STATEMENT OF RETRACTION

We are informed by Dr. Graeme Fairweather, Executive Editor of Mathematical Reviews, that the paper by Asit Kumar Sarkar entitled "On partial bilateral and improper partial bilateral generating functions" which appears in Mathematica Montisnigri 17(2004), 57-66, is essentially identical to the paper with the same title and by the same author which appears in Filomat 18 (2004), 41-49.

We sent an email to Mr. Sarkar about this situation and we demanded an explanation from him but we haven't received any response.

The author did not respect following requirements:

1. Authors of articles submitted to Mathematica Montisnigri are required that the same or similar article has not been published previously and is not simultaneously considered for publication elsewhere.

2. Authors of articles published in Mathematica Montisnigri should not publish the same or similar article elsewhere, unless an agreement from Mathematica Montisnigri is obtained.

Consequently, Editorial Board of Matematica Montisnigri made a decision to retract paper "On partial bilateral and improper partial bilateral generating functions" by Asit Kumar Sarkar and Mr. Sarkar will be denied to publish in Mathematica Montisnigri in the future.

Editor in Chief of Mathematica Montisnigri

Žarko Pavićević

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ON PARTIAL BILATERAL AND IMPROPER PARTIAL BILATERAL GENERATING FUNCTIONS INVOLVING SOME SPECIAL FUNCTIONS

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Abstract. A group-theoretic method of obtaining more general class of generating function from a given class of improper partial bilateral generating functions involving Hermite, Laguerre and Gegenbauer polynomials are discussed.

1. Introduction

The usual generating relation involving one special function may be called linear or unilateral generating relation. By the term usual (proper) bilateral generating function we mean a function $G(x, z, w)$ which can be expanded in powers of w in the following relation

$$G(x, z, w) = \sum_{n=0}^{\infty} a_n f_n(x) g_n(z) w^n,$$

where a_n is arbitrary that is independent of x and z and $f_n(x)$, $g_n(z)$ are two different special functions.

In particular, when two special functions are same that is $f_n \equiv g_n$, we call the generating relation as bilinear generating relation.

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Unlike the usual (proper) bilateral or bilinear generating relations [5], we shall introduce the concepts of usual (proper) partial bilateral generating relation and improper partial bilateral generating relation.

Definition 1.1. By the term usual (proper) partial bilateral generating relation for two classical polynomials, we mean the relation:

$$G(x,z,w)=\sum_{n=0}^{\infty} a_n w^n p_{n+m}^{(\alpha)}(x) q_{m+n}^{(\beta)}(z), \tag{1.1.1}$$

where the coefficients a_n's are quite arbitrary and $p_{n+m}^{(\alpha)}(x)$, $q_{m+n}^{(\beta)}(z)$ are any two classical polynomials of order $(m+n)$ and of parameters α and β respectively.

Definition 1.2. By the term improper partial bilateral generating relation for two classical polynomials, we mean the relation:

$$G(x,z,w)=\sum_{n=0}^{\infty} a_n w^n p_{n+m}^{(\alpha)}(x) q_{k+n}^{(\beta)}(z), \tag{1.2.1}$$

where the coefficients a_n's are quite arbitrary and $p_{n+m}^{(\alpha)}(x)$, $q_{k+n}^{(\beta)}(z)$ are any two classical polynomials of order $(m+n)$, $(k+n)$ and of parameters α, β respectively.

The object of this paper is establish some general class of generating functions from a given class of improper partial bilateral generating functions.

2. Main results

a) For improper partial bilateral generating functions.

Theorem 2.1. *I there exist the following class of improper partial bilateral generating functions for the Hermite and Laguerre polynomials by means of the relation*

$$G(x,z,w)=\sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) L_{k+n}^{(\alpha)}(z), \tag{2.1.1}$$

where a_n is arbitrary, then the following general class of generating functions hold:

$$\exp\left(2wx-w^2\right)(1-v)^{-(\alpha+k+1)}\exp\left(-\frac{vz}{1-v}\right)$$

$$\times G\left(x-w, \frac{z}{1-v}, \frac{wv}{1-v}\right)$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n w^{n+s} v^{n+r} \frac{(k+n+1)!}{s!r!} H_{m+n+s}(x) L_{k+n+r}^{(\alpha)}(z),$$

where $|v|<1$.

Proof. Multiplying both sides of (2.1.1) by $y^m t^k$, we get

$$y^m t^k G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n \left(H_{m+n}(x) y^m\right)\left(L_{k+n}^{(\alpha)}(z) t^k\right). \tag{2.1.2}$$

Now replacing w by $wvyt$ in (2.1.2) we get

$$\begin{aligned} y^m t^k G(x,z,wvyt) \\ &= \sum_{n=0}^{\infty} a_n (wv)^n \left(H_{m+n}(x) y^{m+n}\right)\left(L_{k+n}^{(\alpha)}(z) t^{k+n}\right). \end{aligned} \tag{2.1.3}$$

We now choose the following two operators R_1 and R_2 of one-parameter groups ([1],[2]) namely

$$R_1 = 2xy - y\frac{\partial}{\partial x} \quad \text{and} \quad R_2 = zt\frac{\partial}{\partial z} + t^2\frac{\partial}{\partial t} + (\alpha+1-z)t$$

so that

$$R_1[H_{m+n}(x) y^{m+n}] = H_{m+n+1}(x) y^{m+n+1},$$
$$R_2[L_{k+n}^{(\alpha)}(z) t^{k+n}] = (k+n+1) L_{k+n+1}^{(\alpha)}(z) t^{k+n+1}$$

and

$$\exp(wR_1) f(x,y) = \exp(2wxy - w^2y^2) f(x-wy, y),$$
$$\exp(vR_2) f(z,t) = (1-vt)^{-\alpha-1}\exp\left(-\frac{vzt}{1-vt}\right) \times f\left(\frac{z}{1-vt}, \frac{t}{1-vt}\right),$$

where $|vt|<1$, ([3], [4]).

We now operate both sides of (2.1.3) by $\exp(wR_1)\exp(vR_2)$ and as a result of it, the relation (2.1.3) becomes:

$$\exp\left(2wxy - w^2y^2\right)(1-vt)^{-\alpha-1}\exp\left(-\frac{vzt}{1-vt}\right)$$

$$\times\, y^m \left(\frac{t}{1-vt}\right)^k G\left(z-wy, \frac{z}{1-vt}, \frac{wvyt}{1-vt}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n (wv)^n \left(\frac{(wR_1)^s}{s!}\left(H_{m+n}(x)y^{m+n}\right)\right)$$
$$\times \left(\frac{(vR_2)^r}{r!}\left(L_{k+n}^{(\alpha)}(z)t^{k+n}\right)\right)$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{w^{n+s}v^{n+r}}{s!r!}(k+n+1)_r$$
$$\times \left(H_{m+n+s}(x)y^{m+n+s}\right)\left(L_{k+n+r}^{(\alpha)}(z)t^{k+n+r}\right).$$

Now putting $y = t = 1$ in the above relation, we get:

$$\exp\left(2wx - w^2\right)(1-v)^{-(\alpha+k+1)}\exp\left(-\frac{vz}{1-v}\right)$$
$$\times\, G\left(z-w, \frac{z}{1-v}, \frac{wv}{1-v}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{w^{n+s}v^{n+r}}{s!r!}(k+n+1)_r$$
$$\times\, H_{m+n+s}(x)L_{k+n+r}^{(\alpha)}(z),$$

where

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x)L_{k+n}^{(\alpha)}(z) \quad \text{and} \quad |v| < 1. \quad \square$$

Theorem 2.2. *If there exist the following class of improper partial bilateral generating functions for Hermite and Gegenbauer polynomials by means of the relation*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x)C_{k+n}^{(\alpha)}(z), \tag{2.2.1}$$

where a_n is arbitrary, then the following general class of generating functions hold:

$$\exp\left(2wx - w^2\right)\left(1 - vz + v^2\right)^{-\alpha-\frac{k}{2}}$$
$$\times\, G\left(z-w, \frac{z-v}{\sqrt{1-vz+v^2}}, \frac{wv}{\sqrt{1-vz+v^2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{w^{n+s}v^{n+r}}{s!r!}(k+n+1)_r \times H_{m+n+s}(x) C_{k+n+r}^{(\alpha)}(z),$$

where $|2vz - v^2| < 1$.

Proof. Multiplying both sides of (2.2.1) by $y^m t^k$, we get:

$$y^m t^k G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n \left(H_{m+n}(x) y^m\right)\left(C_{k+n}^{(\alpha)}(z) t^k\right). \tag{2.2.2}$$

Now replacing w by $wvyt$ in (2.2.2), we get

$$y^m t^k G(x,z,wvyt) = \sum_{n=0}^{\infty} a_n (wv)^n \times \left(H_{m+n}(x) y^{m+n}\right)\left(C_{k+n}^{(\alpha)}(z) t^{k+n}\right). \tag{2.2.3}$$

We now choose the following two operators R_1 and R_2 of one-parameters groups ([1],[2]) namely

$$R_1 = 2xy - y\frac{\partial}{\partial x} \quad \text{and} \quad R_2 = \left(z^2-1\right)t\frac{\partial}{\partial z} + tz^2\frac{\partial}{\partial t} + (2\alpha+k)zt$$

so that

$$R_1[H_{m+n}(x)y^{m+n}] = H_{m+n+1}(x)y^{m+n+1},$$
$$R_2[C_{k+n}^{(\alpha)}(z)t^{k+n}] = (k+n+1)C_{k+n+1}^{(\alpha)}(z)t^{k+n+1}$$

and

$$\exp(wR_1)f(x,y) = \exp\left(2wxy - w^2y^2\right)f(x-wy,y),$$
$$\exp(vR_2)f(z,t) = \left(1-2vzt+v^2t^2\right)^{-\alpha} \times f\left(\frac{z-vt}{\sqrt{1-2vzt+v^2t^2}}, \frac{t}{\sqrt{1-2vzt+v^2t^2}}\right),$$

where $\left|2vzt - v^2t^2\right| < 1$, ([3],[4]).

We now operate both sides of (2.2.3) by $\exp(wR_1)\exp(vR_2)$ and as a result of it, the relation (2.2.3) becomes

$$\exp\left(2wxy - w^2y^2\right)\left(1-2vzt+v^2t^2\right)^{-\alpha} y^m \left(\frac{t}{\sqrt{1-2vzt+v^2t^2}}\right) \times G\left(x-wy, \frac{z-vt}{\sqrt{1-2vzt+v^2t^2}}, \frac{wvyt}{\sqrt{1-2vzt+v^2t^2}}\right)$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n (wv)^n \left(\frac{(wR_1)^s}{s!} H_{m+n}(x) y^{m+n} \right) \\
&\quad \times \left(\frac{(vR_2)^r}{r!} C_{k+n}^{(\alpha)}(z) t^{k+n} \right) \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{w^{n+s} v^{n+r}}{s!r!} (k+n+1)_r \\
&\quad \times \left(H_{m+n+s}(x) y^{m+n+s} \right) \left(C_{k+n+r}^{(\alpha)}(z) t^{k+n+r} \right).
\end{aligned}
$$

Now putting $y = t = 1$ in the above relation, we get:

$$
\begin{aligned}
&\exp\left(2wxy - w^2\right)\left(1 - 2vz + v^2\right)^{-\alpha - \frac{k}{2}} \\
&\quad \times G\left(x - w, \frac{z - vt}{\sqrt{1 - 2vz + v^2}}, \frac{wvt}{\sqrt{1 - 2vz + v^2}} \right) \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{w^{n+s} v^{n+r}}{s!r!} (k+n+1)_r \\
&\quad \times H_{m+n+s}(x) C_{k+n+r}^{(\alpha)}(z),
\end{aligned}
$$

where

$$
G(x, z, w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) C_{k+n}^{(\alpha)}(z)
$$

and $|2vz - v^2| < 1$. □

Theorem 2.3. *If there exist the following class of improper partial bilateral generating functions for Laguerre and Gegenbauer polynomials by means of the relation*

$$
G(x, z, w) = \sum_{n=0}^{\infty} a_n w^n L_{m+n}^{(\alpha)}(x) C_{k+n}^{(\beta)}(z), \tag{2.3.1}
$$

where a_n is arbitrary, then the following general class of generating functions hold:

$$
\begin{aligned}
&(1 - w)^{-\alpha - m - 1}\left(1 - 2vz + v^2\right)^{-\beta - \frac{k}{2}} \exp\left(-\frac{wx}{1 - w} \right) \\
&\quad \times G\left(\frac{x}{1 - w}, \frac{z - v}{\sqrt{1 - 2vz + v^2}}, \frac{wv}{(1 - w)\sqrt{1 - 2vz + v^2}} \right)
\end{aligned}
$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n w^{n+s} v^{n+r} \frac{(m+n+1)_s}{s!}\frac{(m+n+1)_r}{r!} \times L_{m+n+s}^{(\alpha)}(x) C_{k+n+r}^{(\beta)}(z),$$

where $|2vz - v^2| < 1$.

Proof. Multiplying both sides of (2.3.1) by $y^m t^k$, we get:

$$y^m t^k G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n \left(L_{m+n}^{(\alpha)}(x) y^m\right)\left(C_{k+n}^{(\beta)}(z) t^k\right). \tag{2.3.2}$$

Now replacing w by $wvyt$ in (2.3.2), we get

$$y^m t^k G(x,z,wvyt) = \sum_{n=0}^{\infty} a_n (wv)^n \times \left(L_{m+n}^{(\alpha)}(x) y^{m+n}\right)\left(C_{k+n}^{(\beta)}(z) t^{k+n}\right). \tag{2.3.3}$$

We now choose the following two operators R_1 and R_2 of one-parameter groups ([1],[2]) namely

$$R_1 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (\alpha + 1 - x)y \quad \text{and}$$

$$R_2 = (z^2-1)t\frac{\partial}{\partial z} + zt^2\frac{\partial}{\partial t} + (2\beta + k)zt,$$

so that

$$R_1\left[L_{m+n}^{(\alpha)}(x) y^{m+n}\right] = (m+n+1) L_{m+n+1}^{(\alpha)}(x) y^{m+n+1},$$

$$R_2\left[C_{k+n}^{(\beta)}(x) t^{k+n}\right] = (k+n+1) C_{k+n+1}^{(\beta)}(z) t^{k+n+1}$$

and

$$\exp(wR_1) f(x,y) = (1-wy)^{-\alpha-1} \exp\left(-\frac{wxy}{1-wy}\right) \times f\left(\frac{x}{1-wy}, \frac{y}{1-wy}\right),$$

$$\exp(vR_2) f(z,t) = \left(1 - 2vzt + v^2t^2\right)^{-\beta} \times f\left(-\frac{z-vt}{\sqrt{1-2vzt+v^2t^2}}, \frac{t}{\sqrt{1-2vzt+v^2t^2}}\right),$$

where $\left|2vzt - v^2t^2\right| < 1$ ([3],[4]).

We now operate both sides of (2.3.3) by $\exp(wR_1)\exp(vR_2)$ and as a result of it, the relation (2.3.3) reduces to

$$
\begin{aligned}
&(1-wy)^{-\alpha-1}(1-2vzt+v^2t^2)^{-\beta}\exp\left(-\frac{wxy}{1-wy}\right)\\
&\times\left(\frac{y}{1-wy}\right)^m\left(\frac{t}{\sqrt{1-2vzt+v^2t^2}}\right)^k\\
&\times G\left(\frac{x}{1-wy},\frac{z-vt}{\sqrt{1-2vzt+v^2t^2}},\frac{wvyt}{(1-wy)\sqrt{1-2vzt+v^2t^2}}\right)\\
&=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}a_n(wv)^n\left(\frac{(wR_1)^s}{s!}L_{m+n}^{(\alpha)}(x)y^{m+n}\right)\\
&\times\left(\frac{(vR_2)^r}{r!}C_{k+n}^{(\beta)}(z)t^{k+n}\right)\\
&=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}a_n w^{n+s}v^{n+r}\frac{(m+n+1)_s}{s!}\frac{(k+n+1)_r}{r!}\\
&\times L_{m+n+s}^{(\alpha)}(x)y^{m+n+s}C_{k+n+r}^{(\beta)}(z)t^{k+n+r}.
\end{aligned}
$$

Now putting $y=t=1$ in the above relation, we get:

$$
\begin{aligned}
&(1-w)^{-\alpha-m-1}(1-2vz+v^2)^{-\beta-\frac{k}{2}}\exp\left(-\frac{wx}{1-w}\right)\\
&\quad\times G\left(\frac{x}{1-w},\frac{z-v}{\sqrt{1-2vz+v^2}},\frac{wv}{(1-w)\sqrt{1-2vz+v^2}}\right)\\
&\quad=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}a_n w^{n+s}v^{n+r}\frac{(m+n+1)_s}{s!}\frac{(k+n+1)_r}{r!}\\
&\quad\times L_{m+n+s}^{(\alpha)}(x)C_{k+n+r}^{(\beta)}(z),
\end{aligned}
$$

where

$$G(x,z,w)=\sum_{n=0}^{\infty}a_n w^n L_{m+n}^{(\alpha)}(x)C_{k+n}^{(\beta)}(z)$$

and $|2vz-v^2|<1$. □

Particular Cases. It may be of interest to point out that for $k=m$, the above **Theorems** 2.1, 2.2 & 2.3 become nice general class of generating functions from the given class of usual (proper) partial bilateral generating functions, which need not be derived independently. We state those results in the following form:

b) For proper partial bilateral generating functions.

Theorem 2′.1. *If there exist the following class of (proper) partial bilateral generating functions for Hermite and Laguerre polynomials by means of the relation*

$$G(x,z,w)=\sum_{n=0}^{\infty}a_nw^nH_{m+n}(x)L_{m+n}^{(\alpha)}(z), \tag{2′.1.1}$$

where a_n is arbitrary, then the following general class of generating functions hold:

$$\begin{aligned}&\exp\left(2wx-w^2\right)(1-v)^{-(\alpha+m+1)}\exp\left(-\frac{vz}{1-v}\right)\\&\times G\left(x-w,\frac{z}{1-v},\frac{wv}{1-v}\right)\\&=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}a_nw^{n+s}v^{n+r}\frac{(m+n+1)_r}{s!r!}H_{m+n+s}(x)L_{m+n+r}^{(\alpha)}(z),\end{aligned}$$

where $|v|<1$.

Theorem 2′.2. *If there exist the following class of (proper) partial bilateral generating functions for Hermite and Gegenbauer polynomials by means of the relation*

$$G(x,z,w)=\sum_{n=0}^{\infty}a_nw^nH_{m+n}(x)C_{m+n}^{(\alpha)}(z), \tag{2′.2.1}$$

where a_n is arbitrary, then the following general class of generating functions hold:

$$\begin{aligned}&\exp\left(2wx-w^2\right)\left(1-2vz+v^2\right)^{-\beta-\frac{m}{2}}\\&\times G\left(x-w,\frac{z-v}{\sqrt{1-2vz+v^2}},\frac{wv}{\sqrt{1-2vz+v^2}}\right)\\&=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}a_nw^{n+s}v^{n+r}\frac{(m+n+1)_r}{s!r!}\\&\times H_{m+n+s}(x)C_{m+n+r}^{(\alpha)}(z),\end{aligned}$$

where $|2vz-v^2|<1$.

Theorem 2′.3. *If there exist the following class of (proper) partial bilateral generating functions for Laguerre and Gegenbauer polynomials by means of the relation*

$$G(x,z,w)=\sum_{n=0}^{\infty} a_n w^n L_{m+n}^{(\alpha)}(x) C_{m+n}^{(\beta)}(z), \tag{2′.3.1}$$

where a_n is arbitrary, then the following general class of generating functions hold:

$$\begin{aligned}
&(1-w)^{-\alpha-m-1}\left(1-2vz+v^2\right)^{-\beta-\frac{m}{2}} \exp\left(-\frac{wx}{1-w}\right)\\
&\quad\times G\left(\frac{x}{1-w}, \frac{z-v}{\sqrt{1-2vz+v^2}}, \frac{wv}{(1-w)\sqrt{1-2vz^2+v^2}}\right)\\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n w^{n+s} v^{n+r} \frac{(m+n+1)_s}{s!}\,\frac{(m+n+1)_r}{r!}\\
&\quad\times L_{m+n+s}^{(\alpha)}(x) C_{m+n+r}^{(\beta)}(z),
\end{aligned}$$

where $|2vz-v^2|<1$.

Remark. In a similar manner some new results can be derived for (proper) partial bilateral as well as improper partial bilinear generating functions.

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