GENERALIZATION OF A DISCRETE OPIAL TYPE INEQUALITY
APPLIED TO THE EIGENVALUES OF GRAPH

D. S. JOKANOVIĆ, T. Z. MIRKOVIĆ

University of East Sarajevo
Production and Management Faculty Trebinje
Trebinje, Republic of Srpska, BiH, e-mail: dusanjok@yahoo.com

Summary. In this paper we present the modifications of a discrete Opial type inequality applied to the eigenvalues of the graph $G$. We are going to derive a new proof of generalized inequality

$$-\cos \frac{\pi}{r+1} \sum_{k=1}^{n} x_k^2 \leq \sum_{k=1}^{n-m} x_k x_{k+m} \leq \cos \frac{\pi}{r+1} \sum_{k=1}^{n} x_k^2,$$

as well as its special case for $m=1$ and $m=2$, whereby use of well-known inequality for real eigenvalues of the graph $\sum_{k=1}^{n} \lambda_k^2 = 2m$, where $n$ is number of vertices, and $m$ is number of edges of graph. This proofs are much simpler and shorter then the one given in [6]. In proofs starting from a classic inequality

$$\left( \sqrt{\frac{\alpha_k}{\beta_k}} \lambda_k \pm \sqrt{\frac{\beta_k}{\alpha_k}} \lambda_{k+m} \right)^2 \geq 0,$$

where $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\tilde{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ are arbitrary n-tuples of real numbers, where $\alpha_k \beta_k > 0$, for each $k = 1, 2, \ldots, n$ and $\tilde{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are eigenvalues of graph $G$. After transformation of these inequalities, depending on the appropriate choice of function $\alpha_k$ and $\beta_k$ obtained above inequalities for different angles. Using trigonometric transformations is obtained their equivalent form

$$8m \sin^2 \frac{\pi}{2(r+1)} \leq \sum_{k=0}^{n} (\lambda_k - \lambda_{k+1})^2 \leq 8m \cos^2 \frac{\pi}{2(r+1)}.$$

It also enables to derive a number of other discrete inequalities related with it.

1 INTRODUCTION

In [6] Opial established the following integral inequality:

Theorem 1.1. (see [6]) Suppose $f \in C^1[0,h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0, h)$. Then the following integral inequality holds

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where the constant \( \frac{h}{4} \) is best possible.

This inequality, its generalizations and modifications play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations, as well as difference equations. These inequalities are also important in solving the problems in polynomial theory, approximation theory, spectral graph theory, and technical applications.

2 MAIN RESULTS

Let us prove the following theorem:

**Theorem 2.1.** If \( \lambda_0, \lambda_1, \ldots, \lambda_n, \lambda_{n+1} \), real eigenvalues of graph \( G = (V, E) \), \( |V| = n > 1 \), \( |E| = m > 1 \), \( \lambda_0 = \lambda_{n+1} = 0 \), then inequality

\[
-2m \cos \frac{\pi}{n+1} \leq \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1} \leq 2m \cos \frac{\pi}{n+1}
\]

holds. Equality in the left (right) side of inequality (3) holds if and only if

\[
\lambda_k = (-1)^{k-1} \sqrt{2m} \sin \frac{k\pi}{n+1} \quad (\lambda_k = \sqrt{2m} \sin \frac{k\pi}{n+1}) \quad k = 1, \ldots, n.
\]

**Proof.** Let \( \mathbf{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), \( \mathbf{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) and \( \mathbf{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \) are arbitrary n-type of real numbers, whereby \( \alpha_k \beta_k > 0 \) for all \( k = 1, \ldots, n \).

From inequality

\[
\left( \frac{\alpha_k \lambda_k + \beta_k \lambda_{k+1}}{\beta_k \alpha_k} \right)^2 \geq 0,
\]

where equality holds if and only if \( \alpha_k \lambda_k \pm \beta_k \lambda_{k+1} = 0 \), we get following inequality:

\[
\frac{\alpha_k}{\beta_k} \lambda_k^2 + \frac{\beta_k}{\alpha_k} \lambda_{k+1}^2 \geq \pm 2 \lambda_k \lambda_{k+1}
\]

\[
\sum_{k=1}^{n} \left( \frac{\alpha_k}{\beta_k} \lambda_k^2 + \frac{\beta_k}{\alpha_k} \lambda_{k+1}^2 \right) \geq \pm 2 \sum_{k=1}^{n} \lambda_k \lambda_{k+1}
\]

\[
\sum_{k=1}^{n} \left( \frac{\alpha_k}{\beta_k} + \frac{\beta_k}{\alpha_k} \lambda_{k+1} \right) \lambda_k^2 \geq \pm 2 \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1} + \frac{\beta_0}{\alpha_0} \lambda_1^2 + \frac{\alpha_n}{\beta_n} \lambda_n^2.
\]
where \( \lambda_{n+1} = 0 \).

By introducing \( \alpha_k = \sin(k+1)t \) and \( \beta_k = \sin kt \), inequality (4) becomes

\[
\sum_{k=1}^{n} \left( \frac{\sin(k+1)t}{\sin kt} + \frac{\sin(k-1)t}{\sin kt} \right) \lambda_k^2 \geq \pm 2 \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1} + \frac{\sin(n+1)t}{\sin nt} \lambda_n^2,
\]

\[
2 \cos t \sum_{k=1}^{n} \lambda_k^2 \geq \pm 2 \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1} + \frac{\sin(n+1)t}{\sin nt} \lambda_n^2.
\]

From \( \sin(n+1)t \geq 0 \), it follows that \( 0 < (n+1)t < \pi \) and \( t < \frac{\pi}{n+1} \) and we take \( t = \frac{\pi}{n+1} \), and the last inequality becomes

\[
\cos \frac{\pi}{n+1} \sum_{k=1}^{n} \lambda_k^2 \geq \pm \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1}.
\]

Based on the known values of equality for their own graph \( \sum_{k=1}^{n} \lambda_k^2 = 2m \), follow by

\[
-2m \cos \frac{\pi}{n+1} \leq \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1} \leq 2m \cos \frac{\pi}{n+1}.
\]

Depending on the choice of values \( \alpha_k \) and \( \beta_k \), as well as parameter \( m \) in inequality (4), leads to discrete inequalities related inequality (1).

**Corollary 2.2.** For real eigenvalues \( \lambda_0, \lambda_1, \ldots, \lambda_n, \lambda_{n+1} \), \( \lambda_0 = \lambda_{n+1} = 0 \) of graph G the inequalities

1. \[
\sum_{k=1}^{n} \left( k + m + \frac{1}{k + m + 1} \right) \lambda_k^2 \geq \pm 2 \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1} + (m+1) \lambda_n^2 + \frac{1}{n + m + 1} \lambda_n^2;
\]

2. \[
(m+1) \lambda_n^2 + \frac{1}{n + m + 1} \lambda_n^2 - \sum_{k=1}^{n} \left( \frac{1}{k + m + 1} + k + m \right) \lambda_k^2 \leq \pm 2 \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1}
\]

hold. Equality in first inequality holds iff \( \lambda_k = \frac{\sqrt{2m}}{(k + m)_{k-1}} \). Equality in second inequality holds iff \( \lambda_k = \frac{(-1)^{k-1} \sqrt{2m}}{(k + m)_{k-1}} \).

**Proof.** If we take in inequality (4) \( \alpha_k = \left( (k + m + 1) \right)^{-1} \) and \( \beta_k = \left( (k + m) \right)^{-1} \), \( m \geq 0 \), we obtain
\[ \sum_{k=1}^{n} \left[ \frac{(k + m + 1)^{-1}}{(k + m)} + \frac{(k + m - 1)^{-1}}{(k + m)} \right] \lambda_k^2 \geq \pm 2 \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1} + (m+1) \lambda_n^2 + \frac{1}{n + m + 1} \lambda_n^2 \]

\[ \sum_{k=1}^{n} \left( \frac{1}{k + m + 1} + k + m \right) \lambda_k^2 \geq \pm 2 \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1} + (m+1) \lambda_n^2 + \frac{1}{n + m + 1} \lambda_n^2 \]

from which follows both inequalities. \(\diamondsuit\)

**Theorem 2.3.** For each sequence of the real eigenvalues \(0 = \lambda_0, \lambda_1, ... , \lambda_n, \lambda_{n+1} = 0\), of a graph \(G\), the following inequalities hold

\[ 8m \sin^2 \frac{\pi}{2(n+1)} \leq \sum_{k=0}^{n} (\lambda_k - \lambda_{k+1})^2 \leq 8m \cos^2 \frac{\pi}{2(n+1)}. \tag{5} \]

**Proof.** From inequality (2) it follows that

\[ -2 \frac{2 \cos^2 \pi}{2(n+1)} - 1 \sum_{k=1}^{n} \lambda_k^2 \leq \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1} \leq \left(1 - 2 \sin^2 \frac{\pi}{2(n+1)}\right) \sum_{k=1}^{n} \lambda_k^2 \]

\[ 2 - 4 \cos^2 \frac{\pi}{2(n+1)} \sum_{k=1}^{n} \lambda_k^2 \leq 2 \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1} \leq \left(2 - 4 \sin^2 \frac{\pi}{2(n+1)}\right) \sum_{k=1}^{n} \lambda_k^2 \]

\[ \sum_{k=1}^{n} \lambda_k^2 - 4 \cos^2 \frac{\pi}{2(n+1)} \sum_{k=1}^{n} \lambda_k^2 \leq -4 \sum_{k=0}^{n} \lambda_k \lambda_{k+1} \leq \sum_{k=1}^{n} \lambda_k^2 - 4 \sin^2 \frac{\pi}{2(n+1)} \sum_{k=1}^{n} \lambda_k^2 \]

whence follows the desired inequality. \(\diamondsuit\)

**Theorem 2.4.** For each sequence of the real eigenvalues \(\lambda_1, ... , \lambda_n, \lambda_{n+1}\), of graph \(G\), the following inequalities hold

\[ -2m \cos \frac{\pi}{r+1} \leq \sum_{k=1}^{n-1} \lambda_k \lambda_{k+m} \leq 2m \cos \frac{\pi}{r+1} \tag{6} \]

where \(n = 2 \cdot r - q, \ 0 \leq q \leq 1\) and \(r = \left\lceil \frac{n-1}{2} \right\rceil + 1\). The equality in the left (right) side of this inequality exists if and only if \(\lambda_{2k+1} = 2 \sin \frac{(k+1) \pi}{r+1} (\lambda_{2k+1} = (-1)^i 2 \sin \frac{(k+1) \pi}{r+1})\). The index \(i\) takes values \(i = 1, 2, k = 0, 1, ..., r - 2\) and \(i = 1, ..., 2 - q\), for \(k = r - 1\).
Proof. From

$$\left( \frac{\alpha_k}{\beta_k} \lambda_k \pm \frac{\beta_k}{\alpha_k} \lambda_{k+2} \right)^2 \geq 0,$$

(7)

where equality holds if and only if $\alpha_k \lambda_k \pm \beta_k \lambda_{k+2} = 0$, we get:

$$\frac{\alpha_k^2}{\beta_k^2} + \frac{\beta_k^2}{\alpha_k^2} \geq \frac{2}{\alpha_k \lambda_{k+2}} \pm \frac{2}{\alpha_k \lambda_{k+2}} \cdot 2 \sum_{k=1}^{n} \lambda_k \lambda_{k+2}, \quad n - 2 > 0$$

where $\lambda_{n+1} = \lambda_{n+2} = 0$.

By substituted $\alpha_k = \sin(k + 2)t$ and $\beta_k = \sin2t$, the last inequality becomes

$$\sum_{k=1}^{n} \left( \frac{\sin(k + 2)t}{\sin t} + \frac{\sin(k - 2)t}{\sin t} \right) \lambda_k^2 \geq \pm 2 \sum_{k=1}^{n} \lambda_k \lambda_{k+2},$$

$$2 \cos 2t \sum_{k=1}^{n} \lambda_k^2 \geq \pm \sum_{k=1}^{n} \lambda_k \lambda_{k+2},$$

$$-2 \cos 2t \sum_{k=1}^{n} \lambda_k^2 \leq 2 \cos 2t \sum_{k=1}^{n} \lambda_k^2$$

Since $\sin(n + 2)t \geq 0$, it follows that

$$0 < (n + 2)t < \pi \Rightarrow 0 < t < \frac{\pi}{n + 2} = \frac{\pi}{2 \cdot r - q + 2} = \frac{\pi}{2(r + 1) - q},$$

(8)

where $n = 2 \cdot r - q$, $0 \leq q \leq 1$, $r = \left\lfloor \frac{n - 1}{2} \right\rfloor + 1$.

$$n = 2 \cdot r \Rightarrow t \leq \frac{\pi}{2r + 2} \Rightarrow \cos 2t = \cos \frac{\pi}{r + 1}$$

$$n = 2 \cdot r - 1 \Rightarrow t \leq \frac{\pi}{2r + 1} \Rightarrow t = \frac{\pi}{2r + 1} \Rightarrow \cos 2t = \cos \frac{\pi}{r + 1}.$$

Substituting these values in the last inequality is obtained desired inequality.

The following Theorem is a generalization of Theorem 1 and Theorem 4.

Theorem 2.5. If $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}$, are real eigenvalues of a graph $G$, then
\[-2m \cos \frac{\pi}{r+1} \leq \sum_{k=1}^{n-m} \lambda_k \lambda_{k+m} \leq 2m \cos \frac{\pi}{r+1}\]  \tag{9}

where \( n = m \cdot r - q, \quad r = \left[ \frac{n-1}{m} \right] + 1, \quad 0 \leq q \leq m - 1 \). The equality in the right (left) side of this inequality exists if and only if 
\[ \lambda_{km+i} = (-1)^{i-1} \sqrt{2m} \sin \frac{(k+1)\pi}{r+1} \]
\[ \lambda_{km+i} = (-1)^i \sqrt{2m} \sin \frac{(k+1)\pi}{r+1} \]. The index \( i \) takes values \( i = 1,...,m \), for \( k = 0,1,...,r-2 \) and \( i = 1,...,m-q \), where \( k = r-1 \).

**Proof.** From
\[
\left( \sqrt{\frac{\alpha_k}{\beta_k} \lambda_k} \pm \sqrt{\frac{\beta_k}{\alpha_k} \lambda_{k+m}} \right)^2 \geq 0, \quad m \geq 1,
\]
where equality holds if and only if \( \alpha_k \lambda_k \pm \beta_k \lambda_{k+m} = 0 \), we get:
\[
\frac{\alpha_k}{\beta_k} \lambda_k^2 + \frac{\beta_k}{\alpha_k} \lambda_{k+m}^2 \geq \pm 2 \lambda_k \lambda_{k+m}
\]
\[
\sum_{k=1}^{n} \left( \frac{\alpha_k}{\beta_k} + \frac{\beta_{k-m}}{\alpha_{k-m}} \right) \lambda_k^2 - \sum_{k=1}^{m} \frac{\beta_{k-m}}{\alpha_{k-m}} \lambda_k^2 \geq \pm 2 \sum_{k=1}^{n-m} \lambda_k \lambda_{k+m},
\]
\[
\sum_{k=1}^{n} \left( \frac{\alpha_k}{\beta_k} + \frac{\beta_{k-m}}{\alpha_{k-m}} \right) \lambda_k^2 \geq \sum_{k=1}^{n-m} \frac{\beta_{k-m}}{\alpha_{k-m}} \lambda_k^2 \pm 2 \sum_{k=1}^{n-m} \lambda_k \lambda_{k+m},
\]
Since \( \alpha_k \beta_k > 0 \) the left side of upper inequality is always positive, so
\[
\sum_{k=1}^{n} \left( \frac{\alpha_k}{\beta_k} + \frac{\beta_{k-m}}{\alpha_{k-m}} \right) \lambda_k^2 \geq \pm 2 \sum_{k=1}^{n-m} \lambda_k \lambda_{k+m}, \quad n - m > 0,
\]  \tag{10}
where \( \lambda_{n+1} = \ldots = \lambda_{n+m} = 0 \).

If \( \alpha_k \) and \( \beta_k \) are substituted by \( \alpha_k = \sin(k + m)t \) and \( \beta_k = \sin kt, \quad t \in \left[ 0, \frac{\pi}{4} \right] \), then (10) becomes
\[
\sum_{k=1}^{n} \left( \frac{\sin(k + m)t}{\sin t} \pm \frac{\sin(k - m)t}{\sin t} \right) \lambda_k^2 \geq \pm 2 \sum_{k=1}^{n-m} \lambda_k \lambda_{k+m},
\]
\[
\cos mt \sum_{k=1}^{n} \lambda_k^2 \geq \pm 2 \sum_{k=1}^{n-m} \lambda_k \lambda_{k+m}.
\]  \tag{11}
In accordance with the requirement \( \sin(n + m)t \geq 0 \), following inequality holds

\[
0, (n + m)t < \pi, \quad 0 < t < \frac{\pi}{n + m} = \frac{\pi}{m \cdot r - q + m} = \frac{\pi}{m(r + 1) - q},
\]

(12)

where \( n = m \cdot r - q, \) \( 0 \leq q \leq m - 1 \).

Based on the property \( 0 \leq q \leq m - 1 \), is valid \( \frac{\pi}{m(r + 1)} \leq \frac{\pi}{m(r + 1) - q} \), for each \( q \). From (12) maximum value for \( t \) is \( t = \frac{\pi}{m(r + 1)} \). If we substitute \( t \) in (11) we obtain

\[
\cos \frac{\pi}{r + 1} \sum_{k=1}^{n} \lambda_k^2 \geq \pm \sum_{k=1}^{n-m} \lambda_k \lambda_{k+m},
\]

i.e.

\[
-\cos \frac{\pi}{r + 1} \sum_{k=1}^{n} \lambda_k^2 \leq \sum_{k=1}^{n-m} \lambda_k \lambda_{k+m} \leq \cos \frac{\pi}{r + 1} \sum_{k=1}^{n} \lambda_k^2,
\]

(13)

whence follows the desired inequality. \( \diamond \)

If inequality (13), when \( m = 1 \) is multiplied \(-2pq\) and term \( p^2 \sum_{k=1}^{n} \lambda_k^2 + q^2 \sum_{k=1}^{n} \lambda_{k+1}, \lambda_{n+1} = 0,\) is added to each part of the inequality, after arranging the following inequality

\[
2m \left( p^2 + q^2 - 2pq \cos \frac{\pi}{n} \right) \leq \sum_{k=1}^{n+1} \left( p\lambda_k - q\lambda_{k+1} \right)^2 \leq 2m \left( p^2 + q^2 + 2pq \cos \frac{\pi}{n} \right).
\]

This inequality it is only equivalent form of the inequality (13) for the case \( m = 1 \).

**Theorem 2.6.** Let \( n = 2m, \ n \geq 4 \) and \( \lambda_1, \ldots, \lambda_n \) are real eigenvalues of a graph \( G \), when \( \lambda_{n+k} = \lambda_k, \ k = 1, \ldots, m. \) Then

\[
\sum_{k=1}^{n} (\lambda_k - \lambda_{k+1})^2 \geq \left( \sin \frac{2\pi}{n} - \sin \frac{\pi}{2n} \right) \sum_{k=1}^{n} (\lambda_k + \lambda_{n+1-k})^2 + 8m \cdot \sin \frac{\pi}{2n}.
\]

Equality holds if and only if \( \lambda_k = \sqrt{2m \left( \cos \frac{2k\pi}{n} + \sin \frac{2k\pi}{n} + \cos \frac{4k\pi}{n} + \sin \frac{4k\pi}{n} \right)}, \ k = 1, \ldots, m. \)

**Theorem 2.7.** Let \( \lambda_1, \ldots, \lambda_n \) are real eigenvalues of a graph \( G, \ n \geq 2. \) Then inequality

\[
\sum_{k=1}^{n} (\lambda_k - \lambda_{k+1})^2 \geq \left( \sin \frac{\pi}{n} - \sin \frac{\pi}{2n} \right) \sum_{k=1}^{n} (\lambda_k + \lambda_{n+1-k})^2 + 8m \cdot \sin \frac{\pi}{2n}
\]
holds. Equality holds if and only if

\[ \lambda_k = \sqrt{2m} \left( \cos \left( \frac{(2k-1)\pi}{2n} \right) + \cos \left( \frac{(2k-1)\pi}{2n} \right) \right), \quad k = 1, \ldots, n. \]

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