MATHEMATICA MONTISNIGRI
Vol XXXI (2014)

AUTOMORPHISM GROUPS OF SOME CLASSES OF GRAPHS

DUŠAN S. JOKANOVIĆ[*] AND MARINA M. ZIROJEVIĆ[†]

[*]University of East Sarajevo
Production and Management Faculty Trebinje
Stepe Stepanovića bb, 89 101 Trebinje, Republic of Srpska, BiH
e-mail: dusanjok@yahoo.com, web page: http://www.fpmtrebinje.com

[†]University of East Sarajevo
Production and Management Faculty Trebinje
Stepe Stepanovića bb, 89 101 Trebinje, Republic of Srpska, BiH
e-mail: marina.zirojevic@live.com, web page: http://www.fpmtrebinje.com

Summary. There is a natural connection between the fields of algebra and graph theory. Both provide interesting ways of studing relationships among elements of a given set. An algebraic approach to graph theory can be useful in numerous ways. These arise from two algebraic objects associated with a graph: its adjacency matrix and its automorphism group. In this paper we investigate automorphism groups of common graphs and introduce a celebrated theorem of Sabidussi. Beside, by finding a graph of a given group, we can define an extremely important family of vertex-transitive graphs. The paper discusses algebraic aspects of Cayley graphs and its group of automorphism. We also give a short presentation of tools which are developed in "Wolfram Mathematica 10.0." to represent finite groups, to perform various group operations.

1 INTRODUCTION

The automorphism group of graph can be naturally defined as a group of permutations of its vertices, and so presents some basic information about permutation group.

A permutation of a set Ω is a bijective mapping $\pi:\Omega\to\Omega$. The composition $\pi_1\pi_2$ of two permutation π_1 and π_2 is the permutation obtained by applying π_1 and then π_2, thus is:

$$v(\pi_1\pi_2)=(v\pi_1)\pi_2 \text{ for each } v\in\Omega. \quad (1)$$

A permutation group on Ω is a set S of permutations of Ω satisfying the following conditions:

- S is closed under composition: if $\pi_1,\pi_2\in S$ than $\pi_1\pi_2\in S$;
- S contains the identity permautation 1 defined by $v1=v$ for $v\in\Omega$.
- S is closed under inversion, where the inverse of π is the permutation π^{-1} defined by the rule that $v\pi^{-1}=w$ if $w\pi=v$.

In this paper $\mathrm{Sym}(\Omega)$ denotes the set of all permutations of Ω, S_n denotes the symmetric group $\mathrm{Sym}(\{1,2,\ldots,n\})$.

Let S be a permutation group on Ω. The relation $\sim$ on Ω, defined by

$$v\sim w \text{ if } w=v\pi \text{ for some } \pi\in S, \quad (2)$$

2010 Mathematics Subject Classification: 11H56, 18A10, 68R10.
Key words and Phrases: Automorphism groups of graphs, Cayley graphs.

is an equivalence relation, and its equivalence classes are the orbits of S. S is transitive if it has just one orbit, thus, S is transitive if, for any $v, w \in \Omega$ there exists $\pi \in \Omega$ such that $v\pi = w$.

The stabilizer S_v of a point $v \in \Omega$ is the set

$$H = \{\pi \in S : v\pi = v\}; \tag{3}$$

it is a subgroup of S. Moreover, if w is a point in the same orbit as v, then the set

$$\{\pi \in S : v\pi = w\} \tag{4}$$

is a right coset of H in S.

2 AUTOMORPHISMS OF TYPICAL GRAPHS

Let $\Gamma := (V, E)$ be a simple, undirected graph. An automorphism of a graph is a permutation of the vertex set that preserves adjacency. The set of all automorphisms of a graph Γ, with the operation of composition of permutations, is a permutation group on vertex set that preserve adjacency. This is the automorphism group of graph Γ denoted by

$$A(\Gamma) := \{\pi \in Sym(V) : \pi(E) = E\}. \tag{5}$$

The automorphism group is an algebraic invariant of a graph. Here are some simple properties (see [1]).

Theorem 1.

(a) A graph and its complement have the same automorphism group.

(b) The automorphism group of *n* disjoint copies of graph Γ is $A(n\Gamma) = S_n[A(\Gamma)]$.

(c) $A(K_n) = S_n$.

Every graph has the trivial automorphism $id : V \to V$ defined by $id(v) = v$. Most graphs have no other automorphisms than, but many interesting graphs have many automorphisms. Erdös and Rényi [3] showed:

Theorem 2. Almost all graphs have no non-trivial automorphisms.

The smallest graph, apart from the one-vertex graph, whose automorphism group is trivial is shown in Figure 1.

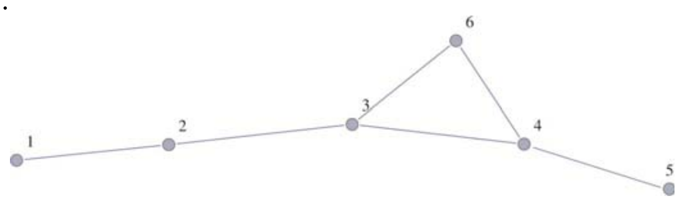


Figure 1: A graph Γ for which $|A(\Gamma)| = 1$

Cyclic graph C_4 is the opposite case, because it has 8 automorphisms. In order to define the possible automorphisms α of C_4, we will define $Z_4 := \{1, 2, 3, 4\}$. First, $\alpha(v_1) = v_i$ for some $i \in Z_4$. Then $\alpha(v_2)$ has to be a neighbour of v_i, so it is either v_{i+1} or v_{i-1}. Now $\alpha(v_4)$ has to be the neighbour of v_i that isn't $\alpha(v_2)$, so it is v_{i-1} or v_{i+1}, whichever one is not $\alpha(v_2)$. Finally, v_{i+2} is the only remaining vertex not yet in the image of α, so it must be equal to $\alpha(v_2)$. We had 4 choices for i, and then we had to choose either $\mathrm{i}+1$ or $\mathrm{i}-1$, which is 2 further choices. That gives us $4\times 2=8$ ways to choose an automorphism of C_4; thus $\left|\mathrm{A}(\mathrm{C}_4)\right|=8$.

Example 1. Let Γ be the octahedron graph. The octahedron graph are shown on next Figure. The complement graph of the octahedron graph is $3\mathrm{K}_2$. We can easily calculate that $\left|\mathrm{A}(3\mathrm{K}_2)\right|=2^3 3!=48$ since there are 3! ways to permute edges and on each edge we can either switch vertices on that edge or leave them fixed what yields another 2^3 automorphisms. Because every automorphism preserves adjacency as well as non-adjacency, a graph and its complement have the same automorphism group, so we have $\left|\mathrm{A}(\Gamma)\right|=2^3 3!=48$.

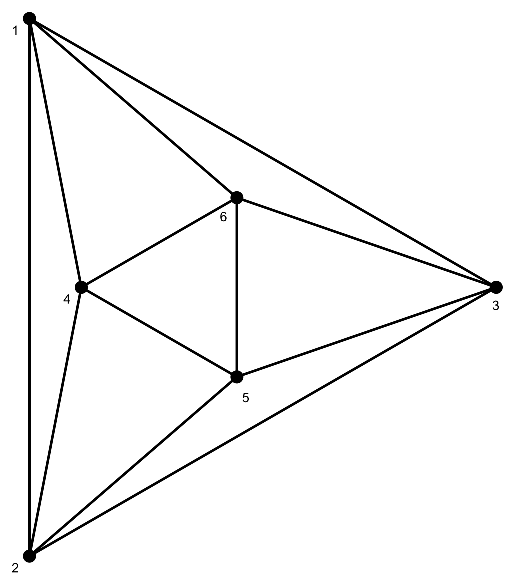


Figure 2: The Octahedron graph

Graph which has more automorphism than any other graph, relative to its size is Petersen graph P. Let $\mathrm{Z}_5=\{1,2,3,4,5\}$ and let $\mathrm{P}_2(\mathrm{Z}_5)$ be the set of all unordered pairs of elements of Z_5. Then $\mathrm{V}(\mathrm{P})=\{\mathrm{v}_{\mathrm{ij}}:\{\mathrm{i},\mathrm{j}\}\in \mathrm{P}_2(\mathrm{Z}_5)\}$. The vertices v_{ji} and v_{ij} are the same vertex. Two vertices in P are adjacent if and only if their labels are disjoint sets, ie $\mathrm{E}(\mathrm{P})=\{\mathrm{v}_{\mathrm{ij}}\mathrm{v}_{\mathrm{ji}}:\{\mathrm{i},\mathrm{j}\}\cap\{\mathrm{k},\mathrm{l}\}=\varnothing\}$. The Petersen graph with the 2-index vertex labelling are shown on the next Figure.

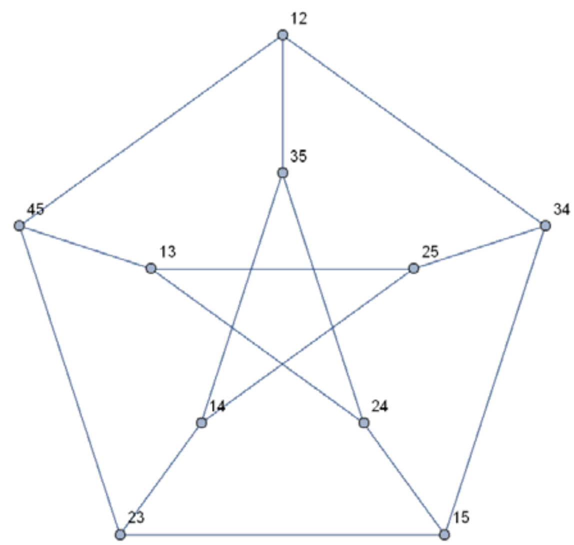


Figure 2: The Petersen graph

Any permutation of Z_5 gives an automorphism of P, and different permutations give different automorphisms. Because there are no other automorphisms of P we can say $|A(P)| = 120$.

Following results holds:

Theorem 3. The full automorphism group of the Petersen graph is isomorphic to S_5.

Proof: Let Γ be the Petersen graph. Every element $\pi \in S_5$ induces a permuatation $\hat{\pi}$ of Γ. Each of these permutations is an automorphism of Γ because for all $\pi \in S_5$, A, B are disjoint if and only if $\pi(A)$ and $\pi(B)$ are disjoint. Thus, the map $\phi : S_5 \mapsto V, \pi \mapsto \hat{\pi}$ is an injective group homomorphism into $A(\Gamma)$, and so $S_5 \cong \phi(S_5) \le A(\Gamma)$.

If we want to show that there is no other automorphisms than $\phi(S_5)$ i.e. that $\phi(S_5)$ is a full automorphism group, it is sufficed to show that $A(\Gamma) \le \phi(S_5)$. If π is any automorphism, then by composing with an appropriate permutation of {1,2,3,4,5} we may assume that the map fixes {1,2}; that means that the vertices adjacent to {1,2}, {3,4}, {3,5}, and {4,5}, must map to each other. To prove this, let $\pi \in A(\Gamma)$. We show that there exists $\hat{g} \in \phi(S_5)$ such that $\pi\hat{g} = 1$. This would imply that $\pi = \hat{g}^{-1} \in \phi(S_5)$.

Suppose $\pi : \{1, 2\} \to \{a, b\}$. Let $g_1 \in S_5$ map $a \mapsto 1$, $b \mapsto 2$. Then $\pi\hat{g}_1$ fixes the vertex 12 and hence permutes its neighbors 34, 45 and 35. We consider a few cases:

1. Suppose $\pi\hat{g}_1$ fixes all three neighbors 34,45,35. So $\pi\hat{g}_1$ permutes the neighbors of 35, and hence fixes 14 and 24, or swaps 14 and 24.

1.1. If $\pi\hat{g}_1 : 14 \mapsto 14$, then 15 is sent to a vertex adjanced to 34 and not adjanced to 14, hence 15 is fixed. Similary, all the vertex are seen to be fixed, and so $\pi\hat{g}_1 = 1$, as was to be shown.

1.2. If $\pi\hat{g}_1 : 14 \mapsto 24$, then it swaps 14 and 24. 15 is sent to a neighbor of 24($\pi\hat{g}_1$)=14, and hence 15 is sent to 25. Thus, we see that $\pi\hat{g}_1$ swaps 14 and 24, 15 and 25, and also 13 and 23, and fixes the remaining vertices. But then $\pi\hat{g}_1 = \hat{g}_2$, where $g_2 = (1,2)$. So, $\pi \in \phi(S_5)$.

2. Suppose $\pi\hat{g}_1$ fixes exactly 2 of the 3 vertices 34, 45 and 35. But this case is not possible, because if it fixes 2 of the 3 vertices, it must also fix the third vertex.
3. Suppose $\pi\hat{g}_1$ fixes exactly 1 of 34, 45 and 35, say 34. So $\pi\hat{g}_1$ swaps 45 and 35. Let $\pi\hat{g}_1$ swaps 45 and 35. Let $g_2 = (3,4)$. Then $\pi\hat{g}_1\hat{g}_2$ satisfies the condition of case (1).
4. Suppose $\pi\hat{g}_1$ fixes none of 34, 45 and 35. Say it has (34, 45, 35) as a 3-cycle. Let $g_2 = (3,4)$. Then $\pi\hat{g}_1\hat{g}_2 = (34,35)(45)$, and we are back to case (3).

In all cases if $\pi \in \mathrm{Aut}(\Gamma)$, then for some nonnegative integer r, there exists $g_1,\ldots,g_r$ such that $\pi g_1,\ldots,g_r = 1$, implying that $\pi \in \phi(S_5)$. □

3 VERTEX-TRANSITIVE GRAPHS

A graph Γ is vertex-transitive if the automorphism group of Γ acts transitively on the vertex-set of Γ. Thus for any two distinct vertices of Γ there is an automorphism mapping one to other.

Common example of vertex-transitive graphs is k-cubes Q_k. The vertex-set of Q_k is the set of all 2^k binary k-tules, with two being adjacent if they differ in precisely one coordinate position.

Theorem 4. The k-cone Q_k is vertex-transitive.

Proof. If we fixed k-tuple, then the mapping $\rho_v : x \mapsto x + v$ is a permutation of the vertices of Q_k. This mapping is an automorphism bacause the k-tuples x and y differ in precisely one coordinate position if and only if x+v and y+v differ in precisely one coordinate position. There are 2^k such permutations, and they form a subgroup H of the automorhism group of Q_k. This subgroup acts transitively on $V(Q_k)$ bacause for any two vertices x and y, the automorhism ρ_{y-x} maps x to y. □

Another example of vertex-transitive graph are circulant graphs.

Definition 1. Let $\mathbb{Z}_n$ denote the additive group of integers modulo n. If C is a subset of $\mathbb{Z}_n \setminus 0$, then construct a direct graph $\Gamma(\mathbb{Z}_n, C)$ as follows. The vertices of Γ are the elements of $\mathbb{Z}_n$ and (i,j) is an arc of Γ if and only if $j-i \in C$. The graph $\Gamma(\mathbb{Z}_n, C)$ is called a circulant graph of order n, and C is called its connection set.

The cycles are special cases of circulant grahs. The cycle C_n is a circulant graph of order n, with conection set $\{1,-1\}$. The complete and empty graphs are also circulant, with connection set $C = \mathbb{Z}_n$ and $C = 0$, respectively.

Let G be a group and let C be a subset of G that is closed under taking inverses and does not contain the identity. Then the Cayley graph $\Gamma(G,C)$ is the graph with vertex set G and edge set

$$E(\Gamma(G,C)) = \{gh : hg^{-1} \in C\}. \tag{6}$$

Theorem 5. The Cayley graph $\Gamma(G,C)$ is vertex transitive.

Proof. For each $g \in G$ the mapping

$$\rho_g : x \mapsto xg \tag{7}$$

is a permuatation of the elements of G. This is an automorphism of $\Gamma(G,C)$ because

$$(yg)(xg)^{-1} = ygg^{-1}x^{-1} = yx^{-1}, \tag{8}$$

and so $xg \sim yg$ if and only if $x \sim y$. The permutations ρ_g form a subgroup of the automorphism group of $\Gamma(G,C)$ isometric to G. This subgroup acts transitively on the vertices of $\Gamma(G,C)$ because for any two vertices g and h, the automorphism $\rho_{g^{-1}h}$ maps g to h.

4 CAYLEY GRAPHS

As we mentioned before, every Cayley graph is vertex-transitive. In fact, most small vertex transitive graphs are Cayley graphs, but there are also many families of vertex transitive graphs that are not Cayley graphs. One example of such graph is Petersen graph. He is vertex-transitive, but not Cayley graph.

Circulant graph on n vertices is a Cayley graph for the cyclic group of order n and k-cube is a Cayley graph for the elementary abelian group $\mathbb{Z}_2^k$.

Some Cayley grahs appear frequently in the literature.

The complete graphs and their complements are Cayley graphs. K_n is a Cayley graph on any group G or order n where connection set is the set of non-identity elements of the group.

The graph formed on the finite field $\mathbb{F}_q$, where $q = 1(\bmod 4)$ and the connection set is the set od quadratic residues in $\mathbb{F}_q$, is also a Cayley graph, called a Paley graph.

Definition 2. Let p be a prime number and n be a positive integer such that $p^n \equiv 1(\bmod 4)$. The graph $P = (V,E)$ with

$$V(P) = \mathbb{F}_{p^n} \text{ and } E(P) = \left\{ \{x, y\} : x, y \in \mathbb{F}_{p^n}, x - y \in \left(\mathbb{F}_{p^n}^* \right)^2 \right\} \tag{10}$$

is called a Paley graph.

The list of integers which can be considered as an order of the Paley graph starts with 5, 9, 13, 17, 25, 29, 37, 41, 49, 53, 61, 73, 81… . The Paley graph of oreder 5 is cycle C_5.

Answer to question whether the arbitrary graph is a Cayley graph gives the theorem of Sabidussi. Before the proceeding to the theorem, a definition is required.

Let G be a transitive permutation group acting on a finite set Ω. Folowing three conditions are equivalent:

1. The only element of G that fixes an element of Ω is the identity permutation;
2. $|G| = |\Omega|$;
3. For any $w_1, w_2 \in \Omega$, there is a unique element $\pi \in G$ satisfying $w_1\pi = w_2$.

A transitive permutation group G that satisfies any of the these conditions is said to be regular. We now state the theorem of Sabidussi.

Theorem 6. A graph Γ is a Cayley graph if and only if A(Γ) contains a regular subgroup.

Some of the most interesting and extensively investigated problem connected with Cayley graph is trying to determine when two graphs are isomorphic.

Let p be a prime and let $\mathbb{Z}_p^*$ denote the multiplicative group of units of $\mathbb{Z}_p$. Define the permutation $T_{a,b}$ on $\mathbb{Z}_p$ for $a \in \mathbb{Z}_p^*$ and $b \in \mathbb{Z}_p$, by $xT_{a,b} = ax + b$. Denoting equivalent permutation group as $G \equiv H$, we state theorem of Burnside.

Theorem 7. A transitive permutation group G of prime degree p is either doubly transitive or $G \equiv \left\{ T_{a,b} : a \in H < \mathbb{Z}_p^*, b \in \mathbb{Z}_p \right\}$.

Burnside's theorem has applications for circulant graphs of prime order. If $A(\Gamma)$ is doubly transitive for a graph Γ, then Γ is either complete or has no edges.

The most important result about isomorphism of Cayley graph is provided by theorem of Turner via next characterization.

Theorem 8. Let p be a prime. Two circulant graphs $\Gamma(p, C)$ and $\Gamma(p, C')$ of order p are isomorphic if and only if $C' = aC$ for some $a \in \mathbb{Z}_p^*$.

It is very difficult to determine the full automorphism group of Cayley graphs. The answer is complete known in special case od prime order circulants.

Suppose that p is a prime, and that we are given the circulant graph $\Gamma(p, C)$, which is Cayley graph on the additive group $\mathbb{Z}_p$. The graph is complete if and only if C is all of $\mathbb{Z}_p^*$ and it is empty graph if and only if C is $\varnothing$. The resulting automorphism group is symmetric group S_p.

When $\varnothing \subset C \subset \mathbb{Z}_p^*$ using Theorem 7 we obtain that $A(\Gamma)$ has the form $\left\{ T_{a,b} : a \in H < \mathbb{Z}_p^*, b \in \mathbb{Z}_p \right\}$. This implies that the stabilizer of the vertex labelled 0 is $T_{a,0}$ with $a \in H < \mathbb{Z}_p$. Thus, if there is an edge joining 0 and k in Γ, than there is an edge joining 0 and all of kH, and so the connection set C is a union of cosets of the multiplicative subgroup H of

$\mathbb{Z}_p^*$. If C is a union of cosets of the subgroup H of $\mathbb{Z}_p^*$, but not an union of cosets of any supergroup of H, then the stabilizer of 0 is $\{T_{a,0} : a \in H < \mathbb{Z}_p^*\}$ and we know precisely what $A(\Gamma)$ is.

If graph Γ is a circulant graph $\Gamma(p, C)$ and $\varnothing \subset C \subset \mathbb{Z}_p^*$, then let $e(C)$ denote the maximum even order subgroup H of $\mathbb{Z}_p^*$ for which C is an union of cosets of H. We now state the folowing result.

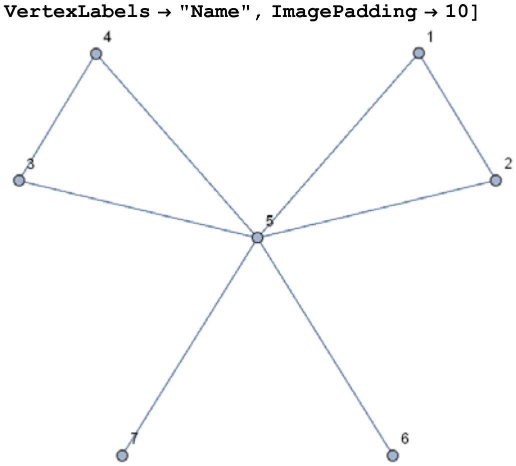
Theorem 9. Let graph Γ be a circulant graph $\Gamma(p, C)$ of prime order. If $C = \varnothing$ or $\mathbb{Z}_p^*$, then $A(\Gamma) = S_p$. Otherwise, $A(\Gamma) = \{T_{a,b} : a \in e(C), b \in \mathbb{Z}_p\}$.

5 APLICATION OF "WOLFRAM MATHEMATICA 10.0" IN ALGEBRAIC GRAPH THEORY

In closing section we investigate „Wolfram Mathematica 10.0" which contains many constructors and tools relating to group theory and algebraic graph theory. The automorphisms group of graph Γ may be computed in Mathematica using GraphAutomorphismGroup[g]. Precomputed automorphisms for many named graphs can be obtained using GraphData[graph, "Automorphisms"]. Just a small part of Wolfram Mathematica utility and his package Combinatorica will be exemplified by the following example. The example shows graph constracting in Wolfram Mathematica, finding his group of automorphisms, group generating set, group orbits and Cayley graph.

Example 2.

```
Graph[{1, 2, 3, 4, 5, 6, 7}, {1 ↔ 5, 1 ↔ 2, 2 ↔ 5, 5 ↔ 3, 3 ↔ 4, 4 ↔ 5, 5 ↔ 6, 5 ↔ 7 }
 VertexLabels → "Name", ImagePadding → 10]
```




```
GraphAutomorphismGroup[%]
PermutationGroup[
 {Cycles[{{6, 7}}], Cycles[{{3, 4}}], Cycles[{{1, 2}}], Cycles[{{1, 3}, {2, 4}}]}
```

```
GroupGenerators[
 PermutationGroup[{System`Cycles[{{6, 7}}], System`Cycles[{{3, 4}}],
   System`Cycles[{{1, 2}}], System`Cycles[{{1, 3}, {2, 4}}]}]]
{System`Cycles[{{6, 7}}], System`Cycles[{{3, 4}}],
 System`Cycles[{{1, 2}}], System`Cycles[{{1, 3}, {2, 4}}]}
GroupOrbits[PermutationGroup[{Cycles[{{6, 7}}],
   Cycles[{{3, 4}}], Cycles[{{1, 2}}], Cycles[{{1, 3}, {2, 4}}]}]]
{{1, 2, 3, 4}, {5}, {6, 7}}
CayleyGraph[PermutationGroup[{Cycles[{{6, 7}}], Cycles[{{3, 4}}],
   Cycles[{{1, 2}}], Cycles[{{1, 3}, {2, 4}}]}], VertexLabels → "Name"]
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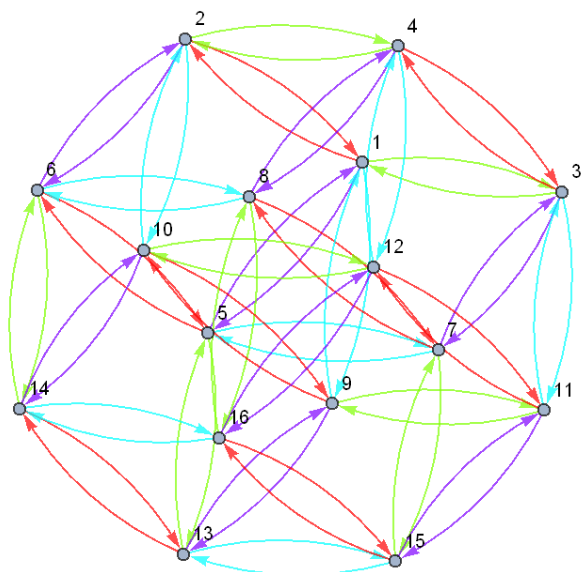

Acknowledgements: This research was supported under the grant by the Ministry of Science and Technology, Government of the Republic of Srpska.

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Received March, 20 2014