MATHEMATICA MONTISNIGRI
Vol XXXII (2015) 119-139

CONSTRUCTION OF THE SOLUTION OF THE BOUNDARY VALUE PROBLEM WITH ONE DELAY AND TWO POTENTIALS AND ASIMPTOTIC OF EIGENVALUES

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Summary. This paper deals with second-order differential operators with one constant delay and two potentials. We consider the boundary value problem $L = L(q_1(x), q_2(x), \tau_2, h, H)$:

$$-y''(x) + q_1(x)y(x) + q_2(x)y(x-\tau_2) = \lambda y(x), x \in [0,\pi]$$
$$y(x-\tau_2) \equiv 0, x \in [0,\tau_2),$$
$$y'(0) - hy(0) = 0$$
$$y'(\pi) + Hy(\pi) = 0,$$

and by the method of successive approximation we construct the solution of the differential equation under the initial condition. Then we determine the characteristic function of the boundary value problem and we study asymptotic of eigenvalues of operator L.

1 INTRODUCTION

Inverse spectral problems for differential operators with delay have not been studied enough, because some of the main methods in the inverse problem theory for classical Sturm-Liouville operators, such as transformation operator method and method of spectral mappings, are not suitable for differential operators with delay. The main results for classical Sturm-Liouville operators are presented in [1]−[3], while some of the results for differential operators with delay can be found in papers [4]−[7].The boundary value problem for the differential operator with two constant delays τ_1 and τ_2 under the condition $\tau_1 = 2\tau_2$, $k_0 = 2$ is studied in [8]. Construction of the solution of the boundary value problem with two constant delays τ_1 and τ_2 under the condition $\tau_1 = k_0\tau_2$, asymptotic of eigenvalues and first regularized trace of the operator are presented in [9].

2010 Mathematics Subject Classification: Primary 34B24; Secondary 34L05,57E05
Key words and Phrases: Differential operators with delay; Boundary value problem; Characteristic function; Asymptotic of eigenvalues

In this paper we deal with the boundary value problem withone delay and two potentials. In Section 2. we construct the solution of the differential equation under the initial condition by the method of successive approximation and determine the characteristic function of the operator L. In Section 3. we study the asymptotic of zeros of the characteristic function in detail. That will be the base for further consideration of the inverse problems for this class of operators by new method based on direct relations between eigenvalues and Fourier's coefficients.

2 CONSTRUCTION OF THE SOLUTION AND DETERMINING OF THE CHARACTERISTIC FUNCTION

We consider the boundary value problem $L = L(q_1(x), q_2(x), \tau_2, h, H)$:

$$-y''(x) + q_1(x)y(x) + q_2(x)y(x-\tau_2) = \lambda y(x), \lambda = z^2 \tag{1}$$

$$y(x-\tau_2) \equiv 0, x \in [0,\tau_2), \tag{2}$$

$$y'(0) - hy(0) = 0 \tag{3}$$

$$y'(\pi) + Hy(\pi) = 0 \tag{4}$$

whereas

$$k_0\tau_2 \leq \pi < (k_0+1)\tau_2 \tag{5}$$

Firstly, we will determine the integral equation equivalent to the boundary value problem.

Lemma 2.1. The boundary value problem (1)-(3) for $x \in (\tau_2, \pi]$ is equivalent to the integral equation

$$y(x,z) = cosxz + \frac{h}{z}sinzx + \frac{1}{z}\int_0^x q_1(t_1)sinz(x-t_1)y(t_1,z)dt_1 + \frac{1}{z}\int_{\tau_2}^x q_2(t_1)sinz(x-t_1)y(t_1-\tau_2,z)dt_1, \tag{6}$$

while for $x \in (0,\tau_2]$, it is equivalent to the integral equation

$$y(x,z) = cosxz + \frac{h}{z}sinzx + \frac{1}{z}\int_0^x q_1(t_1)sinz(x-t_1)y(t_1,z)dt_1. \tag{7}$$

Proof. Using the method of variation of constants we get the integral equation (6), and then from (6) and (2), we get (7).

Let us introduce the following:

$$b_{s^2}(x,z) = \int_0^x q_1(t_1)sinz(x-t_1)sinzt_1dt_1;\ b_{s,c}(x,z) = \int_0^x q_1(t_1)sinz(x-t_1)coszt_1dt_1,$$

$$b_{s^{l+1}}(x,z)=\int_0^x q_1(t_1)\sin z(x-t_1)b_{s^l}(t_1,z)dt_1,\ l=2,3,\ldots$$
$$b_{s^l,c}(x,z)=\int_0^x q_1(t_1)\sin z(x-t_1)b_{s^{l-1},c}(t_1,z)dt_1,\ l=2,3,\ldots$$
$$b_{s^2,\tau_2}(x,z)=\int_{\tau_2}^x q_2(t_1)\sin z(x-t_1)\sin z(t_1-\tau_2)dt_1,$$
$$b_{s,c,\tau_2}(x,z)=\int_{\tau_2}^x q_2(t_1)\sin z(x-t_1)\cos z(t_1-\tau_2)dt_1,$$
$$b_{s^l,c,l\tau_2}(x,z)=\int_{l\tau_2}^x q_2(t_1)\sin z(x-t_1)b_{s^{l-1},c,(l-1)\tau_2}(t_1-\tau_2,z)dt_1,\qquad l=2,3,\ldots$$
$$b_{s^{l+1},l\tau_2}(x,z)=\int_{l\tau_2}^x q_2(t_1)\sin z(x-t_1)b_{s^l,(l-1)\tau_2}(t_1-\tau_2,z)dt_1,\qquad l=2,3,\ldots$$
$$b^{(1,2)}_{s^2,c,\tau_2}(x,z)=\int_{\tau_2}^x q_1(t_1)\sin z(x-t_1)b_{s,c,\tau_2}(t_1,z)dt_1,$$
$$b^{(1,2)}_{s^3,\tau_2}(x,z)=\int_{\tau_2}^x q_1(t_1)\sin z(x-t_1)b_{s^2,\tau_2}(t_1,z)dt_1,$$
$$b^{(2,1)}_{s^2,c,\tau_2}(x,z)=\int_{\tau_2}^x q_2(t_1)\sin z(x-t_1)b_{s,c}(t_1-\tau_2,z)dt_1,$$
$$b^{(2,1)}_{s^3,\tau_2}(x,z)=\int_{\tau_2}^x q_2(t_1)\sin z(x-t_1)b_{s^2}(t_1-\tau_2,z)dt_1, \tag{8}$$

Let $S_l(l-k,k)$ denote a set of all permutations with repetition with $l-k$ 1's and k 2's, and $S_l^{(i)}(l-k,k)$ denote a subset of $S_l(l-k,k)$ of permutations beginning with $i, i=1,2$.
Let us introduce:

$$b^P_{s^l,c,k\tau_2}(x,z)\Big|_{P\in S^{(1)}{}_l(l-k,k)}=\int_{k\tau_2}^x q_1(t_1)\sin z(x-t_1)b^P_{s^{l-1},c,k\tau_2}(t_1,z)dt_1,$$
$$b^P_{s^{l+1},k\tau_2}(x,z)\Big|_{P\in S^{(1)}{}_l(l-k,k)}=\int_{k\tau_2}^x q_1(t_1)\sin z(x-t_1)b^P_{s^l,k\tau_2}(t_1,z)dt_1,\ k=1,2,\ldots k_0,\ l=k+1,\ldots \tag{8_1}$$

The permutation P in $b^P_{s^l,c,k\tau_2}(x,z)\Big|_{P\in S^{(1)}{}_l(l-k,k)}$ $\left(\text{in } b^P_{s^{l+1},k\tau_2}(x,z)\Big|_{P\in S^{(1)}{}_l(l-k,k)}\right)$ in (8_1) is formed by adding 1 at the beginning of the permutation P in $b^P_{s^{l-1},c,k\tau_2}(x,z)$ $\left(\text{in} b^P_{s^l,k\tau_2}(x,z)\right)$, where we consider that

$$b^P_{s^{l-1},c,(l-1)\tau_2}(x,z)=b_{s^{l-1},c,(l-1)\tau_2}(x,z);\ \ b^P_{s^l,(l-1)\tau_2}(x,z)=b_{s^l,(l-1)\tau_2}(x,z),\ l=2,\ldots k_0.$$

Let us also introduce:

$$b^P_{s^l,c,k\tau_2}(x,z)\Big|_{P\in S^{(2)}{}_l(l-k,k)}=\int_{k\tau_2}^x q_2(t_1)\sin z(x-t_1)b^P_{s^{l-1},c,(k-1)\tau_2}(t_1-\tau_2,z)dt_1,$$
$$b^P_{s^{l+1},k\tau_2}(x,z)\Big|_{P\in S^{(2)}{}_l(l-k,k)}=\int_{k\tau_2}^x q_2(t_1)\sin z(x-t_1)b^P_{s^l,(k-1)\tau_2}(t_1-\tau_2,z)dt_1, \tag{8_2}$$

$$k=2,\ldots k_0,\ l=k+1,\ldots$$

The permutation P in $b^P_{s^l,c,k\tau_2}(x,z)\Big|_{P\in S^{(2)}{}_l(l-k,k)}$ $\left(\text{in } b^P_{s^{l+1},k\tau_2}(x,z)\Big|_{P\in S^{(2)}{}_l(l-k,k)}\right)$ in (8_2) is formed by adding 2 at the beginning of the permutation P in $b^P_{s^{l-1},c,(k-1)\tau_2}(x,z)$ $\left(\text{in} b^{\,P}_{s^l,(k-1)\tau_2}(x,z)\right)$, where we consider that

$$b^P_{s^l,c}(x,z)=b_{s^l,c}(x,z);\ b^P_{s^{l+1}}(x,z)=b_{s^{l+1}}(x,z), l=1,2,...$$

Theorem 1.1. If $q_1,q_2\in L_2[0,\pi]$ and $k_0\tau_2\le\pi<(k_0+1)\tau_2$, then the solution of the boundary value problem (1)-(3) within the interval $(k_0\tau_2,\pi]$ has the form:

$$y(x,z)=coszx+\frac{h}{z}sinzx+\sum_{l=1}^{\infty}\frac{1}{z^l}\left(b_{s^l,c}(x,z)+\frac{h}{z}b_{s^{l+1}}(x,z)\right)+\frac{1}{z}\left(b_{s,c,\tau_2}(x,z)+\frac{h}{z}b_{s^2,\tau_2}(x,z)\right)$$
$$+\sum_{l=2}^{k_0}\frac{1}{z^l}\left(b_{s^l,c,l\tau_2}(x,z)+\frac{h}{z}b_{s^{l+1},l\tau_2}(x,z)\right)+\sum_{i=1}^{k_0}\sum_{k=i+1}^{\infty}\sum_{P\in S_k(k-i,i)}\frac{1}{z^k}\left(b^P_{s^k,c,i\tau_2}(x,z)+\frac{h}{z}b^P_{s^{k+1},i\tau_2}(x,z)\right)\tag{9}$$

Proof. We solve integral equations (6) and (7) by the method of successive aproximations, using the following reccurent formula:

$$y^{(i)}_k(x,z)=\frac{1}{z}\int_{i\tau_2}^{x}q_1(t_1)sinz(x-t_1)y^{(i)}_{k-1}(t_1,z)dt_1+$$
$$\frac{1}{z}\int_{i\tau_2}^{x}q_2(t_1)sinz(x-t_1)y^{(i-1)}_k(t_1-\tau_2,z)dt_1\,,x>i\tau_2,\tag{10}$$

$$y^{(i)}_k(x,z)=0\ \text{for}\,x\le i\tau_2,\ \ i=0,1,2,...k_0,\ \ k=0,1,2,...,$$

where

$$y^{(0)}_0(x,z)=y_0(x,z)=coszx+\frac{h}{z}sinzx;\ \ y^{(0)}_k(x,z)=y_k(x,z),k=1,2,...,$$

$$y^{(i-1)}_k(x,z)=0\ \text{for}\,i=0,k=1,2,..;\ \ y^{(i)}_{k-1}(x,z)=0\ \text{for}\,k=0,i=1,2,3...k_0.$$

In order to simplify, hereinafter we will write the values of the functions from the recurrent formula (10) only for $x>i\tau_2$, assuming that their value is 0 for $x\le i\tau_2$.
From the reccurent formula (10) it is obvious that for $i=0$ we get the solution of the integral equation within the interval $(0,\tau_2]$, for $i=0,1$ we get the solution within the interval $(\tau_2,2\tau_2]$, and for $i=0,1,...n$ we get the solution of the integral equation within the interval $(n\tau_2,(n+1)\tau_2], n=2,3...k_0$.

1. Firstly, we will determine the solution within the interval $(0,\tau_2]$. For $i=0$ from (10) we get

$$y_k(x,z)=\frac{1}{z}\int_0^x q_1(t_1)sinz(x-t_1)y_{k-1}(t_1,z)dt_1,x>\ 0,k=1,2,...$$
$$y_0(x,z)=coszx+\frac{h}{z}sinzx.$$

Then we have

$$y_1(x,z) = \frac{1}{z}\int_0^x q_1(t_1) sinz(x-t_1) y_0(t_1,z)dt_1 =$$

$$\frac{1}{z}\int_0^x q_1(t_1) sinz(x-t_1)\left(coszt_1 + \frac{h}{z} sinzt_1\right)dt_1 = \frac{1}{z}\left(b_{s,c}(x,z) + \frac{h}{z} b_{s^2}(x,z)\right),$$

and easily show that

$$y_l(x,z) = \frac{1}{z^l}\left(b_{s^l,c}(x,z) + \frac{h}{z} b_{s^{l+1}}(x,z)\right), l = 2,3,... \qquad (11)$$

From (10) and (11) we get the solution of the boundary value problem within the interval $(0,\tau_2]$ in the form:

$$y(x,z) = \sum_{k=0}^{\infty} y_k(x,z) = coszx + \frac{h}{z} sinzx + \sum_{l=1}^{\infty} \frac{1}{z^l}\left(b_{s^l,c}(x,z) + \frac{h}{z} b_{s^{l+1}}(x,z)\right). \qquad (12)$$

2. Let us now determine functions $y_k^{(1)}(x,z)$, $k = 0,1,2$ ... Since from (10) $y_{k-1}^{(1)}(x,z)$=0 for $k = 0$, we get

$$y_0^{(1)}(x,z) = \frac{1}{z}\int_{\tau_2}^x q_2(t_1) sinz(x-t_1) y_0(t_1-\tau_2,z)dt_1 =$$

$$\frac{1}{z}\int_{\tau_2}^x q_2(t_1) sinz(x-t_1)\left(cosz(t_1-\tau_2) + \frac{h}{z} sinz(t_1-\tau_2,z)\right)dt_1 = \frac{1}{z}\left(b_{s,c,\tau_2}(x,z) + \frac{h}{z} b_{s^2,\tau_2}(x,z)\right).$$

Then

$$y_1^{(1)}(x,z) = \int_{\tau_2}^x q_1(t_1) sinz(x-t_1) y_0^{(1)}(t_1,z)dt_1 + \frac{1}{z}\int_{\tau_2}^x q_2(t_1) sinz(x-t_1) y_1(t_1-\tau_2,z)dt_1 =$$

$$\frac{1}{z}\int_{\tau_2}^x q_1(t_1) sinz(x-t_1)\frac{1}{z}\left(b_{s,c,\tau_2}(t_1,z) + \frac{h}{z} b_{s^2,\tau_2}(t_1,z)\right)dt_1 +$$

$$+\frac{1}{z}\int_{\tau_2}^x q_2(t_1) sinz(x-t_1)\frac{1}{z}\left(b_{s,c}(t_1-\tau_2,z) + \frac{h}{z} b_{s^2}(t_1-\tau_2,z)\right)dt_1 =$$

$$\frac{1}{z^2}\left(b_{s^2,c,\tau_2}^{(1,2)}(x,z) + \frac{h}{z} b_{s^3,\tau_2}^{(1,2)}(x,z)\right) + \frac{1}{z^2}\left(b_{s^2,c,\tau_2}^{(2,1)}(x,z) + \frac{h}{z} b_{s^3,\tau_2}^{(2,1)}(x,z)\right) =$$

$$\sum_{P\in S_2(1,1)} \frac{1}{z^2}\left(b_{s^2,c,\tau_2}^{P}(x,z) + \frac{h}{z} b_{s^3,\tau_2}^{P}(x,z)\right),$$

and

$$y_2^{(1)}(x,z) = \frac{1}{z}\int_{\tau_2}^x q_1(t_1) sinz(x-t_1) y_1^{(1)}(t_1,z)dt_1 + \frac{1}{z}\int_{\tau_2}^x q_2(t_1) sinz(x-t_1) y_2(t_1-\tau_2,z)dt_1 =$$

$$\frac{1}{z}\int_{\tau_2}^{x} q_1(t_1) sinz(x-t_1) \sum_{P\in S_2(1,1)} \frac{1}{z^2}\left(b^P_{s^2,c,\tau_2}(t_1,z)+\frac{h}{z}b^P_{s^3,\tau_2}(t_1,z)\right)dt_1 +$$

$$+\frac{1}{z}\int_{\tau_2}^{x} q_2(t_1) sinz(x-t_1) \frac{1}{z^2}\left(b_{s^2,c}(t_1-\tau_2,z)+\frac{h}{z}b_{s^3}(t_1-\tau_2,z)\right)dt_1 =$$

$$\sum_{P\in {S_3}^{(1)}(2,1)} \frac{1}{z^3}\left(b^P_{s^3,c,\tau_2}(x,z)+\frac{h}{z}b^P_{s^4,\tau_2}(x,z)\right)+\frac{1}{z^3}\left(b^P_{s^3,c,\tau_2}(x,z)+\frac{h}{z}b^P_{s^4,\tau_2}(x,z)\right)\Bigg|_{P\in {S_3}^{(2)}(2,1)} =$$

$$\sum_{P\in S_3(2,1)} \frac{1}{z^3}\left(b^P_{s^3,c,\tau_2}(x,z)+\frac{h}{z}b^P_{s^4,\tau_2}(x,z)\right).$$

By the method of mathematical induction it is easily shown that functions $y^{(1)}_k(x,z), k = 1,2,3, ...$have the form:

$$y^{(1)}_k(x,z) = \sum_{P\in S_{k+1}(k,1)} \frac{1}{z^{k+1}}\left(b^P_{s^{k+1},c,\tau_2}(x,z)+\frac{h}{z}b^P_{s^{k+2},\tau_2}(x,z)\right), k = 1,2,3, ... \tag{12}$$

Then, from (10), (11) and (12) we get the solution within $(\tau_2, 2\tau_2]$ in the form:

$$y(x,z) = \sum_{k=0}^{\infty} y_k(x,z) + \sum_{k=0}^{\infty} y^{(1)}_k(x,z) = coszx + \frac{h}{z}sinzx + \frac{1}{z}\left(b_{s,c,\tau_2}(x,z)+\frac{h}{z}b_{s^2,\tau_2}(x,z)\right)+$$

$$\sum_{l=1}^{\infty}\frac{1}{z^l}\left(b_{s^l,c}(x,z)+\frac{h}{z}b_{s^{l+1}}(x,z)\right)+\sum_{l=1}^{\infty}\sum_{P\in S_{l+1}(l,1)}\frac{1}{z^{l+1}}\left(b^P_{s^{l+1},c,\tau_2}(x,z)+\frac{h}{z}b^P_{s^{l+2},\tau_2}(x,z)\right)$$

3. From (10) for $i = 2$ we get

$$y^{(2)}_0(x,z) = \frac{1}{z}\int_{2\tau_2}^{x} q_2(t_1) sinz(x-t_1) y^{(1)}_0(t_1-\tau_2,z)dt_1 =$$

$$\frac{1}{z}\int_{2\tau_2}^{x} q_2(t_1) sinz(x-t_1) \frac{1}{z}\left(b_{s,c,\tau_2}(t_1-\tau_2,z)+\frac{h}{z}b_{s^2,\tau_2}(t_1-\tau_2,z)\right)dt_1 =$$

$$\frac{1}{z^2}\left(b_{s^2,c,2\tau_2}(x,z)+\frac{h}{z}b_{s^3,2\tau_2}(x,z)\right),$$

and

$$y^{(2)}_1(x,z) = \int_{2\tau_2}^{x} q_1(t_1) sinz(x-t_1) y^{(2)}_0(t_1,z)dt_1 + \frac{1}{z}\int_{2\tau_2}^{x} q_2(t_1) sinz(x-t_1) y^{(1)}_1(t_1-\tau_2,z)dt_1 =$$

$$\frac{1}{z}\int_{2\tau_2}^{x} q_1(t_1)sinz(x-t_1)\frac{1}{z^2}\left(b_{s^2,c,2\tau_2}(t_1,z)+\frac{h}{z}b_{s^3,2\tau_2}(t_1,z)\right)dt_1+$$

$$+\frac{1}{z}\int_{2\tau_2}^{x} q_2(t_1)sinz(x-t_1)\sum_{P\in S_2(1,1)}\frac{1}{z^2}\left(b^P_{s^2,c,\tau_2}(t_1-\tau_2,z)+\frac{h}{z}b^P_{s^3,\tau_2}(t_1-\tau_2,z)\right)dt_1$$

$$=\frac{1}{z^3}\left(b^P_{s^3,c,2\tau_2}(x,z)+\frac{h}{z}b^P_{s^4,2\tau_2}(x,z)\right)\Bigg|_{P\in S_3^{(1)}(1,2)}+\sum_{P\in S_3{}^{(2)}(1,2)}\frac{1}{z^3}\left(b^P_{s^3,c,2\tau_2}(x,z)+\frac{h}{z}b^P_{s^4,2\tau_2}(x,z)\right)=$$

$$\sum_{P\in S_3(1,2)}\frac{1}{z^3}\left(b^P_{s^3,c,2\tau_2}(x,z)+\frac{h}{z}b^P_{s^4,2\tau_2}(x,z)\right)$$

By the method of mathematical induction, we prove that

$$y_k^{(2)}(x,z)=\sum_{P\in S_{k+2}(k,2)}\frac{1}{z^{k+2}}\left(b^P_{s^{k+2},c,2\tau_2}(x,z)+\frac{h}{z}b^P_{s^{k+3},2\tau_2}(x,z)\right),k=1,2,\dots$$

so, the solution within the interval $(2\tau_2,3\tau_2]$ has the form:

$$y(x,z)=\sum_{k=0}^{\infty}y_k(x,z)+\sum_{k=0}^{\infty}y_k^{(1)}(x,z)+\sum_{k=0}^{\infty}y_k^{(2)}(x,z)=coszx+\frac{h}{z}sinzx+$$

$$\sum_{l=1}^{2}\frac{1}{z^l}\left(b_{s^l,c,l\tau_2}(x,z)+\frac{h}{z}b_{s^{l+1},l\tau_2}(x,z)\right)+\sum_{l=1}^{\infty}\frac{1}{z^l}\left(b_{s^l,c}(x,z)+\frac{h}{z}b_{s^{l+1}}(x,z)\right)+$$

$$\sum_{i=1}^{2}\sum_{k=i+1}^{\infty}\sum_{P\in S_k(k-i,i)}\frac{1}{z^k}\left(b^P_{s^k,c,i\tau_2}(x,z)+\frac{h}{z}b^P_{s^{k+1},i\tau_2}(x,z)\right).$$

4. Now by the method of mathematical induction,we will prove that functions $y_n^{(k)}(x,z)$ for every $k=1,2,\dots k_0$ have the form:

$$y_n^{(k)}(x,z)=\sum_{P\in S_{n+k}(n,k)}\frac{1}{z^{n+k}}\left(b^P_{s^{n+k},c,k\tau_2}(x,z)+\frac{h}{z}b^P_{s^{n+k+1},k\tau_2}(x,z)\right),n=1,2,\dots \tag{13}$$

From (12) we get that (13) is correct for $k=1$. Let us assume that (13) is valid for $k<K, 1\le K\le k_0$ and then we show that (13) is correct for $k=K$.From (10), for $k=1$ and $i=K$, we have

$$y_1^{(K)}(x,z)=\int_{K\tau_2}^{x}q_1(t_1)sinz(x-t_1)y_0^{(K)}(t_1,z)dt_1+\frac{1}{z}\int_{K\tau_2}^{x}q_2(t_1)sinz(x-t_1)y_1^{(K-1)}(t_1-\tau_2,z)dt_1.$$

Taking into account that $y_{k-1}^{(K)}=0$ for $k=0$, it is obvious from (10) that functions $y_0^{(K)}(x,z), K=1,2,\dots k_0$, have the form:

$$y_0^{(K)}(x,z)=\frac{1}{z^K}\left(b_{s^K,c,K\tau_2}(x,z)+\frac{h}{z}b_{s^{K+1},K\tau_2}(x,z)\right). \tag{14}$$

From (13) for $n=1$ and $k=K-1$, and from (14) we have

$$y_1^{(K)}(x,z)=\frac{1}{z}\int_{K\tau_2}^{x}q_1(t_1)sinz(x-t_1)\frac{1}{z^K}\left(b_{s^K,c,K\tau_2}(t_1,z)+\frac{h}{z}b_{s^{K+1},K\tau_2}(t_1,z)\right)dt_1+$$

$$\frac{1}{z}\int_{K\tau_2}^{x}q_2(t_1)sinz(x-t_1)\sum_{P\in S_K(1,K-1)}\frac{1}{z^K}\left(b^P_{s^K,c,(K-1)\tau_2}(t_1-\tau_2,z)+\frac{h}{z}b^P_{s^{K+1},(K-1)\tau_2}(t_1-\tau_2,z)\right)dt_1=$$

$$\frac{1}{z^{K+1}}\left(b^P_{s^{K+1},c,K\tau_2}(x,z)+\frac{h}{z}b^P_{s^{K+2},K\tau_2}(x,z)\right)\Bigg|_{P\in S_{K+1}^{(1)}(1,K)}+$$

$$\sum_{P\in S_{K+1}^{(2)}(1,K)}\frac{1}{z^{K+1}}\left(b^P_{s^{K+1},c,K\tau_2}(x,z)+\frac{h}{z}b^P_{s^{K+2},K\tau_2}(x,z)\right)=$$

$$\sum_{P\in S_{K+1}(1,K)}\frac{1}{z^{K+1}}\left(b^P_{s^{K+1},c,K\tau_2}(x,z)+\frac{h}{z}b^P_{s^{K+2},K\tau_2}(x,z)\right),$$

so (13) is correct for $n=1$. Then, using (13) for $y_{n+1}^{(K)}(x,z)$, we get

$$y_{n+1}^{(K)}(x,z)=\frac{1}{z}\int_{K\tau_2}^{x}q_1(t_1)sinz(x-t_1)y_n^{(K)}(t_1,z)dt_1+\frac{1}{z}\int_{K\tau_2}^{x}q_2(t_1)sinz(x-t_1)y_{n+1}^{(K-1)}(t_1-\tau_2,z)dt_1$$

$$=\frac{1}{z^{n+K+1}}\int_{K\tau_2}^{x}q_1(t_1)sinz(x-t_1)\sum_{P\in S_{n+K}(n,K)}\left(b^P_{s^{n+K},c,K\tau_2}(t_1,z)+\frac{h}{z}b^P_{s^{n+K+1},K\tau_2}(t_1,z)\right)dt_1+$$

$$\frac{1}{z^{n+K+1}}\int_{K\tau_2}^{x}q_2(t_1)sinz(x-t_1)\sum_{P\in S_{n+K}(n+1,K-1)}\left(b^P_{s^{n+K},c,(K-1)\tau_2}(t_1-\tau_2,z)+\frac{h}{z}b^P_{s^{n+K+1},(K-1)\tau_2}(t_1-\tau_2,z)\right)dt_1$$

$$=\sum_{P\in S^{(1)}_{n+K+1}(n+1,K)}\frac{1}{z^{n+K+1}}\left(b^P_{s^{n+K+1},c,K\tau_2}(x,z)+\frac{h}{z}b^P_{s^{n+K+2},K\tau_2}(x,z)\right)+$$

$$+\sum_{P\in S^{(2)}_{n+K+1}(n+1,K)}\frac{1}{z^{n+K+1}}\left(b^P_{s^{n+K+1},c,K\tau_2}(x,z)+\frac{h}{z}b^P_{s^{n+K+2},K\tau_2}(x,z)\right)=$$

$$\sum_{P\in S_{n+K+1}(n+1,K)}\frac{1}{z^{n+K+1}}\left(b^P_{s^{n+K+1},c,K\tau_2}(x,z)+\frac{h}{z}b^P_{s^{n+K+2},K\tau_2}(x,z)\right)$$

so (13) is correct for $n+1$.

5. Now we can determine the solution of the boundary value problem within the interval $(n\tau_2,(n+1)\tau_2]$, $n=1,2,\dots k_0$. From (10) we have

$$y(x,z)=\sum_{k=0}^{\infty} y_k(x,z)+\sum_{i=1}^{n}\sum_{k=0}^{\infty} y_k^{(i)}(x,z).$$

and using(13), we get the solution within the interval$(n\tau_2,(n+1)\tau_2]$ in the form:

$$y(x,z)=coszx+\frac{h}{z}sinzx+\sum_{l=1}^{n}\frac{1}{z^l}\left(b_{s^l,c,l\tau_2}(x,z)+\frac{h}{z}b_{s^{l+1},l\tau_2}(x,z)\right)+$$

$$+\sum_{l=1}^{\infty}\frac{1}{z^l}\left(b_{s^l,c}(x,z)+\frac{h}{z}b_{s^{l+1}}(x,z)\right)+\sum_{i=1}^{n}\sum_{k=i+1}^{\infty}\sum_{P\in S_k(k-i,i)}\frac{1}{z^k}\left(b^P_{s^k,c,i\tau_2}(x,z)+\frac{h}{z}b^P_{s^{k+1},i\tau_2}(x,z)\right)$$

From here, for$n=k_0$, we get that the solution within the interval $(k_0\tau_2,\pi]$ has the form of (9),thus proving the theorem.

Let us now determine the characteristic function of the boundary value problem(1)-(4). From (9), we have

$$\frac{dy}{dx}(x,z)=-zsinxz+hcoszx+b_{c^2}(x,z)+\frac{h}{z}b_{c,s}(x,z)+\sum_{l=2}^{\infty}\frac{1}{z^{l-1}}\left(b_{c,s^{l-1},c}(x,z)+\frac{h}{z}b_{c,s^l}(x,z)\right)+$$

$$+b_{c^2,\tau_2}(x,z)+\frac{h}{z}b_{c,s,\tau_2}(x,z)++\sum_{l=2}^{k_0}\frac{1}{z^{l-1}}\left(b_{c,s^{l-1},c,l\tau_2}(x,z)+\frac{h}{z}b_{c,s^l,l\tau_2}(x,z)\right)$$

$$+\sum_{i=1}^{k_0}\sum_{k=i+1}^{\infty}\sum_{P\in S_k(k-i,i)}\frac{1}{z^{k-1}}\left(b^P_{c,s^{k-1},c,i\tau_2}(x,z)+\frac{h}{z}b^P_{c,s^k,i\tau_2}(x,z)\right) \tag{15}$$

where

$$b_{c^2}(x,z)=\int_0^x q_1(t_1)cosz(x-t_1)coszt_1dt_1;\quad b_{c,s}(x,z)=\int_0^x q_1(t_1)cosz(x-t_1)sinzt_1dt_1,$$

$$b_{c^2,\tau_2}(x,z)=\int_{\tau_2}^x q_2(t_1)cosz(x-t_1)coszt_1dt_1,$$

$$b_{c,s,\tau_2}(x,z)=\int_{\tau_2}^x q_2(t_1)cosz(x-t_1)sinz(t_1-\tau_2,z)dt_1,$$

$$b_{c,s^l}(x,z)=\int_0^x q_1(t_1)cosz(x-t_1)b_{s^l}(t_1,z)dt_1,\qquad l=2,3,\dots,$$

$$b_{c,s^{l-1},c}(x,z)=\int_0^x q_1(t_1)cosz(x-t_1)b_{s^{l-1},c}(t_1,z)dt_1,\qquad l=2,3,\dots,$$

$$b_{c,s^{l-1},c,l\tau_2}(x,z)=\int_{l\tau_2}^x q_2(t_1)cosz(x-t_1)b_{s^{l-1},c,(l-1)\tau_2}(t_1-\tau_2,z)dt_1,\qquad l=2,3,\dots$$

$$b_{c,s^l,l\tau_2}(x,z)=\int_{l\tau_2}^x q_2(t_1)cosz(x-t_1)b_{s^l,(l-1)\tau_2}(t_1-\tau_2,z)dt_1,\qquad l=2,3,\dots$$

$$b^{P}_{c,s^{l-1},c,k\tau_2}(x,z)\Big|_{P\in S^{(1)}{}_{l}(l-k,k)} = \int_{k\tau_2}^{x} q_1(t_1)cosz(x-t_1)b^{P}_{s^{l-1},c,k\tau_2}(t_1,z)dt_1, k=1,\dots k_0, l=k+1,\dots$$

$$b^{P}_{c,s^{l-1},c,k\tau_2}(x,z)\Big|_{P\in S^{(2)}{}_{l}(l-k,k)} = \int_{k\tau_2}^{x} q_2(t_1)cosz(x-t_1)b^{P}_{s^{l-1},c,(k-1)\tau_2}(t_1-\tau_2,z)dt_1, k=2,..k_0, l=k+1,\dots$$

$$b^{P}_{c,s^{l},k\tau_2}(x,z)\Big|_{P\in S^{(1)}{}_{l}(l-k,k)} = \int_{k\tau_2}^{x} q_1(t_1)cosz(x-t_1)b^{P}_{s^{l},k\tau_2}(t_1,z)dt_1, k=1,2,\dots k_0, l=k+1,\dots$$

$$b^{P}_{c,s^{l},k\tau_2}(x,z)\Big|_{P\in S^{(2)}{}_{l}(l-k,k)} = \int_{k\tau_2}^{x} q_2(t_1)cosz(x-t_1)b^{P}_{s^{l},(k-1)\tau_2}(t_1-\tau_2,z)dt_1, k=2,\dots k_0, l=k+1,\dots$$

Using the condition (4), now we can determine the characteristic function and, in order to simplify, hereinafter we will write $b(z)$ instead of $b(\pi,z)$.

Teorema 2.2. The characteristic function of the boundary value problem (1)-(4) has the form:

$$F(z) = \left(-z+\frac{Hh}{z}\right)sin\pi z + (h+H)cosz\pi + b_{c^2}(z) + \frac{h}{z}b_{c,s}(z) + b_{c^2,\tau_2}(z) + \frac{h}{z}b_{c,s,\tau_2}(z) +$$

$$+\frac{H}{z}\left(b_{s,c}(z) + \frac{h}{z}b_{s^2}(z) + b_{s,c,\tau_2}(z) + \frac{h}{z}b_{s^2,\tau_2}(z)\right) +$$

$$\frac{1}{z}\left(b_{c,s,c,2\tau_2}(z) + \frac{h}{z}b_{c,s^2,2\tau_2}(z) + +b_{c,s,c}(z) + \frac{h}{z}b_{c,s^2}(z) + \sum_{P\in S_2(1,1)}\left(b^{P}_{c,s,c,\tau_2}(z) + \frac{h}{z}b^{P}_{c,s^2,\tau_2}(z)\right)\right) +$$

$$\sum_{k=3}^{\infty}\sum_{P\in S_k(k-1,1)}\frac{1}{z^{k-1}}\left(b^{P}_{c,s^{k-1},c,\tau_2}(z) + \frac{h}{z}b^{P}_{c,s^{k},\tau_2}(z)\right) +$$

$$\sum_{l=2}^{\infty}\frac{1}{z^{l}}\left(Hb_{s^{l},c}(z) + \frac{Hh}{z}b_{s^{l+1}}(z) + b_{c,s^{l},c}(z) + \frac{h}{z}b_{c,s^{l+1}}(z)\right) +$$

$$\sum_{l=2}^{k_0}\frac{1}{z^{l}}\left(Hb_{s^{l},c,l\tau_2}(z) + \frac{Hh}{z}b_{s^{l+1},l\tau_2}(z) + b_{c,s^{l},c,(l+1)\tau_2}(z) + \frac{h}{z}b_{c,s^{l+1},(l+1)\tau_2}(z)\right) +$$

$$\sum_{i=1}^{k_0}\sum_{k=i+1}^{\infty}\sum_{P\in S_k(k-i,i)}\frac{H}{z^{k}}\left(b^{P}_{s^{k},c,i\tau_2}(z) + \frac{h}{z}b^{P}_{s^{k+1},i\tau_2}(z)\right) +$$

$$\sum_{i=2}^{k_0}\sum_{k=i+1}^{\infty}\sum_{P\in S_k(k-i,i)}\frac{1}{z^{k-1}}\left(b^{P}_{c,s^{k-1},c,i\tau_2}(z) + \frac{h}{z}b^{P}_{c,s^{k},i\tau_2}(z)\right). \tag{16}$$

3. ASYMPTOTIC OF EIGENVALUES

It is known that eigenvalues λ_n of the operator L are squares of zeros of the characteristic function. It is also known that zeros of the characteristic function have the form:

$$z_n = n + \varkappa_n,\ \varkappa_n \in l_2.$$

In this paper we will study the asymptotic of eigenvalues in detail because it will be the base for further consideration of the inverse problems for this class of operators by new method based on direct relations between eigenvalues and Fourier's coefficients. Because of that, we will determine the asymptotic of zeros of the characteristic functionin the form:

$$z_n = n + \frac{C_1(n)}{n} + \frac{C_2(n)}{n^2} + \frac{C_3(n)}{n^3} + o\left(\frac{1}{n^3}\right)(n \to \infty) \tag{17}$$

In order to simplify, hereinafter we will write C_1, C_2, C_3 instead of $C_1(n)$, $C_2(n)$, $C_3(n)$, respectively.From (16),we get asymptotic of the characteristic function in the form:

$$\begin{aligned} F(z) &= \left(-z + \frac{hH}{z}\right) sin\pi z + (h+H)cos z\pi + b_{c^2}(z) + b_{c^2,\tau_2}(z) + \\ &\frac{1}{z}\left[h\left(b_{c,s}(z) + b_{c,s,\tau_2}(z)\right) + H\left(b_{s,c}(z) + b_{s,c,\tau_2}(z)\right) + b_{c,s,c}(z) + b_{c,s,c,2\tau_2}(z) + \sum_{P\in S_2(1,1)} b^{P}_{c,s,c,\tau_2}(z)\right] + \\ &\frac{1}{z^2}\left[Hh\left(b_{s^2}(z) + b_{s^2,\tau_2}(z)\right) + h\left(b_{c,s^2}(z) + b_{c,s^2,2\tau_2}(z) + \sum_{P\in S_2(1,1)} b^{P}_{c,s^2,\tau_2}(z)\right)\right] + \\ &\frac{1}{z^2}\left[H\left(b_{s^2,c}(z) + b_{s^2,c,2\tau_2}(z) + \sum_{P\in S_2(1,1)} b^{P}_{s^2,c,\tau_2}(z)\right)\right] + \\ &\frac{1}{z^2}\left[b_{c,s^2,c}(z) + b_{c,s^2,c,3\tau_2}(z) + \sum_{P\in S_3(2,1)} b^{P}_{c,s^2,c,\tau_2}(z) + \sum_{P\in S_3(1,2)} b^{P}_{c,s^2,c,2\tau_2}(z)\right] + O\left(\frac{b_{s^3}(z)}{z^3}\right), z \to \infty \end{aligned} \tag{18}$$

Let us define so called *transitional function* $\tilde{q}$

$$\tilde{q}(t_1) = \begin{cases} q_1(t_1), t_1 \in \left[0, \frac{\tau_2}{2}\right) \cup \left(\pi - \frac{\tau_2}{2}, \pi\right] \\ q_1(t_1) + q_2\left(t_1 + \frac{\tau_2}{2}\right), t_1 \in \left[\frac{\tau_2}{2}, \pi - \frac{\tau_2}{2}\right] \end{cases} \tag{19}$$

and introduce the following notations:

$$J_1^1 = \int_0^\pi q_1(t_1)dt_1;\ J_1^2 = \int_{\tau_2}^\pi q_2(t_1)dt_1;\ \ J_2^1 = \int_0^\pi q_1(t_1)\int_0^{t_1} q_1(t_2)dt_2dt_1,$$

$$J_2^2 = \int_{2\tau_2}^\pi q_2(t_1)\int_{\tau_2}^{t_1-\tau_2} q_2(t_2)\, dt^2dt_1;\ \ J_2^{12} = \int_{\tau_2}^\pi q_1(t_1)\int_{\tau_2}^{t_1} q_2(t_2)\, dt_2dt_1,$$

$$J_2^{21} = \int_{\tau_2}^\pi q_2(t_1)\int_0^{t_1-\tau_2} q_1(t_2)\, dt_2dt_1;\ J_3^1 = \int_0^\pi q_1(t_1)\int_0^{t_1} q_1(t_2)\int_0^{t_2} q_1(t_3)dt_3dt_2dt_1,$$

$$J_3^2 = \int_{3\tau_2}^\pi q_2(t_1)\int_{2\tau_2}^{t_1-\tau_2} q_2(t_2)\int_{\tau_2}^{t_2-\tau_2} q_2(t_3)dt_3dt_2dt_1,$$

$$J_3^{112} = \int_{\tau_2}^{\pi} q_1(t_1) \int_{\tau_2}^{t_1} q_1(t_2) \int_{\tau_2}^{t_2} q_2(t_3) dt_3 dt_2 dt_1,$$
$$J_3^{121} = \int_{\tau_2}^{\pi} q_1(t_1) \int_{\tau_2}^{t_1} q_2(t_2) \int_{0}^{t_2-\tau_2} q_1(t_3) dt_3 dt_2 dt_1,$$
$$J_3^{211} = \int_{\tau_2}^{\pi} q_2(t_1) \int_{0}^{t_1-\tau_2} q_1(t_2) \int_{0}^{t_2} q_1(t_3) dt_3 dt_2 dt_1,$$
$$J_3^{122} = \int_{2\tau_2}^{\pi} q_2(t_1) \int_{2\tau_2}^{t_1} q_2(t_2) \int_{\tau_2}^{t_2-\tau_2} q_1(t_3) dt_3 dt_2 dt_1,$$
$$J_3^{212} = \int_{2\tau_2}^{\pi} q_2(t_1) \int_{\tau_2}^{t_1-\tau_2} q_1(t_2) \int_{\tau_2}^{t_2} q_2(t_3) dt_3 dt_2 dt_1,$$
$$J_3^{221} = \int_{2\tau_2}^{\pi} q_2(t_1) \int_{\tau_2}^{t_1-\tau_2} q_2(t_2) \int_{0}^{t_2-\tau_2} q_1(t_3) dt_3 dt_2 dt_1,$$
$$\tilde{a}_c(z) = \int_0^{\pi} \tilde{q}(t_1) cosz(\pi - 2\,t_1) dt_1;\ \tilde{a}_s(z) = \int_0^{\pi} \tilde{q}(t_1) sinz(\pi - 2\,t_1) dt_1. \tag{20}$$

Then we have

$$b_{c^2}(z) = \frac{J_1^1}{2} cos\pi z + \frac{1}{2}\int_0^{\pi} q_1(t_1) cosz(\pi - 2t_1) dt_1$$
$$b_{c^2,\tau_2}(z) = \frac{J_1^2}{2} cosz(\pi - \tau_2) + \frac{1}{2}\int_{\tau_2}^{\pi} q_2(t_1) cosz(\pi - 2t_1 + \tau_2) dt_1$$

and

$$b_{c^2}(z) + b_{c^2,\tau_2}(z) = \frac{1}{2}\tilde{a}_c(z) + \frac{J_1^1}{2} cos\pi z + \frac{J_1^2}{2} cosz(\pi - \tau_2). \tag{21}$$

In the same way we get

$$b_{c,s}(z) + b_{c,s,\tau_2}(z) = \frac{J_1^1}{2} sin\pi z + \frac{J_1^2}{2} sinz(\pi - \tau_2) - \frac{1}{2}\tilde{a}_s(z),$$
$$b_{s,c}(z) + b_{s,c,\tau_2}(z) = \frac{J_1^1}{2} sin\pi z + \frac{J_1^2}{2} sinz(\pi - \tau_2) + \frac{1}{2}\tilde{a}_s(z)$$
$$b_{s^2}(z) + b_{s^2,\tau_2}(z) = -\frac{J_1^1}{2} cos\pi z + \frac{J_1^2}{2} cosz(\pi - \tau_2) + \frac{1}{2}\tilde{a}_c(z). \tag{22}$$

Using trigonometric identities for transformation a product of trigonometric functions into a sum, we get

$$b_{c,s,c}(z) = \int_0^{\pi} q_1(t_1) cosz(\pi - t_1) \int_0^{t_1} q_1(t_2)\, sinz(t_1 - t_2) coszt_2 dt_2 dt_1 =$$
$$= \frac{J_2^1}{4} sin\pi z - \frac{1}{4}\beta_1^{(1)}(z) + \frac{1}{4}\beta_2^{(1)}(z) - \frac{1}{4}\beta_3^{(1)}(z)$$
$$\beta_i^{(1)}(z) = \int_0^{\pi} q_1(t_1) \int_0^{t_1} q_1(t_2) sinz(\pi - 2t_i)\, dt_2 dt_1, i = 1{,}2$$
$$\beta_3^{(1)}(z) = \int_0^{\pi} q_1(t_1) \int_0^{t_1} q_1(t_2)\, sinz(\pi - 2t_1 + 2t_2) dt_2 dt_1.$$

Now, changing the order of integration in $\beta_2^{(1)}(z)$, and using the method of substitution of variables in $\beta_3^{(1)}(z)$, we get

$$\beta_2^{(1)}(z) = \int_0^\pi \left(q_1(t_1)\int_{t_1}^\pi q_1(t_2)dt_2\right) sinz(\pi - 2t_1)dt_1,$$
$$\beta_3^{(1)}(z) = -\int_0^\pi \left(\int_{t_1}^\pi q_1(t_2)q_1(t_2 - t_1)dt_2\right) sinz(\pi - 2t_1)dt_1,$$

so we can present $b_{c,s,c}(z)$ in the form:

$$b_{c,s,c}(z) = \frac{J_2^1}{4} sin\pi z - \frac{1}{4}\int_0^\pi K^1\big(t_1, q_1(t_1)\big) sinz(\pi - 2t_1)dt_1,$$
$$K^1\big(t_1, q_1(t_1)\big) = q_1(t_1)\int_0^{t_1} q_1(t_2)dt_2 - q_1(t_1)\int_{t_1}^\pi q_1(t_2)dt_2 - \int_{t_1}^\pi q_1(t_2)q_1(t_2 - t_1)dt_2\,, t_1 \in [0,\pi]$$

Analogously, for $b_{c,s,c,2\tau_2}(z)$ we have

$$b_{c,s,c,2\tau_2}(z) = \frac{J_2^2}{4} sinz(\pi - 2\tau_2) - \frac{1}{4}\int_0^\pi K^2(t_1, q_2(t_1)) sinz(\pi - 2t_1)dt_1$$
$$K^2\big(t_1, q_2(t_1)\big) = q_2(t_1 + \tau_2)\int_{\tau_2}^{t_1} q_2(t_2)dt_2 - q_2(t_1)\int_{t_1+\tau_2}^\pi q_2(t_2)\,dt_2 - \int_{t_1+\tau_2}^\pi q_2(t_2)q_2(t_2 - t_1)dt_2,$$
$$t_1 \in [\tau_2, \pi - \tau_2],\ \ K^2\big(t_1, q_2(t_1)\big) = 0, t_1 \in [0,\tau_2) \cup (\pi - \tau_2, \pi]$$

and for integrals $b_{c,s,c,\tau_2}^{(1,2)}(z)$ and $b_{c,s,c,\tau_2}^{(2,1)}(z)$ we have

$$b_{c,s,c,\tau_2}^{(1,2)}(z) = \int_{\tau_2}^\pi q_1\,(t_1) cosz(\pi - t_1)\int_{\tau_2}^{t_1} q_2(t_2) sinz(t_1 - t_2)\, cosz(t_2 - \tau_2)dt_1dt_2 =$$
$$\frac{J_2^{12}}{4} sinz(\pi - \tau_2) - \frac{1}{4}\int_0^\pi K^{12}(t_1, q_1(t_1), q_2(t_1)) sinz(\pi - 2t_1)dt_1$$
$$b_{c,s,c,\tau_2}^{(2,1)}(z) = \int_{\tau_2}^\pi q_1\,(t_1) cosz(\pi - t_1)\int_{\tau_2}^{t_1} q_2(t_2) sinz(t_1 - t_2)\, cosz(t_2 - \tau_2)dt_1dt_2 =$$
$$\frac{J_2^{21}}{4} sinz(\pi - \tau_2) - \frac{1}{4}\int_0^\pi K^{21}(t_1, q_1(t_1), q_2(t_1)) sinz(\pi - 2t_1)dt_1$$
$$K^{12}\big(t_1, q_1(t_1), q_2(t_1)\big) = q_1\left(t_1 + \frac{\tau_2}{2}\right)\int_{\tau_2}^{t_1+\frac{\tau_2}{2}} q_2(t_2)dt_2 - q_2\left(t_1 + \frac{\tau_2}{2}\right)\int_{t_1+\frac{\tau_2}{2}}^\pi q_1(t_2)\,dt_2 -$$
$$\int_{t_1+\frac{\tau_2}{2}}^\pi q_1(t_2)q_2\left(t_2 - t_1 - \frac{\tau_2}{2}\right)dt_2, t_1 \in \left[\frac{\tau_2}{2}, \pi - \frac{\tau_2}{2}\right]$$
$$K^{12}\big(t_1, q_1(t_1), q_2(t_1)\big) = 0, t_1 \in [0, \frac{\tau_2}{2}) \cup (\pi - \frac{\tau_2}{2}, \pi]$$
$$K^{21}\big(t_1, q_1(t_1), q_2(t_1)\big) = q_2\left(t_1 + \frac{\tau_2}{2}\right)\int_0^{t_1-\frac{\tau_2}{2}} q_1(t_2)dt_2 - q_1\left(t_1 - \frac{\tau_2}{2}\right)\int_{t_1+\frac{\tau_2}{2}}^\pi q_2(t_2)\,dt_2 -$$
$$\int_{t_1+\frac{\tau_2}{2}}^\pi q_2(t_2)q_1\left(t_2 - t_1 - \frac{\tau_2}{2}\right)dt_2, t_1 \in \left[\frac{\tau_2}{2}, \pi - \frac{\tau_2}{2}\right],$$

$$K^{21}\big(t_1, q_1(t_1), q_2(t_1)\big) = 0, t_1 \in [0, \frac{\tau_2}{2}) \cup (\pi - \frac{\tau_2}{2}, \pi]$$

From relations above, we get

$$b_{c,s,c}(z) + b_{c,s,c,2\tau_2}(z) + b^{(1,2)}_{c,s,c,\tau_2}(z) + b^{(2,1)}_{c,s,c,\tau_2}(z) =$$

$$\frac{J_2^1}{4} sinz\pi + \frac{J_2^2}{4} sinz(\pi - 2\tau_2) + \frac{J_2^{12} + J_2^{21}}{4} sinz(\pi - \tau_2) - \frac{1}{4} a_s^{c,s,c}(z), \tag{23}$$

where

$$a_s^{c,s,c}(z) = \int_0^\pi K^{c,s,c}\big(t_1, q_1(t_1), q_2(t_1)\big) sinz(\pi - 2t_1) dt_1$$

$$K^{c,s,c}\big(t_1, q_1(t_1), q_2(t_1)\big) = K^1\big(t_1, q_1(t_1)\big) + K^2\big(t_1, q_2(t_1)\big) +$$

$$K^{12}\big(t_1, q_1(t_1), q_2(t_1)\big) + K^{21}(t_1, q_1(t_1), q_2(t_1)).$$

In the same way we prove that

$$b_{c,s^2}(z) + b_{c,s^2,2\tau_2}(z) + b^{(1,2)}_{c,s^2,\tau_2}(z) + b^{(2,1)}_{c,s^2,\tau_2}(z) =$$

$$-\frac{J_2^1}{4} cosz\pi - \frac{J_2^2}{4} cosz(\pi - 2\tau_2) - \frac{J_2^{12} + J_2^{21}}{4} cosz(\pi - \tau_2) - \frac{1}{4} a_c^{c,s^2}(z), \tag{24}$$

whereas

$$a_s^{c,s^2}(z) = \int_0^\pi M^{c,s^2}\big(t_1, q_1(t_1), q_2(t_1)\big) cosz(\pi - 2t_1) dt_1$$

$$M^{c,s^2}\big(t_1, q_1(t_1), q_2(t_1)\big) = M^1\big(t_1, q_1(t_1)\big) + M^2\big(t_1, q_2(t_1)\big) +$$

$$M^{12}\big(t_1, q_1(t_1), q_2(t_1)\big) + M^{21}(t_1, q_1(t_1), q_2(t_1))$$

$$M^1\big(t_1, q_1(t_1)\big) = q_1(t_1) \int_0^{t_1} q_1(t_2) dt_2 - q_1(t_1) \int_{t_1}^{\pi} q_1(t_2) dt_2 + \int_{t_1}^{\pi} q_1(t_2) q_1(t_2 - t_1) dt_2, t_1 \in [0, \pi]$$

$$M^2\big(t_1, q_2(t_1)\big) = q_2(t_1 + \tau_2) \int_{\tau_2}^{t_1} q_2(t_2) dt_2 - q_2(t_1) \int_{t_1 + \tau_2}^{\pi} q_2(t_2)\, dt_2 +$$

$$\int_{t_1 + \tau_2}^{\pi} q_2(t_2) q_2(t_2 - t_1) dt_2, t_1 \in [\tau_2, \pi - \tau_2], M^2\big(t_1, q_2(t_1)\big) = 0, t_1 \in [0, \tau_2) \cup (\pi - \tau_2, \pi]$$

$$M^{12}\big(t_1, q_1(t_1), q_2(t_1)\big) = q_1\left(t_1 + \frac{\tau_2}{2}\right) \int_{\tau_2}^{t_1 + \frac{\tau_2}{2}} q_2(t_2) dt_2 -$$

$$q_2\left(t_1 + \frac{\tau_2}{2}\right) \int_{t_1 + \frac{\tau_2}{2}}^{\pi} q_1(t_2)\, dt_2 + \int_{t_1 + \frac{\tau_2}{2}}^{\pi} q_1(t_2) q_2\left(t_2 - t_1 - \frac{\tau_2}{2}\right) dt_2, t_1 \in \left[\frac{\tau_2}{2}, \pi - \frac{\tau_2}{2}\right],$$

$$M^{12}\big(t_1, q_1(t_1), q_2(t_1)\big) = 0, t_1 \in [0, \frac{\tau_2}{2}) \cup (\pi - \frac{\tau_2}{2}, \pi]$$

$$M^{21}\big(t_1, q_1(t_1), q_2(t_1)\big) = q_2\left(t_1 + \frac{\tau_2}{2}\right) \int_0^{t_1 - \frac{\tau_2}{2}} q_1(t_2) dt_2 -$$

$$q_1\left(t_1 - \frac{\tau_2}{2}\right) \int_{t_1 + \frac{\tau_2}{2}}^{\pi} q_2(t_2)\, dt_2 + \int_{t_1 + \frac{\tau_2}{2}}^{\pi} q_2(t_2) q_1\left(t_2 - t_1 - \frac{\tau_2}{2}\right) dt_2, t_1 \in \left[\frac{\tau_2}{2}, \pi - \frac{\tau_2}{2}\right],$$

$$M^{21}\big(t_1, q_1(t_1), q_2(t_1)\big) = 0, t_1 \in [0, \frac{\tau_2}{2}) \cup (\pi - \frac{\tau_2}{2}, \pi]$$

Also, we show that the sum: $b_{s^2,c}(z) + b_{s^2,c,2\tau_2}(z) + b^{(1,2)}_{s^2,c,\tau_2}(z) + b^{(2,1)}_{c,s^2,\tau_2}(z)$ can be presented in the form:

$$b_{s^2,c}(z) + b_{s^2,c,2\tau_2}(z) + b^{(1,2)}_{s^2,c,\tau_2}(z) + b^{(2,1)}_{c,s^2,\tau_2}(z) =$$

$$-\frac{J_2^1}{4} cosz\pi - \frac{J_2^2}{4} cosz(\pi - 2\tau_2) - \frac{J_2^{12} + J_2^{21}}{4} cosz(\pi - \tau_2) + \frac{1}{4} a_c^{c,s,c}(z) \tag{25}$$

$$a_c^{c,s,c}(z) = \int_0^\pi K^{c,s,c}\big(t_1, q_1(t_1), q_2(t_1)\big)\, cosz(\pi - 2t_1) dt_1.$$

For integral $b_{c,s^2,c}(z)$ we have

$$b_{c,s^2,c}(z) = \int_0^\pi q_1(t_1) cosz(\pi - t_1) \int_0^{t_1} q_1(t_2)\, sinz(t_1 - t_2) \int_0^{t_2} q_1(t_3)\, sinz(t_2 - t_3) coszt_3 dt_3 dt_2 dt_1$$

$$= -\frac{cos\pi z}{8} \int_0^\pi q_1(t_1) \int_0^{t_1} q_1(t_2) \int_0^{t_2} q_1(t_3)\, dt_3 dt_2 dt_1 -$$

$$\frac{1}{8} \int_0^\pi q_1(t_1) \int_0^{t_1} q_1(t_2) \int_0^{t_2} q_1(t_3)\, cosz(\pi - 2t_1) dt_3 dt_2 dt_1 +$$

$$\frac{1}{8} \int_0^\pi q_1(t_1) \int_0^{t_1} q_1(t_2) \int_0^{t_2} q_1(t_3)\, cosz(\pi - 2t_2) dt_3 dt_2 dt_1 -$$

$$\frac{1}{8} \int_0^\pi q_1(t_1) \int_0^{t_1} q_1(t_2) \int_0^{t_2} q_1(t_3)\, cosz(\pi - 2t_3) dt_3 dt_2 dt_1 +$$

$$\frac{1}{8} \int_0^\pi q_1(t_1) \int_0^{t_1} q_1(t_2) \int_0^{t_2} q_1(t_3)\, cosz(\pi - 2t_1 + 2t_2) dt_3 dt_2 dt_1 -$$

$$\frac{1}{8} \int_0^\pi q_1(t_1) \int_0^{t_1} q_1(t_2) \int_0^{t_2} q_1(t_3)\, cosz(\pi - 2t_1 + 2t_3) dt_3 dt_2 dt_1 +$$

$$\frac{1}{8} \int_0^\pi q_1(t_1) \int_0^{t_1} q_1(t_2) \int_0^{t_2} q_1(t_3)\, cosz(\pi - 2t_2 + 2t_3) dt_3 dt_2 dt_1 +$$

$$\frac{1}{8} \int_0^\pi q_1(t_1) \int_0^{t_1} q_1(t_2) \int_0^{t_2} q_1(t_3)\, cosz(\pi - 2t_1 + 2t_2 - 2t_3) dt_3 dt_2 dt_1 \tag{26}$$

Changing the order of integration and/or using the method of substitution of variables in last seven integrals in (26),we get

$$b_{c,s^2,c}(z) = -\frac{J_3^1}{8} cos\pi z + \frac{1}{8} \int_0^\pi T^1\big(t_1, q_1(t_1)\big) cosz(\pi - 2t_1) dt_1$$

$$T^1\big(t_1, q_1(t_1)\big) = -q_1(t_1) \int_0^{t_1} q_1(t_2) \int_0^{t_2} q_1(t_3)\, dt_3 dt_2 +$$

$$q_1(t_1) \int_{t_1}^\pi q_1(t_2) \int_0^{t_1} q_1(t_3)\, dt_3 dt_2 - q_1(t_1) \int_{t_1}^\pi q_1(t_3) \int_{t_1}^{t_3} q_1(t_2)\, dt_3 dt_2 -$$

$$\int_{t_1}^\pi q_1(t_2) q_1(t_2 - t_1) \int_0^{t_2 - t_1} q_1(t_3)\, dt_3 dt_2 + \int_{t_1}^\pi q_1(t_3) q_1(t_3 - t_1) \int_{t_3 - t_1}^\pi q_1(t_2)\, dt_3 dt_2 -$$

$$\int_{t_1}^\pi \int_{t_1}^{t_3} q_1(t_3) q_1(t_2)\, q_1(t_2 - t_3)\, dt_3 dt_2 - q_1(t_1) \int_{t_1}^\pi \int_{t_3 - t_1}^\pi q_1(t_2) q_1(t_1 - t_3 + t_2)\, dt_3 dt_2$$

Now, in the similar way, we can determine functions $T^2(t_1, q_2(t_1))$and$T^P(t_1, q_1(t_1), q_2(t_1)), P \in S_3(2,1) \cup S_3(1,2)$with characteristics:

$$b_{c,s^2,c,3\tau_2}(z) = -\frac{J_3^2}{8}cosz(\pi - 3\tau_2) + \frac{1}{8}\int_0^\pi T^2\left(t_1, q_2(t_1)\right)cosz(\pi - 2t_1)dt_1$$

$$b_{c,s^2,c,\tau_2}{}^P(z) = -\frac{J_3^P}{8}cosz(\pi - \tau_2) + \frac{1}{8}\int_0^\pi T^P\left(t_1, q_1(t_1), q_2(t_1)\right)cosz(\pi - 2t_1)dt_1, P \in S_3(2,1)$$

$$b_{c,s^2,c,2\tau_2}{}^P(z) = -\frac{J_3^P}{8}cosz(\pi - 2\tau_2) + \frac{1}{8}\int_0^\pi T^P\left(t_1, q_1(t_1), q_2(t_1)\right)cosz(\pi - 2t_1)dt_1, P \in S_3(1,2)$$

From relations above, we get

$$b_{c,s^2,c}(z) + b_{c,s^2,c,3\tau_2}(z) + \sum_{P \in S_3(2,1)} b_{c,s^2,c,\tau_2}{}^P(z) + \sum_{P \in S_3(1,2)} b_{c,s^2,c,2\tau_2}{}^P(z) =$$

$$= -\frac{J_3^1}{8}cos\pi z - \frac{1}{8}(J_3^{211} + J_3^{121} + J_3^{112})cosz(\pi - \tau_2) - \frac{1}{8}(J_3^{221} + J_3^{122} + J_3^{212})cosz(\pi - 2\tau_2) -$$
$$\frac{J_3^2}{8}cosz(\pi - 3\tau_2) + \frac{1}{8}a_c^{c,s^2,c}(z) \quad (27)$$

where

$$a_c^{c,s^2,c}(z) = \int_0^\pi T^{c,s^2,c}\left(t_1, q_1(t_1), q_2(t_1)\right)cosz(\pi - 2t_1)dt_1,$$

and the function $T^{c,s^2,c}(t_1, q_1(t_1), q_2(t_1))$ is the sum of functions $T^1(t_1, q_1(t_1))$, $T^2(t_1, q_2(t_1))$ and $T^P(t_1, q_1(t_1), q_2(t_1)), P \in S_3(2,1) \cup S_3(1,2)$.
Now, substituting relations (21)-(27) in (18), we get

$$F(z) = -zsin\pi z + R_1cosz\pi + R_2cosz(\pi - \tau_2) + \frac{1}{2}\tilde{a}_c(z) +$$
$$\frac{1}{z}[R_3sin\pi z + R_4sinz(\pi - \tau_2) + R_5sinz(\pi - 2\tau_2) + \frac{H-h}{2}\tilde{a}_s(z) - \frac{1}{4}a_s^{c,s,c}(z)] +$$
$$\frac{1}{z^2}[R_6cos\pi z + R_7cosz(\pi - \tau_2) + R_8cosz(\pi - 2\tau_2) + R_9cosz(\pi - 3\tau_2)] +$$
$$+\frac{1}{z^2}\left[\frac{Hh}{2}\tilde{a}_c(z) + u_c(z)\right] + O\left(\frac{b_{s^3}(z)}{z^3}\right) \quad (28)$$

where

$$R_1 = h + H + \frac{J_1^1}{2}, R_2 = \frac{J_1^2}{2}, R_3 = \frac{h+H}{2}J_1^1 + \frac{J_2^1}{4} + Hh, R_4 = \frac{h+H}{2}J_1^2 + \frac{J_2^{12} + J_2^{21}}{4}$$
$$R_5 = \frac{J_2^2}{4}, R_6 = -\frac{Hh}{2}J_1^1 - \frac{H+h}{4}J_2^1 - \frac{1}{8}J_3^1,$$
$$R_7 = -\frac{Hh}{2}J_1^2 - \frac{H+h}{4}(J_2^{12} + J_2^{21}) - \frac{1}{8}(J_3^{211} + J_3^{121} + J_3^{112})$$
$$R_8 = -\frac{H+h}{4}J_2^2 - \frac{1}{8}(J_3^{122} + J_3^{212} + J_3^{221}), R_9 = -\frac{1}{8}J_3^2$$
$$u_c(z) = \int_0^\pi \left(\left(-\frac{h}{4}M^{c,s^2} + \frac{H}{4}K^{c,s,c} + \frac{1}{8}T^{c,s^2,c}\right)\left(t_1, q_1(t_1), q_2(t_1)\right)\right)cosz(\pi - 2t_1)dt_1 \quad (29)$$

From (17) we get

$$sin\pi z_n = (-1)^n\left(\frac{\pi C_1}{n}+\frac{\pi C_2}{n^2}+\frac{6\pi C_3-\pi^3 {C_1}^3}{6n^3}\right)+o\left(\frac{1}{n^3}\right)$$

$$-z_n sin\pi z_n = (-1)^{n+1}\left(\pi C_1+\frac{\pi C_2}{n}+\frac{6\pi C_3-\pi^3 {C_1}^3+6\pi {C_1}^2}{6n^2}\right)+o\left(\frac{1}{n^2}\right)$$

$$\frac{1}{z_n}=\frac{1}{n}+O\left(\frac{1}{n^3}\right),\frac{sin\pi z_n}{z_n}=(-1)^n\frac{\pi C_1}{n^2}+o\left(\frac{1}{n^2}\right),cos\pi z_n=(-1)^n\left(1-\frac{\pi^2 C_1^2}{2n^2}\right)+o\left(\frac{1}{n^2}\right)$$

$$cosz_n(\pi-\tau_2)=(-1)^n\left(\left(1-\frac{(\pi-\tau_2)^2C_1^2}{2n^2}\right)cosn\tau_2+\left(\frac{(\pi-\tau_2)C_1}{n}+\frac{(\pi-\tau_2)C_2}{n^2}\right)sinn\tau_2\right)+o\left(\frac{1}{n^2}\right)$$

$$sinz_n(\pi-\tau_2)=(-1)^{n+1}sinn\tau_2+(-1)^n\frac{(\pi-\tau_2)C_1}{n}cosn\tau_2+O\left(\frac{1}{n^2}\right)$$

$$\tilde{a}_c(z_n)=(-1)^n\tilde{a}_{2n}+(-1)^n\pi\frac{C_1}{n}\tilde{b}_{2n}-2(-1)^n\frac{C_1}{n}\tilde{b}^*_{2n}+o\left(\frac{1}{n^2}\right)$$

$$\tilde{a}_s(z_n)=(-1)^{n+1}\tilde{b}_{2n}+o\left(\frac{1}{n}\right),a_s^{c,s,c}(z_n)=(-1)^{n+1}{b_{2n}}^{c,s,c}+o\left(\frac{1}{n}\right),u_c(z_n)=o(1) \tag{30}$$

where

$$\tilde{a}_{2n}=\int_0^\pi\tilde{q}(t_1)cos2nt_1dt_1,\tilde{b}_{2n}=\int_0^\pi\tilde{q}(t_1)sin2nt_1dt_1,\tilde{b}^*_{2n}=\int_0^\pi t\tilde{q}(t_1)sin2nt_1dt_1,$$

$${b_{2n}}^{c,s,c}=\int_0^\pi K^{c,s,c}\big(t_1,q_1(t_1),q_2(t_1)\big)sin2nt_1dt_1,$$

Inserting asymptotic relations (30) in the equation $F(z_n)=0$,we get the system of equations:

$$\pi C_1-R_1-R_2cosn\tau_2-\frac{1}{2}\tilde{a}_{2n}=0,$$

$$\pi C_2-R_2(\pi-\tau_2)C_1sinn\tau_2-\frac{\pi C_1}{2}\tilde{b}_{2n}+C_1\tilde{b}^*_{2n}+R_4sinn\tau_2+R_5sin2n\tau_2+\frac{H-h}{2}\tilde{b}_{2n}-\frac{1}{4}b_{2n}^{c,s,c}=0,$$

$$\left(\pi C_3+\pi {C_1}^2-\frac{\pi^3{C_1}^3}{6}\right)+R_1\frac{\pi^2C_1^2}{2}+R_2\frac{(\pi-\tau_2)^2C_1^2}{2}cosn\tau_2-R_2(\pi-\tau_2)C_2sinn\tau_2-R_3\pi C_1-$$
$$R_4(\pi-\tau_2)C_1cosn\tau_2-R_5(\pi-2\tau_2)C_1cos2n\tau_2-R_6-R_7cosn\tau_2-R_8cos2n\tau_2-R_9cos3n\tau_2=0 \tag{31}$$

From the first equation of the system (31), we get

$$C_1=\frac{R_1}{\pi}+\frac{R_2}{\pi}cosn\tau_2+\frac{\tilde{a}_{2n}}{2\pi}=\frac{h+H}{\pi}+\frac{J_1^1}{2\pi}+\frac{J_1^2}{2\pi}cosn\tau_2+\frac{\tilde{a}_{2n}}{2\pi} \tag{32}$$

From the second equation of the system (31), we get

$$C_2=\left(\frac{\pi-\tau_2}{\pi}R_2sinn\tau_2+\frac{\tilde{b}_{2n}}{2}-\frac{\tilde{b}^*_{2n}}{\pi}\right)C_1-\frac{1}{\pi}R_4sinn\tau_2-\frac{1}{\pi}R_5sin2n\tau_2-\frac{H-h}{2\pi}\tilde{b}_{2n}+\frac{b_{2n}^{c,s,c}}{4\pi} \tag{33}$$

or

$$C_2=l_0\tilde{b}_{2n}+l_1sinn\tau_2+l_2sin2n\tau_2+l_3 \tag{34}$$

where

$$l_0=\frac{R_1}{2\pi}-\frac{H-h}{2\pi}+\frac{R_2}{2\pi}cosn\tau_2=\frac{h}{\pi}+\frac{J_1^1}{4\pi}+\frac{J_1^2}{4\pi}cosn\tau_2$$
$$l_1=\frac{\pi-\tau_2}{\pi^2}R_1R_2+\frac{\pi-\tau_2}{2\pi^2}R_2\tilde{a}_{2n}-\frac{1}{\pi}R_4=$$
$$-\frac{\tau_2(h+H)}{2\pi^2}\ J_1^2+\frac{\pi-\tau_2}{4\pi^2}\ J_1^1\ J_1^2-\frac{J_2^{12}+\ J_2^{21}}{4\pi}+\frac{\pi-\tau_2}{4\pi^2}\ J_1^2\tilde{a}_{2n}$$
$$l_2=\frac{\pi-\tau_2}{2\pi^2}(R_2)^2-\frac{1}{\pi}R_5=\frac{\pi-\tau_2}{8\pi^2}(\ J_1^2)^2-\frac{J_2^2}{4\pi}$$
$$l_3=-\frac{\tilde{b}_{2n}^*}{\pi}\left(\frac{R_1}{\pi}+\frac{R_2}{\pi}cosn\tau_2+\frac{\tilde{a}_{2n}}{2\pi}\right)+\frac{\tilde{b}_{2n}\tilde{a}_{2n}}{4\pi}+\frac{b_{2n}^{c,s,c}}{4\pi}=$$
$$-\tilde{b}_{2n}^*\left(\frac{h+H}{\pi^2}+\frac{J_1^1}{2\pi^2}+\frac{J_1^2}{2\pi^2}cosn\tau_2+\frac{\tilde{a}_{2n}}{2\pi^2}\right)+\frac{\tilde{b}_{2n}\tilde{a}_{2n}}{4\pi}+\frac{b_{2n}^{c,s,c}}{4\pi}. \tag{35}$$

From the third equation of the system (31), we get

$$C_3=\frac{\pi^2}{6}{C_1}^3-\left(1+\frac{\pi}{2}R_1+\frac{(\pi-\tau_2)^2}{2\pi}R_2cosn\tau_2\right)C_1^2+$$
$$\left(R_3+\frac{\pi-\tau_2}{\pi}R_4cosn\tau_2+\frac{\pi-2\tau_2}{\pi}R_5cos2n\tau_2\right)C_1+\frac{\pi-\tau_2}{\pi}R_2sinn\tau_2C_2+$$
$$\frac{R_6}{\pi}+\frac{R_7}{\pi}cosn\tau_2+\frac{R_8}{\pi}cos2n\tau_2+\frac{R_9}{\pi}cos3n\tau_2+o(1) \tag{36}$$

Using trigonometric identities

$$cos^3x=\frac{1}{4}(cos3x+3cosx),cos^2x=\frac{1}{2}(1+cos2x),sin^2x=\frac{1}{2}(1-cos2x)$$
$$sinxsin2x=\frac{1}{2}(cosx-cos3x),cosxcos2x=\frac{1}{2}(cosx+cos3x)$$

from (32), we get

$$C_1^3=\frac{R_1^3}{\pi^3}+\frac{3R_1R_2^2}{2\pi^3}+\left(\frac{3R_2R_1^2}{\pi^3}+\frac{3R_2^3}{4\pi^3}\right)cosn\tau_2+\frac{3R_1R_2^2}{2\pi^3}cos2n\tau_2+\frac{R_2^3}{4\pi^3}cos3n\tau_2+o(1)$$
$$C_1^2=\frac{R_1^2}{\pi^2}+\frac{R_2^2}{2\pi^2}+\frac{2R_2R_1}{\pi^2}cosn\tau_2+\frac{R_2^2}{2\pi^2}cos2n\tau_2+o(1) \tag{37}$$

Substituting (32), (33) and (37) in (36), we get

$$C_3=d_0+d_1cosn\tau_2+d_2cos2n\tau_2+d_3cos3n\tau_2+o(1) \tag{38}$$

where

$$d_0=-\frac{R_1^3}{3\pi}-\frac{R_1^2}{\pi^2}-\frac{R_2^2}{2\pi^2}+\frac{R_1R_3}{\pi}+\frac{R_6}{\pi},$$
$$d_1=\frac{\pi^2-(\pi-\tau_2)^2}{8\pi^3}R_2^3-\frac{\pi^2+(\pi-\tau_2)^2}{2\pi^3}R_1^2R_2-\frac{2R_1R_2}{\pi^2}+\frac{\pi-\tau_2}{\pi^2}R_1R_4+\frac{R_3R_2}{\pi}-\frac{\tau_2}{2\pi^2}R_2R_5+\frac{R_7}{\pi},$$
$$d_2=-\frac{R_2^2}{2\pi^2}-\frac{(\pi-\tau_2)^2}{\pi^3}R_1R_2^2+\frac{\pi-\tau_2}{\pi^2}R_2R_4+\frac{\pi-2\tau_2}{\pi^2}R_1R_5+\frac{R_8}{\pi},$$
$$d_3=\frac{\pi^2-9(\pi-\tau_2)^2}{24\pi^3}R_2^3+\frac{2\pi-3\tau_2}{2\pi^2}R_2R_5+\frac{R_9}{\pi} \tag{39}$$

Now, we can formulate the theorem about the asymptotic of eigenvalues of the operator L.

Theorem 3.1. If $q_i \epsilon L_2[0,\pi], i = 1,2$, then the asymptotic of eigenvalues λ_n of the operator L have the representation in the form:

$$\lambda_n = z_n^2 = n^2 + r_0 + r_1 cosn\tau_2 + \frac{\tilde{a}_{2n}}{\pi} + r_2\frac{\tilde{b}_{2n}}{n} + r_3\frac{sinn\tau_2}{n} + r_4\frac{sin2n\tau_2}{n} + \frac{r_5}{n} + \frac{r_6}{n^2} + r_7\frac{cosn\tau_2}{n^2} + r_8\frac{cos2n\tau_2}{n^2} + r_9\frac{cos3n\tau_2}{n^2} + o\left(\frac{1}{n^2}\right), \tag{40}$$

with coefficients

$$r_0 = \frac{2h + 2H + J_1^1}{\pi}, r_1 = \frac{J_1^2}{\pi}, r_2 = \frac{2h}{\pi} + \frac{J_1^1}{2\pi} + \frac{J_1^2}{2\pi}cosn\tau_2$$

$$r_3 = -\frac{\tau_2(h+H)}{\pi^2}J_1^2 + \frac{\pi - \tau_2}{2\pi^2}J_1^1 J_1^2 - \frac{J_2^{12} + J_2^{21}}{2\pi} + \frac{\pi - \tau_2}{2\pi^2}J_1^2\tilde{a}_{2n},$$

$$r_4 = \frac{\pi - \tau_2}{4\pi^2}(J_1^2)^2 - \frac{J_2^2}{2\pi}, r_5 = -\frac{\tilde{b}_{2n}^*}{\pi}\left(\frac{2h + 2H + J_1^1}{\pi} + \frac{J_1^2}{\pi}cosn\tau_2 + \frac{\tilde{a}_{2n}}{\pi}\right) + \frac{\tilde{b}_{2n}\tilde{a}_{2n}}{2\pi} + \frac{b_{2n}^{c,s,c}}{2\pi}$$

$$r_6 = -\frac{2\pi(h^3 + H^3) + 3(h+H)^2}{3\pi^2} - \frac{(J_1^1)^3}{12\pi} - \frac{(J_1^1)^2}{4\pi^2} - \frac{(h+H)J_1^1}{\pi^2} - \frac{(J_1^2)^2}{8\pi^2} + \frac{J_1^1 J_2^1}{4\pi} - \frac{J_3^1}{4\pi}$$

$$r_7 = \frac{\pi^2 - (\pi - \tau_2)^2}{32\pi^3}(J_1^2)^3 - \frac{\tau_2{}^2(h+H)^2 + 2\pi(h + H + \pi hH)}{2\pi^3}J_1^2 + \frac{\tau_2(\pi - \tau_2)(h+H) - \pi}{2\pi^3}J_1^1 J_1^2 - \frac{\pi^2 + (\pi - \tau_2)^2}{8\pi^3}(J_1^1)^2 J_1^2 - \frac{1}{4\pi}J_1^2 J_2^1 - \frac{\tau_2}{8\pi^2}J_1^2 J_2^2 + \left(\frac{(\pi - \tau_2)}{4\pi^2}J_1^1 - \frac{\tau_2(h+H)}{2\pi^2}\right)(J_2^{12} + J_2^{21}) - \frac{1}{4\pi}(J_3^{211} + J_3^{121} + J_3^{112})$$

$$r_8 = -\frac{(\pi - \tau_2)^2}{4\pi^3}J_1^1(J_1^2)^2 + \frac{4\tau_2(\pi - \tau_2)(h+H) - \pi}{8\pi^3}(J_1^2)^2 + \frac{\pi - \tau_2}{4\pi^2}J_1^2(J_2^{12} + J_2^{21}) + \frac{\pi - 2\tau_2}{4\pi^2}J_1^1 J_2^2 - \frac{\tau_2(h+H)}{2\pi^2}J_2^2 - \frac{1}{4\pi}(J_3^{122} + J_3^{212} + J_3^{221})$$

$$r_9 = \frac{\pi^2 - 9(\pi - \tau_2)^2}{96\pi^3}(J_1^2)^3 + \frac{2\pi - 3\tau_2}{8\pi^2}J_1^2 J_2^2 - \frac{J_3^2}{4\pi}. \tag{41}$$

Proof. Since

$$F(z) = F(-z)$$

the zeros of the characteristic function appear in pairs, i.e.

$$z_n = \pm\left[n + \frac{C_1}{n} + \frac{C_2}{n^2} + \frac{C_3}{n^3} + o\left(\frac{1}{n^3}\right)\right].$$

Then, we get

$$z_n{}^2 = n^2 + 2C_1 + \frac{2C_2}{n} + \frac{2C_3 + C_1{}^2}{n^2} + o\left(\frac{1}{n^2}\right),$$

or

$$z_n^2 = n^2 + \frac{2R_1}{\pi} + \frac{2R_2}{\pi} cosn\tau_2 + \frac{\tilde{a}_{2n}}{\pi} + \frac{2}{n}\left(l_0\tilde{b}_{2n} + l_1 sinn\tau_2 + l_2 sin2n\tau_2 + l_3\right) +$$
$$\frac{1}{n^2}\left(\frac{{R_1}^2}{\pi^2} + \frac{{R_2}^2}{2\pi^2} + 2d_0\right) + \left(\frac{2R_1R_2}{\pi^2} + 2d_1\right)\frac{cosn\tau_2}{n^2} + \left(\frac{R_2^2}{2\pi^2} + 2d_2\right)\frac{cos2n\tau_2}{n^2} + 2d_3\frac{cos3n\tau_2}{n^2} + o\left(\frac{1}{n^2}\right).$$

From here, we get

$$r_0 = \frac{2R_1}{\pi}, r_1 = \frac{2R_2}{\pi}, r_2 = 2l_0, r_3 = 2l_1, r_4 = 2l_2, r_5 = 2l_3,$$
$$r_6 = \frac{{R_1}^2}{\pi^2} + \frac{{R_2}^2}{2\pi^2} + 2d_0 = -\frac{2R_1^3}{3\pi} - \frac{R_1^2}{\pi^2} - \frac{R_2^2}{2\pi^2} + \frac{2R_1R_3}{\pi} + \frac{2R_6}{\pi},$$
$$r_7 = \frac{2R_1R_2}{\pi} + 2d_1 = \frac{\pi^2 - (\pi-\tau_2)^2}{4\pi^3}R_2^3 - \frac{\pi^2 + (\pi-\tau_2)^2}{4\pi^3}R_2R_1^2 + \frac{1}{\pi^2}R_2R_1 +$$
$$\frac{2(\pi-\tau_2)}{\pi^2}R_1R_4 + \frac{2R_3R_2}{\pi} - \frac{\tau_2}{\pi^2}R_2R_5 + \frac{2R_7}{\pi}$$
$$r_8 = \frac{R_2^2}{2\pi^2} + 2d_2 = -\frac{R_2^2}{2\pi^2} - \frac{2(\pi-\tau_2)^2}{\pi^3}R_1R_2^2 + \frac{2(\pi-\tau_2)}{\pi^2}R_2R_4 + \frac{2(\pi-2\tau_2)}{\pi^2}R_1R_5 + \frac{2R_8}{\pi}.$$
$$r_9 = 2d_3 \tag{42}$$

Inserting (29)and (39) in (42), we get coefficients of representation from (41), thus proving the theorem.

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Received January 25, 2015.