substack{p|n \\ \alpha_p > k}} \log_2 \left(e^k \left(\frac{\alpha_p}{k} + \frac{1}{2} \right)^k \right) = \frac{k}{\ln 2} \omega^k(n) + k \sum_{\substack{p|n \\ \alpha_p > k}} \log_2 \left(\frac{\alpha_p}{k} + \frac{1}{2} \right).$$

Let us denote $\hat{n}=\prod_{\substack{p|n\\ \alpha_p>k}} p^{\alpha_p}$ and $N=\prod_{p|\hat{n}} p^{\beta_p}$, where $\beta_p=\left[\frac{\alpha_p}{k}\right]+1$. Then we get

$$\log_2 \Pi_2 \le \frac{k}{\ln 2}\omega^k(n)+k\sum_{p|N}\log_2\left(\beta_p+\frac{1}{2}\right)\le \frac{k}{\ln 2}\omega_k(n)+k\cdot\log_2\tau(N),$$

and $\hat{n}<N^k<\hat{n}^2$ inasmuch as $\alpha_p<k\left(\left[\frac{\alpha_p}{k}\right]+1\right)<\alpha_p+k<2\alpha_p$. From the relation (1) we have

$$\log_2\tau(N)\le \frac{\log_2 N}{\log_2\log_2 N}\big(1+\mathrm{o}(1)\big)\le \frac{2\log_2\hat{n}}{k}.$$

Lemma 3 implies that

$$\log_2\Pi_2=\mathrm{o}\left(\frac{\log_2 n\cdot\log_2 k}{\log_2\log_2 n}\right). \tag{7}$$

So, putting together estimates (6) and (7), we obtain the desired inequality (4).

The theorem is proved.

THEOREM 2. *Let* $k=k(n)\to\infty$ *and* $\frac{\log_2\log_2 n}{\log_2 k}\to 0$ *when* $n\to\infty$. *Then the following equality holds*

$$\limsup_{n\to\infty}\left(\frac{\log_2\tau_k(n)}{\log_2 k\cdot\log_2 n}\right)=1.$$

PROOF. First of all, we note that under the conditions of the theorem for any prime divisor $p|n$ inequality $\alpha_p\le k$ holds. Therefore, to estimate the divisor function, we can simply use only the second inequality in Lemma 1:

$$\log_2\tau_k(n)\le\sum_{p|n}\alpha_p\left(\frac{1}{\ln 2}+\log_2\left(\frac{k}{\alpha_p}+\frac{1}{2}\right)\right).$$

It is sufficient to prove that

$$\log_2\tau_k(n)\le\log_2 k\cdot\log_2 n\cdot\big(1+\mathrm{o}(1)\big).$$

Indeed, using the obvious inequality $\sum_{p|n}\alpha_p\le\log_2 n,$ we have

$$\sum_{p|n}\alpha_p\cdot\log_2\left(\frac{k}{\alpha_p}+\frac{1}{2}\right)\le\log_2(k+1)\cdot\sum_{p|n}\alpha_p\le\log_2 k\cdot\log_2 n\cdot\big(1+\mathrm{o}(1)\big).$$

It remains to show that the upper limit specified in the theorem is obtained. For this purpose we consider the sequence $n_m=2^m$. Then we get

$$\log_2 \tau_k(n_m) = \log_2 \binom{k+m-1}{m} = \log_2 \frac{k^m}{m!} + \sum_{j=1}^{m-1} \log_2\left(1+\frac{j}{k}\right) \geq$$

$$\geq \log_2 \frac{k^m}{m!} \geq m\log_2\left(\frac{ek}{m}\right) - \log_2 m = m\log_2 k - m\log_2 m + \frac{m}{\ln 2} - \log_2 m.$$

Since $\frac{\log_2 m}{\log_2 k} = \frac{\log_2 \log_2 n_m}{\log_2 k} \to 0$ as $m \to \infty$ then

$$\lim_{m\to\infty}\left(\frac{\log_2 \tau_k(n_m)}{m\cdot \log_2 k}\right) \geq \lim_{m\to\infty}\left(\frac{m\log_2 k - m\log_2 m + \frac{m}{\ln 2} - \log_2 m}{m\cdot \log_2 k}\right) = 1.$$

Taking into account that $m = \log_2 n_m$, we get

$$\lim_{m\to\infty}\left(\frac{\log_2 \tau_k(n_m)}{\log_2 k\cdot \log_2 n}\right) \geq 1.$$

The theorem is proved.

4. ACKNOWLEDGEMENTS

The author wishes to express deep gratitude to Professor V.N. Chubarikov for his careful attention of this work.

This research was partly supported by RFFI (grant no. 13-01-00835).

REFERENCES

[1]. S. Wigert, *Sur l'ordre de grandeur du nombre des diviseurs d'un entier*, Arkiv for Matematik, Astronomi och Fysik 3 (1907), 1-9.

[2]. S. Ramanuman, *Highly composite numbers*, Proc. London Math. Soc. (2), Vol. 14 (1915), 347-409.

[3]. J.L. Nicolas and G. Robin, *Majorations explicites pour le nombre de diviseurs de* n, Canad. Math. Bull. **26** (1983), 485-492.

[4]. G. Robin, *These d'etat*, Universite de Limoges, France, 1983.

[5]. G.V. Fedorov, *On the number of divisors of binomial coefficients*, Mathematical Notes, **93** №2 (2013), 308-316 (Original Russian Text: G. V. Fedorov, 2013, published in Matematicheskie Zametki, 2013, Vol. 93, №2, pp. 276-285).

[6]. G.V. Fedorov, *On a theorem of A.I. Pavlov*, Doklady Mathematics, **86** №2 (2012), 648-649 (Original Russian Text: G.V. Fedorov, 2012, published in Doklady Akademii Nauk, 2012, Vol. 446, №3, pp. 269-270).

Received October 1, 2013