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A TOPOLOGICAL PROPERTY OF PRIVALOV SPACES ON THE UNIT DISK

ROMEO MEŠTROVIĆ* AND ŽARKO PAVIĆEVIĆ**

* Maritime Faculty
University of Montenegro
85330 Kotor, Montenegro
e-mail: romeo@ac.me

** Faculty of Science
University of Montenegro
81000 Podgorica, Montenegro
e-mail: zarkop@ac.me

Abstract. For $1 < p < \infty$, the Privalov class N^p consists of all holomorphic functions f on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ such that

$$\sup_{0\le r<1}\int_0^{2\pi}(\log^+|f(re^{i\theta})|)^p\,\frac{d\theta}{2\pi}<+\infty.$$

M. Stoll [19] showed that the space N^p with the topology given by the metric d_p defined as

$$d_p(f,g)=\Big(\int_0^{2\pi}\big(\log(1+|f^*(e^{i\theta})-g^*(e^{i\theta})|)\big)^p\,\frac{d\theta}{2\pi}\Big)^{1/p},\quad f,g\in N^p,$$

becomes an F-algebra, that is, an F-space (a complete metrizable topological vector space with the invariant metric) in which multiplication is continuous. In this paper we prove that for any $1 < p < \infty$ the space N^p is not locally bounded with respect to the topology induced by the metric d_p. The proof of this result is based on a characterization of multipliers from the spaces N^p $(1 < p < \infty)$ to the Hardy spaces $H^q(0 < q \le \infty)$.

1 INTRODUCTION AND THE MAIN RESULT

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let $\mathbb{T}$ denote the boundary of $\mathbb{D}$. Let $L^p(\mathbb{T})$ $(0 < p \le \infty)$ be the familiar Lebesgue spaces on the unit circle $\mathbb{T}$. The

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Privalov class N^p $(1 < p < \infty)$ consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\sup_{0<r<1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p \, \frac{d\theta}{2\pi} < +\infty,$$

where $\log^+ a = \max\{\log a, 0\}$ for $a > 0$ and $\log^+ 0 = 0$. These classes were first introduced by I.I. Privalov [17, p. 93], where N^p is denoted as A_q.

Notice that the above condition with $p = 1$ defines the *Nevanlinna class* N of holomorphic functions in $\mathbb{D}$ (see, e.g., [3]). Furthermore, the *Smirnov class* N^+ is the set of all functions f holomorphic on $\mathbb{D}$ such that

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f^*(e^{i\theta})| \, \frac{d\theta}{2\pi} < +\infty,$$

where f^* is the boundary function of f on T, i.e.,

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

is the radial limit of f which exists for almost every $e^{i\theta}$.

Recall that we denote by H^q $(0 < q \leq \infty)$ the classical *Hardy space* on $\mathbb{D}$, defined as the set of all holomorphic functions f on $\mathbb{D}$ for which

$$\|f\|_q^{\max\{1,q\}} := \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^q \, \frac{d\theta}{2\pi} < +\infty.$$

Further, H^∞ is the *space of all bounded holomorphic functions* on $\mathbb{D}$ with the supremum norm $\|\cdot\|_\infty$ defined as

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|, \quad f \in H^\infty.$$

We refer [3] for a good reference on the spaces H^q and N^+.

It is known (see [16]) that

$$N^q \subset N^p \ (q > p), \quad \cup_{p>0} H^p \subset \cap_{p>1} N^p, \quad \text{and} \quad \cup_{p>1} N^p \subset N^+,$$

where the above containment relations are proper.

The study of the spaces N^p $(1 < p < \infty)$ was continued in 1977 by M. Stoll [19] (with the notation $(\log^+ H)^\alpha$ in [19]). Further, the topological and functional properties of these spaces were studied by C.M. Eoff ([4] and [5]), N. Mochizuki [16], Y. Iida and N. Mochizuki [6], Y. Matsugu [7], J.S. Choa [1], J.S. Choa and H.O. Kim [2], A.K. Sharma and S.-I. Ueki [18], and in works [8]–[15] of authors of this paper; typically, the notation varied and Privalov was mentioned in [7], [13], [14], [15] and [18]. For example, it is proved in [9, Theorem] that the space N^p $(1 < p < \infty)$ does not have the Hahn-Banach approximation property, and hence it does not have the Hahn-Banach separation property. Furthermore, the spaces N^p are not locally convex [9, Corollary].

In particular, the functional, topological and algebraic properties of the spaces N^p and their Fréchet envelopes were recently investigated in [10], [13] and [15].

Stoll [19, Theorem 4.2] showed that the space N^p (with the notation $(\log^+ H)^\alpha$ in [19]) with the topology given by the metric d_p defined by

$$d_p(f,g) = \Big(\int_0^{2\pi} \big(\log(1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|)\big)^p \frac{d\theta}{2\pi}\Big)^{1/p}, \quad f,g \in N^p, \tag{1}$$

becomes an F-algebra, that is, a topological vector space whose topology is given by a complete, translation invariant metric in which multiplication is continuous.

It was also investigated in [19] the containing Fréchet space $F_{1/p}$ for N^p with $p > 1$. It is proved in [19, Corollary 4.4] that the restriction of every continuous linear functional on $F_{1/p}$ to N^p forms a continuous linear functional on N^p. It is remained an open question whether the spaces N^p and $F_{1/p}$ have the same dual spaces, in the sense that every continuous linear functional on N^p is a restriction to one on $F_{1/p}$. In 1999 R. Meštrović and A.V. Subbotin [14, Theorem 2] gave a positive answer to this question. In order to prove a complete characterization of a *topological dual space* of N^p (the set of all linear functionals that are continuous with respect to the metric topology d_p) it was used a description of multipliers from the spaces N^p to the Hardy spaces H^q $(0 < q \leq \infty)$ established in [14, Theorem 1].

Let $p > 1$ and $0 < q \leq \infty$ be arbitrary fixed. A sequence $\{\lambda_n\}_{n=0}^\infty := \{\lambda_n\}$ of complex numbers is said to be a *multiplier* from the space N^p to the Hardy space H^q if for each function $f \in N^p$ with the Taylor expansion $f(z) = \sum_{n=0}^\infty a_n z^n$, the function g defined on $\mathbb{D}$ as $g(z) = \sum_{n=0}^\infty \lambda_n a_n z^n$ belongs to H^q. According to this definition, every multiplier $\{\lambda_n\}$ from N^p to H^q can be considered as an induced linear operator Λ from N^p to H^q defined as

$$\Lambda : \sum_{n=0}^\infty a_n z^n \longmapsto \sum_{n=0}^\infty \lambda_n a_n z^n. \tag{2}$$

Theorem A ([14, Theorem 1]; also see [8, Chapter 2, Section 2.2, Theorem 2.3]). *Suppose $\{\lambda_n\}$ is a multiplier from N^p to the Hardy space H^q $(0 < q \leq \infty)$. Then the linear operator Λ defined from N^p to H^q by (2) is continuous. Thus, Λ maps bounded subsets of N^p into bounded subsets of H^q.*

The characterization of multipliers from the spaces N^p to the spaces H^q given in [14] can be reformulated as follows.

Theorem B ([14, Theorem 1]). *Let $0 < q \leq \infty$ and $1 < p < \infty$. In order that a sequence $\{\lambda_k\}$ of complex numbers to be a multiplier from N^p into H^q, it is necessary and sufficient that*

$$\lambda_k = O\left(\exp(-ck^{1/(p+1)})\right) \tag{3}$$

for some positive constant c.

Remark. Note that the assumption of Theorem B contains q, but this is not the case for the growth estimate (3).

Notice that the proof of Theorem A given in [14] is based on Lemmas 1–3 from [14] by using Yanagigara's technique applied in [20] for characterizing multipliers from N^+ to the Hardy spaces H^q.

Recall that a subset L of a topological vector space X is *bounded* if for every neighborhood V of zero there is a $\alpha_0 > 0$ such that $\alpha L \subset V$ for all $\alpha \in \mathbb{C}$ such that $|\alpha| \leq \alpha_0$. Furthermore, a topological vector space X is *locally bounded* if it does not contain none base of neighborhood of zero consisting only bounded sets. Theorems A and B are used here to prove the following result (see [8, Chapter 2, Section 2.3, Corollary 3.1]).

Theorem 1.1. *The space N^p is not locally bounded. This means that none ball $B(c) = \{f \in N^p : d_p(f, 0) < c\}$ is not bounded subset of N^p.*

2 PROOF OF THEOREM 1.1

Proof of Theorem 1.1 is based on the following six lemmas.

Lemma 2.1. *If a sequence of functions in N^p converges with respect to the metric d_p, then this sequence converges uniformly on each compact subset of the unit disk $\mathbb{D}$.*

Proof. The assertion immediately follows from the inequality (2) in [16] given for $f \in N^p$ by

$$\log(1 + |f(z)|) \leq 2^{1/p} d_p(f, 0)(1 - |z|)^{-1/p} \quad (z \in \mathbb{D}).$$

□

Lemma 2.2. ([3, Theorem 6.4, p. 98]). *Suppose*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^q, \quad 0 < q \leq 1.$$

Then

$$a_n = o(n^{1/q-1}), \tag{4}$$

as well as

$$|a_n| \leq C n^{1/q-1} \|f\|_q. \tag{5}$$

Lemma 2.3. *Suppose $\{\lambda_n\}$ is a multiplier from N^p to the Hardy space H^q $(0 < q \leq \infty)$. Then the linear operator Λ defined from N^p to H^q by (2) is continuous. Thus, Λ maps bounded subsets of N^p into bounded subsets of H^q.*

Proof. According to Lemma 2.1, if a sequence $\{f_n\}$ in N^p converges to some fubction $f \in N^p$ in N^p, then $\{f_n(z)\}$ converges uniformly to $f(z)$ on each compact subset $|z| \leq r < 1$. Hence, if $f_n(z) = \sum_{k=0}^{\infty} a_k^{(n)} z^k$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then

$$a_k^{(n)} \to a_k \ \ (k = 0, 1, \ldots), \ \text{ if } \ f_n \to f \ \text{ in } \ N^p \ \text{ as } \ n \to \infty. \tag{6}$$

Let $g_n(z) = \sum_{k=0}^{\infty} b_k^{(n)} z^k$ be a sequence in H^q and let $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be a function in H^q such that $g_n \to g$ in H^q as $n \to \infty$. By [2, the inequality (9) in Theorem 6.4], we see that

$b_k^{(n)} \to b_k$ $(k = 0, 1, \ldots)$ as $n \to \infty$. This together with (6) immediately yields that Λ is a closed operator. Hence, by the Closed Graph Theorem, Λ is a continuous operator, and so Λ maps bounded subsets of N^p onto bounded subsets of H^q. □

Lemma 2.4. ([20, Lemma 2 and Remark 3]). *Let*

$$\exp\left(\frac{c}{2} \cdot \frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} a_n(c) z^n, \quad 0 < c \leq 1.$$

Then

$$\log |a_n(c)| \geq \sqrt{cn} + O(\log n) + O(\log c)$$

In particular, if $\{c_k^*\}$ *is a sequence of positive numbers such that*

$$\frac{1}{k^{1/(p+1)}} \leq c_k^* \leq 1,$$

then

$$\log |a_k(c_k^*)| \geq \sqrt{c_k^* k}(1 + o(1)). \tag{7}$$

Lemma 2.5. *Let* $\{c_k\}$ *and* $\{r_k\}$ *be sequences of positive numbers such that* $c_k \downarrow 0$ *and* $r_k \uparrow 1$ *as* $k \to \infty$ *and* $r_k \geq 1/2$, $k = 1, 2, \ldots$. *Define*

$$f_k(z) = \exp\left(c_k(1-r_k)^{1-1/p}\frac{1+r_k z}{1-r_k z}\right), \quad z \in \mathbb{D},\ k = 1, 2, \ldots.$$

Then a sequence of functions $\{f_k\}$ $(k = 1, 2, \ldots)$, *is a bounded subset of* N^p.

Proof. Let $\{\varepsilon_k\}$ and $\{\delta_k\}$ be sequences of positive numbers such that $\varepsilon_k \downarrow 0$, $\delta_k \downarrow 0$ as $k \to \infty$ and

$$\frac{1-r_k^2}{1+r_k^2 - 2r\cos\theta} \leq 1 \quad \text{for} \quad |\theta| \geq \varepsilon_k \quad \text{and} \quad r \geq r_k, \quad \text{for all} \quad k = 1, 2, \ldots. \tag{8}$$

For given neighborhood

$$V = \{g \in N^p : \, d_p(g, 0) < \eta\}$$

of zero in N^p, choose $m \in \mathbb{N}$ for which

$$\log^p(1+\delta_m) + 2^p \pi^{-1} \varepsilon_m \log^p 2 + 2^{p-1} C c_m^p < \eta^p, \tag{9}$$

where C is a positive constant also satisfying (11). Next assume α_0, $0 < \alpha_0 < 1$, such that

$$\alpha_0 \exp\frac{1+r_m}{1-r_m} \leq \delta_m, \quad \text{and thus} \quad \alpha_0 e \leq \delta_m. \tag{10}$$

Then for all $k \in \mathbb{N}$ with $k \leq m$ holds

$$|\alpha_0 f_k^*(e^{i\theta})| \leq \alpha_0 \exp\frac{1+r_k}{1-r_k} \leq \delta_m,$$

whence by (10) and (9), for $0 < \alpha \leq \alpha_0$ and $k \leq m$ we obtain

$$d_p(\alpha f_k, 0) \leq \log(1+\delta_m) < \eta.$$

Therefore, $\alpha f_k \in V$ for all $k \leq m$ and $0 < \alpha \leq \alpha_0$. By the inequality $\sin x \geq (2/\pi)x$ for $0 \leq x \leq \pi/2$, we have

$$\begin{aligned} 1 - 2r\cos\theta + r^2 &= (1-r)^2 + 4r\sin^2\frac{\theta}{2} \\ &\geq (1-r)^2 + (4r/\pi^2)\theta^2. \end{aligned}$$

Hence, for $r_k \geq 1/2$ we obtain

$$\begin{aligned} &\int\limits_{|\theta|<\varepsilon_k} \left(\log^+ |f_k^*(e^{i\theta})|\right)^p \frac{d\theta}{2\pi} \\ &= c_k^p(1-r_k)^{p-1} \int\limits_{|\theta|<\varepsilon_k} \left(\frac{1-r_k^2}{1+r_k^2-2r_k\cos\theta}\right)^p \frac{d\theta}{2\pi} \\ &< 2^p\pi^{-1}c_k^p \int\limits_0^\infty \frac{dt}{(1+2\pi^{-2}t^2)^p} \qquad \left(t = \frac{\theta}{1-r_k}\right) \\ &= Cc_k^p, \end{aligned} \tag{11}$$

where the constant C does not depend on k. Now from (8), (9), (11) and the inequality $\log^p(1+|x|) \leq 2^{p-1}\left((\log 2)^p + (\log^+|x|)^p\right)$, we find that for all $k > m$ and $0 < \alpha \leq \alpha_0$

$$\begin{aligned} \left(d_p(\alpha f_k, 0)\right)^p &= \int\limits_0^{2\pi} \log^p\left(1 + |\alpha f_k^*(e^{i\theta})|\right) \frac{d\theta}{2\pi} \\ &= \int\limits_{|\theta|\geq\varepsilon_k} + \int\limits_{|\theta|<\varepsilon_k} \\ &\leq \log^p(1+\alpha e) + 2^{p-1} \int\limits_{|\theta|<\varepsilon_k} \left(\log^p 2 + \left(\log^+|f_k^*(e^{i\theta})|\right)^p\right) \frac{d\theta}{2\pi} \\ &\leq \log^p(1+\delta_m) + 2^p\pi^{-1}\varepsilon_m \log^p 2 + 2^{p-1}Cc_m^p \\ &< \eta^p. \end{aligned}$$

Therefore, $\{\alpha f_k\} \subset V$ for every $0 < \alpha < \alpha_0$. This shows that the sequence $\{f_k\}$ forms a bounded set in N^p. □

Remark. Similarly, we can prove the converse of Lemma 2.5, i.e., if a sequence $\{f_k\}$ is a bounded subset of N^p and $r_k \uparrow 1$ as $k \to \infty$ and $c_k > 0$, then $c_k \to 0$ as $k \to \infty$.

Lemma 2.6. *Let $0 < q \leq \infty$ and $1 < p < \infty$. Let $\{c_k\}$ and $\{r_k\}$ be sequences of positive numbers such that $c_k \downarrow 0$ and $r_k \uparrow 1$ as $k \to \infty$ and $r_k \geq 1/2$ fo all $k = 1, 2, \ldots$. Let $\{f_k\}$*

be a sequence of functions defined as

$$\begin{aligned} f_k(z) &= \exp\left(c_k(1-r_k)^{1-1/p}\frac{1+r_kz}{1-r_kz}\right) \\ &= \sum_{n=0}^{\infty} a_n^{(k)} r_k^n z^n, \quad k=1,2,\dots. \end{aligned}$$

Suppose that $\{\lambda_k\}$ is a multiplier from N^p into H^q, and let Λ be a linear operator from N^p to H^q defined by (2). If a sequence $\{\Lambda(f_k)\}$ is a bounded set in H^q by a constant L, then for all $n=1,2,\dots$

$$|\lambda_n a_n^{(k)}|r_k^n \le \begin{cases} C_q L n^{-1+1/q} & \textit{for} \quad 0<q<1, \\ C_q L & \textit{for} \quad 1\le q\le\infty, \end{cases} \tag{12}$$

where C_q is a positive constant depending only on q.

Proof. Under conditions on sequences $\{c_k\}$ and $\{r_k\}$ from Lemma 2.6, it follows by Lemma 2.5 that the sequence of functions $\{f_k\}$ defined as

$$\begin{aligned} f_k(z) &= \exp\left(c_k(1-r_k)^{1-1/p}\frac{1+r_kz}{1-r_kz}\right) \\ &= \sum_{n=0}^{\infty} a_n^{(k)} r_k^n z^n, \quad k=1,2,\dots. \end{aligned}$$

forms a bounded subset of N^p. Since by Lemma 2.3, the operator Λ is continuous, we conclude that the sequence $\{\Lambda(f_k)\}$ must be a bounded set in H^q. Assume that the sequence $\{\Lambda(f_k)\}$ is a bounded set in the space H^q by a constant L. As

$$\Lambda(f_k) = \sum_{n=0}^{\infty} \lambda_n a_n^{(k)} r_k^n z^n,$$

from Lemma 2.2 and ([3, Theorem 6.1, p. 94]), for all $n=0,1,2\dots$ we obtain

$$|\lambda_n a_n^{(k)}|r_k^n \le \begin{cases} C_q L n^{-1+1/q} & \text{for} \quad 0<q<1, \\ C_q L & \text{for} \quad 1\le q\le\infty, \end{cases}$$

where C_q is a positive constant depending only on q. The above inequality is in fact the desired inequality (12). □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. From the proof of the inequality (11) of Lemma 2.5 it follows that there is a positive constant b depending only on p, such that for the Poisson kernel $P_r(\theta,t) = (1-r^2)/(1+r^2-2r\cos(\theta-t))$ holds

$$\int_0^{2\pi} (P_r(\theta,t))^p \,\frac{d\theta}{2\pi} \le \frac{b}{(1-r)^{p-1}}\,. \tag{13}$$

Suppose that a ball $B(c)$ with radius c is bounded in N^p. Then choose numbers $\varepsilon > 0$, $\delta > 0$ and $a > 0$ such that

$$0 < \varepsilon^p + 2^{p-1}(\log 2)^p \varepsilon \pi^{-1}(1 + 2^{p-1}) + 4^{p-1} a^{2p} b < c^p, \tag{14}$$

$$|e^{\xi} - 1| < \varepsilon \quad \text{whenever} \quad |\xi| < \delta, \tag{15}$$

and

$$\frac{2a^2}{\sin \varepsilon} < \delta. \tag{16}$$

Define the function f_r on $\mathbb{D}$ as

$$f_r(z) = \exp\left(a^2(1-r)^{\frac{p-1}{p}} \frac{1+rz}{1-rz}\right) - 1 \quad \text{for each} \quad 0 < r < 1. \tag{17}$$

It is obvious that each function f_r with $0 < r < 1$ is a bounded holomorphic function on $\mathbb{D}$, and hence f_r belongs to N^p. Moreover, if $z = \rho e^{i\theta}$ for $|\theta| \geq \varepsilon$, then $|1-rz| \geq \sin|\theta| \geq \sin \varepsilon$. From this together with (15) we obtain

$$a^2(1-r)^{\frac{p-1}{p}} \left|\frac{1+rz}{1-rz}\right| < \frac{2a^2}{|1-rz|} \leq \frac{2a^2}{\sin \varepsilon} < \delta. \tag{18}$$

Using the inequality $(\log^+|x-1|)^p \leq 2^{p-1}((\log^+|x|)^p + (\log 2)^p)$ and the fact that

$$\begin{aligned} \mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) &= \frac{1-r^2}{1-2r\cos\theta + r^2} \\ &= P_r(\theta, 0), \end{aligned}$$

we find that

$$(\log^+|f_r(e^{i\theta})|)^p \leq 2^{p-1}\left(a^{2p}(1-r)^{p-1}\left(P_r(\theta,0)\right)^p + (\log 2)^p\right). \tag{19}$$

Further, by (13)–(19) and the inequality

$$\log^p(1+|x|) \leq 2^{p-1}\left((\log 2)^p + (\log^+|x|)^p\right),$$

we obtain

$$\begin{aligned}
(d_p(f_r,0))^p &= \int_0^{2\pi} \log^p\left(1+|f_r(e^{i\theta})\right)\frac{d\theta}{2\pi}\\
&= \int_{|\theta|\geq\varepsilon} + \int_{|\theta|<\varepsilon}\\
&< \log^p(1+\varepsilon)\\
&\quad +2^{p-1}\times\left(\int_{|\theta|<\varepsilon}(\log 2)^p\frac{d\theta}{2\pi}+\int_{|\theta|<\varepsilon}(\log^+|f_r(e^{i\theta})|)^p\frac{d\theta}{2\pi}\right)\\
&\leq \varepsilon^p+2^{p-1}(\log 2)^p\varepsilon\pi^{-1}+2^{p-1}2^{p-1}\\
&\quad\times\left(a^{2p}(1-r)^{p-1}\int_{|\theta|<\varepsilon}(P_r(\theta,0))^p\frac{d\theta}{2\pi}+\int_{|\theta|<\varepsilon}(\log 2)^p\frac{d\theta}{2\pi}\right)\\
&\leq \varepsilon^p+2^{p-1}(\log 2)^p\varepsilon\pi^{-1}(1+2^{p-1})+4^{p-1}a^{2p}b<c^p.
\end{aligned}$$

From the above inequality we see that $\{f_r : 0 \leq r < 1\} \subset B(c)$. Therefore, the assumption that the ball $B(c)$ is bounded in N^p, by Lemma 2.3, it follows that every multiplier $\Lambda = \{\lambda_n\}$ maps the set $\{f_r : 0 \leq r < 1\}$ to some bounded subset of H^∞. Hence, if $f_r(z) = \sum_{n=0}^{\infty} a_n r^n z^n$, $z \in \mathbb{D}$, then the estimate (12) yields

$$|\lambda_n a_n r^n| \leq L = L(\Lambda) \tag{20}$$

for each r with $0 \leq r < 1$, where L is a positive constant depending on Λ. Using the notations from Lemma 2.4, by this lemma we have

$$|a_n| = a_n\left(2a^2(1-r)^{\frac{p-1}{p}}\right) \geq \exp\left(a(1-r)^{\frac{p-1}{2p}}\sqrt{2n}\,(1+o(1))\right) \tag{21}$$

for a constant a satisfying the conditions (13)–(15). Therefore, (20) and (21) immediately imply that

$$|\lambda_n| \leq Lr^{-n}\exp\left(-a(1-r)^{\frac{p-1}{2p}}\sqrt{2n}(1+o(1))\right) \quad \text{for each} \quad 0 \leq r < 1.$$

By setting $r = 1 - a^2/n^{\frac{p}{p+1}}$, from the previous inequality we obtain

$$\begin{aligned} |\lambda_n| &\leq L\left(1-\frac{a^2}{n^{\frac{p}{p+1}}}\right)^{-n}\exp\left(-a\frac{a\sqrt{2}}{n^{\frac{p-1}{2(p+1)}}}\sqrt{n}(1+o(1))\right)\\ &\leq L\left(\left(1-\frac{a^2}{n^{\frac{p}{p+1}}}\right)^{-\frac{n^{\frac{p}{p+1}}}{a^2}}\right)^{a^2n^{\frac{1}{p+1}}}\exp\left(-a^2\sqrt{2}n^{\frac{1}{p+1}}\right)\\ &\leq L\exp\left(a^2n^{\frac{1}{p+1}(1+o(1))}\right)\exp\left(-a^2\sqrt{2}n^{\frac{1}{p+1}}\right)\\ &= L\exp\left(-a^2(\sqrt{2}-1)n^{\frac{1}{p+1}}(1+o(1))\right)\\ &< L\exp\left(-0.3a^2n^{\frac{1}{p+1}}\right). \end{aligned}$$

This shows that every multiplier $\Lambda = \{\lambda_n\}$ from N^p into H^q satisfies the condition

$$\lambda_n = O\left(\exp\left(-0.3a^2n^{\frac{1}{p+1}}\right)\right). \tag{22}$$

On the other hand, the sequence $\Lambda^* = \{\lambda_n^*\}$ defined as $\lambda_n^* = \exp\left(-0.2a^2n^{\frac{1}{p+1}}\right)$ ($n = 0, 1, 2, \ldots$) is by Theorem B, also a multiplier from N^p into H^q. This contradicts (22), and the proof of Theorem 1.1 is now complete. □

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