MULTIGRID METHOD FOR DIFFUSION EQUATIONS BASED ON ADAPTIVE SMOOTHING

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Summary. We focus on the study of multigrid method for the solution of the finite volume scheme for the diffusion equation. The features of the multigrid method and the adaptation of the smoothing procedures to the spectrum of the discrete operators are presented. The smoothers are based on Chebyshev's polynomials and allow us to achieve automatically the prescribed convergence rate of the multigrid solver even in the presence of a strong anisotropy in the problems.

1 INTRODUCTION

We present some results of development of multigrid method for solving three-dimensional diffusion equations with anisotropic discontinuous coefficients. For discretization it is used a convenient seven-point stencil on a Cartesian grid. In the main features the proposed algorithm represents an efficient parallel implementation of the geometrical multigrid method. A detailed presentation of the multigrid algorithm, which is a version of Fedorenko's method [1–2], can be found in [3–6].

The algorithm is intended for solving the stationary diffusion equations and it is able to solve the boundary-value problems including semi-definite Neumann problem. Scalability to a large number of processors is based on the use of the Chebyshev's iterations for solution of the coarsest grid equations and for construction of the smoothing procedures.

Two smoothing operators are considered here; see [4–6]. The first one is the Chebyshev polynomial, while the second operator is a rational function of the discrete operator. Presently special attention focuses on polynomial smoothers due to their high efficiency in parallel computing. In addition it is possible to provide desired damping on a preset interval of high-frequency components by using a sufficiently large degree in the polynomial smoothing operators [7–9].

For equations with discontinuous coefficients we use so called problem-dependent intergrid transfer operators.

We developed a special procedure to adjust smoothers for achieving the prescribed rate of multigrid iteration convergence and present examples, which show that proposed adaptation improves the efficiency of the multigrid method even in the presence of a strong anisotropy in the problem.

The need for efficient solutions of such problems is caused by their prevalence in mathematical models. Increased interest in this problem is motivated by the development of parallel computations: codes for multiprocessor supercomputers are required. The developed code ensures scalable simulation on grids with billion (and more) nodes.

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2 FORMULATION OF THE PROBLEM

In three-dimensional domain Ω with boundary $\partial\Omega$ we consider a boundary value problem

$$\begin{aligned} &L\,u = f \quad \text{in } \Omega\,, \\ &-(\kappa\,\nabla u)\cdot n = \sigma\,u + \gamma \quad \text{on } \partial\Omega\,. \end{aligned} \tag{1}$$

Here L is a linear elliptic self-adjoint positive semi-definite operator

$$L\,u = -\nabla\cdot(\kappa\,\nabla u) + a_0 \cdot u \tag{2}$$

with the coefficients $\kappa = diag\{k_1, k_2, k_3\}$, $k_i \geq 0$, $a_0 \geq 0$. Strong anisotropy $k_1 >> k_2 >> k_3$ is admissible. The boundary condition (Dirichlet, Neumann, or Robin) is set on $\partial\Omega$, n is outward normal vector to $\partial\Omega$. The coefficients $\kappa(r)$, $f(r)$, $a_0(r) \geq 0$, $\sigma(r) \geq 0$, $\gamma(r)$ are given functions, and $u(r)$ is the sought function; $r = (x, y, z) \in \Omega$. The input data are assumed to ensure the existence and uniqueness of a solution of the required smoothness.

The approximation of problem (1) is not discussed here. Accordingly, for simplicity, we use Ω as a right parallelepiped, in which we introduce a Cartesian grid $\Omega_h = \{x_n \in \Omega,\ 0 \leq n \leq N\}$ with grid boundary Γ_h; the grid Ω_h is non-uniform in each coordinate direction and depends on a parameter h (mesh size) characterizing the average cell size.

The space U_h of functions on Ω_h is defined in the standard manner (with the L_2 – grid inner product and the corresponding norm). On the space U_h we define a difference operator L_h approximating the operator L with the second order of accuracy. To be more specific, it can be assumed that the grid functions are given at the nodes of the grid and L_h arises from 7-point finite volume discretization of the problem (1). The operator L_h is self-adjoint, and its eigenvalues λ are nonnegative and belong to the real interval $[\lambda_{\min};\ \lambda_{\max}]$. Assume that the estimates for the bounds $\lambda_{min} \geq 0$ and λ_{max} of the spectrum of are known; in some cases, they can be calculated (see [6]). For the degenerate Neumann problem, the null space (kernel) of the operator is a one-dimensional subspace consisting of constant grid functions. In this case, $\lambda_{min} = 0$. Then, by λ_{min}, we mean the lower estimate $\lambda^d_{min} > 0$ for the operator L_h on the subspace orthogonal to the nullspace. An estimate for λ_{max} is fairly easy to obtain with the help of the Gershgorin theorem [10].

3 FEATURES OF MULTIGRID METHODS

3.1 General scheme

The discrete approximation of problem (1) is written taking into account the boundary conditions in the operator form:

$$A_h\,u_h = g_h, \tag{3}$$

where A_h is an $N \times N$ – matrix, u_h is the desired grid function, and g_h is a given grid function. We define the operator $D_h = -\Delta_h$, where Δ_h is the difference Laplacian (with the same boundary conditions as for operator A_h). Assume that, for the operators A_h and D_h, the constants $0 < \zeta_1 \le \zeta_2$ in the inequality

$$\zeta_1(D_h u_h, u_h) \le (A_h u_h, u_h) \le \zeta_2(D_h u_h, u_h), \tag{4}$$

are known. This means the spectral equivalence of A_h and D_h.

Let us briefly describe the main elements of the multigrid method (MM). To solve a system of the grid equations (3), we use the MM in a form close to Fedorenko's original method [1–2]. Every multigrid iteration step consists in the transition from a fine grid to one of the next level up to the coarsest grid and back (the so called V – cycle). It is convenient to describe the main features of the method in a two-grid representation, when there are only two grids: fine h and coarse H. Then the corresponding error propagation operator of the two-level method has the form [11]:

$$Q = S_p \cdot (I - P \cdot A_H^{-1} \cdot R \cdot A_h) \cdot S_p . \tag{5}$$

Here, A_h and A_H are the operators on the fine and coarse grids, respectively; P and R are the intergrid transfer operators from the coarse to fine grid (prolongation or interpolation P) and from the fine to coarse grid (restriction $R = P^*$); and S_p is a smoothing operator with p pre- and post-smoothing steps. In the algorithm under consideration, the operator A_H is constructed via re-discretization of the problem on the coarse grid. The efficiency of the algorithm depends on the multigrid triad: the intergrid transfer operators, the algorithm for solving the coarsest grid equations, and the smoothing operator S_p. The above two-level algorithm can be generalized to multilevel ones by recursion. The number of grid levels in the MM algorithm is usually rather small (five or six, even we use a large grid with billions of nodes).

The operators P and R are described in [8], where along with a trilinear interpolation operator P, an interpolation operator based on an approximate solution of local discrete boundary value problem is presented. Such operators P and $R = P^*$ are called problem-dependent and they provide robustness of the multigrid algorithm for equations with discontinuous coefficients, see [3]).

3.2 Coarse grid solution

On the coarsest grid, a system of linear equations $A_H \cdot y = g_H$ with $g_H = R \cdot (g_h - A_h u_h)$ on the right-hand side is solved using the Chebyshev iterative method [12] with parameters ω_j (reordered for stability [12, 13]):

$$y^j = y^{j-1} - \omega_j (A_H \cdot y^{j-1} - g_H), \quad j = 1, ..., p \ , \tag{6}$$

where y^0 is an initial guess, usually, $y^0 \equiv 0$. A number p of iterations is determined by the condition of achieving the prescribed accuracy ε according the criterion $\| r_p \| < \varepsilon \| r_0 \|$,

where r_0 and $r_p = g_H - A_H \cdot y^p$ are the initial and final residuals. An estimate for p has the form (see [12])]:

$$p = p(\varepsilon, \eta) \approx \frac{\ln\ \varepsilon^{-1} + \sqrt{\varepsilon^{-2} - 1}}{\ln \rho}, \qquad \rho = \frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}, \qquad \eta = \frac{\lambda_{\min}}{\lambda_{\max}}\ , \tag{7}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are estimated minimal and maximal eigenvalues of the operator A_H.

This iteration process is defined by the Chebyshev polynomial F_p of degree p, that deviates least from zero on the interval $[\lambda_{min}; \lambda_{max}]$ and is normalized by the condition $F_p(0) = 1$.

3.3 Smoothing procedures

The smoothing (relaxation) operator S_p specifies the transition from an iterative approximation v to a "smoother" approximation v_{new}. Two smoothing operators $S_p = S_p(A_h)$ are presented here (see [4–6]). The first one is a Chebyshev polynomial of degree p, while the second one is a rational function constructed in [14] and called the LIM. Note in the multigrid theory a smoothing operator (called the smoother error propagator) is defined by

$$S = I - M^{-1} A_h,$$

where M is a smoother. But we work directly with the smoother error propagator and will call it as the relaxation or smoothing operator.

Each such operator is a self-adjoint, and its eigenvalues are the values of the function $S_p(\lambda)$ on the spectrum of A_h. These operators serve to reduce the high-frequency components of an initial vector e^0 by the iteration (or smoothing step)

$$e^1 = S_p \cdot e^0\ .$$

The function $S_p(\lambda)$ must be small on the high frequency subinterval $[\lambda^*_{min}; \lambda_{max}]$. We distinguish high- and low-frequency components of grid functions on Ω_h. The separation of high- and low-frequencies is done with respect to the coarse grid, and therefore high-frequency components are not represented on the coarse grid. More strictly, there is a decomposition of the space U_h into a direct sum of two invariant subspaces associated to a two groups of eigenvalues placed on the low-and high-frequency spectral subintervals s $[\lambda_{min}; \lambda^*_{min})$ and $[\lambda^*_{min}; \lambda_{max}]$ respectively. The left end λ^*_{min} of the high-frequency spectral range $[\lambda^*_{min}; \lambda_{max}]$ is an unknown (in general). The quality of the smoothing procedures depends on the choice of λ^*_{min} and the construction of a smoothing operator S.

How to define the bound λ^*_{min} and the smoothing operator $S_p = S_p(A_h)$ optimally are the key questions and this is addressed this paper.

Sometimes we can obtain the exact value λ^*_{min} of left end of the high frequency interval. To begin with, we consider the classical case when A_h is a discrete Laplacian on a cartesian grid in the unit cube and the grid is uniform in each direction. Then the bound separating all the eigenfunctions into smooth and nonsmooth ones is associated with eigenfunctions indexed by $(0.5\cdot N_1,\ 1,\ 1),\ (1,\ 0.5\cdot N_2,\ 1),\ (1,\ 1,\ 0.5\cdot N_3)$; here, N_α is the number of mesh spacings in each coordinate direction, $\alpha = 1,2,3$. The eigenvalues associated with these eigenfunctions have the form

$$\lambda^*_1 = \frac{1}{2}\lambda^{(1)}_{max} + \lambda^{(2)}_{min} + \lambda^{(3)}_{min},\quad \lambda^*_2 = \frac{1}{2}\lambda^{(2)}_{max} + \lambda^{(1)}_{min} + \lambda^{(3)}_{min},\quad \lambda^*_3 = \frac{1}{2}\lambda^{(3)}_{max} + \lambda^{(1)}_{min} + \lambda^{(2)}_{min} \tag{8}$$

where $\lambda^{(\alpha)}_{min}$ and $\lambda^{(\alpha)}_{max}$ are the bounds of the difference analogues of the one-dimensional operators in the coordinate directions $\alpha = 1,2,3$. This exact bound is $\lambda^*_{min} = min(\lambda^*_1, \lambda^*_2, \lambda^*_3)$. As a relative bound, we take

$$\eta = min(\lambda^*_1, \lambda^*_2, \lambda^*_3)/\lambda_{max}\,. \tag{9}$$

This bound is exact and takes into account the difference between the spectral intervals in the coordinate directions. Such simplest example of such anisotropy can arise, for example, when problems are solved in a parallelepiped with sizes differing in different directions by many times. Estimate (9) is an extension, to the three-dimensional case, of a rule according to which the frequencies $\lambda > 0.5\lambda_{max}$ in the simplest one-dimensional case belong to the high frequency range. In the isotropic d–dimensional case of the Laplacian discretization the estimate (9) reads $\lambda^*_{min} = \lambda_{max}/(2d)$.

The estimation (9) is also exact in the case of the considered discretiazion of the problem (1), (2) with the constant coefficients (k_1, k_2, k_3 and a_0, σ). Note that the bound λ^*_{min} can be obtained if we specify the constants $0 < \zeta_1 \le \zeta_2$ in inequality (7). Let λ^0_{max} denote the upper bound for D_h. In view of the relation $\lambda_{max} = \zeta_2 \cdot \lambda^0_{max}$, we can estimate λ^*_{min} and η:

$$\lambda^*_{min} \;=\; \frac{1}{6}\cdot\zeta_1\cdot\lambda^0_{max} \;\equiv\; \eta\cdot\lambda_{max}, \qquad \eta = \frac{1}{6}\cdot\frac{\zeta_1}{\zeta_2}.$$

If we do not know λ^*_{min} in advance we take the bound $\lambda^*_{min} = \lambda_{max}/6$ as an initial guess and correct automatically this bound in the course of multigrid iterations. We verified (see [7, 8]) such an adaptation procedure and describe algorithm of adaptation in this paper.

We propose two smoothing operators which are related to a first kind Chebyshev polynomial $T_p(x)$ of degree p that deviates least from zero on the interval [–1; 1] (see [12]):

$$T_p(x) = \cos(p\,\arccos x),\quad |x| \le 1; \qquad T_p(x) = \cosh(p\,\mathrm{Acosh}x),\quad |x| > 1. \tag{10}$$

To construct such an optimal polynomial for an arbitrary interval $a;b$, we make a linear change of variables mapping the interval $[-1;1]$ to an interval $[a;b]$.

When a smoothing operator is applied, the parameters of the method (degree p of the polynomial and the corresponding set of Chebyshev parameters) are chosen to suppress the high-frequency residual components with a prescribed accuracy ε_{smooth}, usually, $\varepsilon_{smooth} = 0.5$. We check the reduction of the norm of the entire residual that provides the desired reduction of the high-frequency residual components. We can estimate the smoothing factor of the operator S_p, i.e. the error reduction in the high-frequency space as follows

$$\rho = \max |S_p(\lambda)| \leq 1 \tag{11}$$

where maximum is taken over the interval $[\lambda^*_{min}; \lambda_{max}]$ and p is degree of the Chebyshev polynomial; this parameter defines the number of iterations of the relaxation process.

3.4 Chebyshev polynomial smoother

Consider explicit iterations of form (6) with a Chebyshev polynomial $F_p(\lambda)$ that deviates least from zero on the high-frequency spectral interval $[\lambda^*_{min}; \lambda_{max}]$ and is normalized by the condition $F_p(0) = 1$. The polynomial F_p can easily be expressed in terms of standard polynomial $T_p(x)$ (10). On the entire spectrum, the inequality $|F_p(\lambda)| < 1$ holds, i.e., this smoothing procedure is a converging iterative method. Let $p = p(\varepsilon, \eta)$ be defined by (7) with a given value $\eta = \lambda^*_{\min} / \lambda_{\max}$ and a prescribed tolerance $\varepsilon = \varepsilon_{smooth}$. Then the error components on the interval $[\lambda^*_{min}; \lambda_{max}]$ are decreased by $\varepsilon^{-1} > 1$ times uniformly over this spectral range and the smoother error reduction function

$$\rho_{Cheb}(\lambda) = F_p(\lambda) \tag{12}$$

takes the values $\pm\varepsilon$ at the extremal points of the polynomial on $[\lambda^*_{\min}; \lambda_{max}]$.

The polynomial $|F_p(\lambda)|$ looks like the ideal smoother. In practice, the situation differs from the ideal one and usually we do not know the value λ^*_{min}. The Chebyshev smoothing operator turns out to be sensitive to the inaccuracy of specifying λ^*_{min} (see [4, 8]). For example, when λ^*_{min} is underestimated then the number of smoothing steps grows without improving the quality of smoothing. For this reason, we examine another smoothing iterative procedure, which is called LIM. In this case, the use of an underestimated value of λ^*_{min} improves smoothing and the overall multigrid convergence rate (and as we expect number of smoothing iterations grows). Note that LIM is reliable in complicated diffusion problems.

3.5 LIM smoothing operator

Define $p = p(\varepsilon, \eta)$ by the formula

$$p = \left\lceil \frac{\pi}{4} \sqrt{(\varepsilon^{-1} - 1)/\eta + 1} \right\rceil \tag{13}$$

with $\eta = \lambda^*_{min}/\lambda_{max}$ and a prescribed tolerance $\varepsilon = \varepsilon_{smooth}$, (in (13), $\lceil x \rceil$ is the least integer greater than or equal to x). Then, to reduce the error by $\varepsilon^{-1} > 1$ times on the high-frequency interval $[\lambda^*_{min}; \lambda_{max}]$, we use the LIM smoothing operator

$$S_\nu = (I - G_p^2) \cdot (I + \tau A_h)^{-1}. \tag{14}$$

This operator is implemented by the explicit iterative algorithm, see below. Here $\nu = 2p-1$, $G_p(\lambda)$ is a Chebyshev polynomial of the first kind that deviates least from zero on the interval $[\lambda_0; \lambda_{max}]$, where

$$\lambda_0 = \lambda_{max} \frac{z_1 - 1}{z_1 + 1} \in [-1/\tau; 0], \qquad \tau = \frac{1}{\lambda_{max}} \, 16p^2/\pi^2 - 1 \, , \qquad z_1 = \cos\frac{\pi}{2p}, \tag{15}$$

with the normalization $G_p(-1/\tau) = 1$. On the interval $[0; \lambda_{max}]$, we have $|G_p(\lambda)| \le 1$ and $|S_\nu(\lambda)| \le 1$, moreover, $G_p(0) = 0$.

A function of form (14) was studied in detail in [14] and its construction was based on the scheme proposed for numerical solution of parabolic equations in [15]. The polynomial $G_p(\lambda)$ can express in terms of the classical Chebyshev polynomial $T_p(x)$ (10) (see [14]):

$$G_p(\lambda) = \frac{H_p(\lambda)}{H_p(-1/\tau)}, \qquad H_p(\lambda) = \prod_{m=1}^{m=p} (a_m - \lambda) \equiv T_p(z_1 - (z_1 + 1) \cdot \lambda/\lambda_{max}).$$

The zeros a_m, $m = 1, ..., p$ of the polynomial H_p are the iteration parameters of the method and can be expressed in terms of the zeros β_m of the Chebyshev polynomial T_p:

$$a_m = \frac{z_1 - \beta_m}{1 + z_1} \lambda_{\max}. \tag{16}$$

The implementation of the operator (14) is similar to the algorithm (6). The transition from an iterative approximation v to a "smoother" approximation v_{new} is executed in $\nu = 2p-1$ explicit steps:

$$y^m = \frac{1}{1 + \tau \cdot b_m} \{v + \tau \cdot b_m \cdot y^{m-1} - \tau \cdot (A_h \cdot y^{m-1} - f_h)\}, \quad m = 1, \ldots, 2p-1$$

where an initial guess $y^0 = v$, the iteration parameters $\{b_1, ..., b_{2p-1}\} \equiv \{a_1, ..., a_p, a_2, ..., a_p\}$ and $\tau = 16p^2/\pi^2 - 1 \, / \lambda_\infty$. The result of this iterative algorithm is a smooth approximation to the solution: $v_{new} = y^{2p-1}$.

The spectrum of smoothing operator (14) is given by the formula

$$\rho_{LIM}(\lambda) = \frac{1 - G_p^2(\lambda)}{1 + \tau\lambda}. \tag{17}$$

By the construction of G_p, we have the following inequality for the damping factor:

$$0 \le \rho_{LIM}(\lambda) \le \frac{1}{1+\tau\lambda}. \tag{18}$$

On the right end of the spectrum, damping is faster than with the Chebyshev smoothing operator (12)and is determined by the factor

$$\rho_{LIM}(\lambda_{max}) \le \frac{\pi^2}{16p^2} \tag{19}$$

On Fig. 1 the damping factors for both smoothing functions (12) and (17) are shown. Spectral interval is normalized in such a manner, the normalized upper bound $\overline{\lambda}_{max} = h^2 \cdot \lambda_{max}$ equals 12. The Chebyshev polynomial Cheb(p) is optimal for $\overline{\lambda}^*_{min} = 0.3$ with $p = 7$ and $\varepsilon_{smooth} = 0.5$. For comparison the LIM function is taken for $p = 4$ in order to be eqivalent to $2p-1 = 7$ explicit steps; damping factors correspond to 0.5 at $\overline{\lambda}^*_{min} = 0.3$ and 0.04 at $\overline{\lambda}_{max}$ (with respect to (19)).

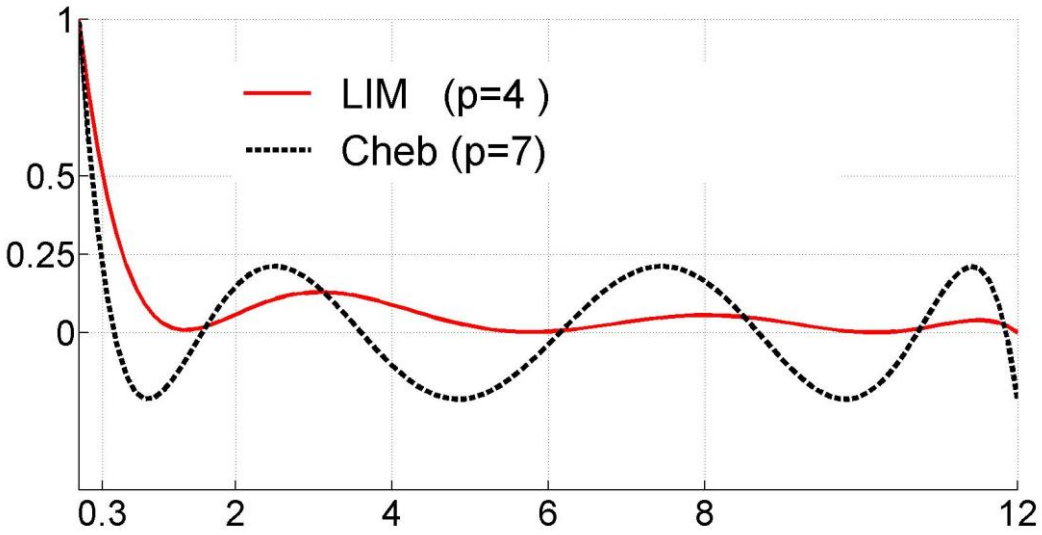

Figure 1. The damping factors for smoothing operators

Example 1. Consider the Poisson equation in a unit cube. Neumann boundary conditions are imposed. The function f (right hand side) is determined by the given solution $u(x,y,z) = (\cos l_1 x + \cos l_2 x) \cdot (\cos m_1 y + \cos m_2 y) \cdot (\cos n_1 z + \cos n_2 z)$, where $l_1 = m_1 = n_1 = 2$, $l_2 = 4$, $m_2 = 8$, $n_2 = 16$.

Five grid levels and an identical number of smoothing steps at each level are used. When comparing the smoothers, we took into account that, for given p in $LIM(p)$, the number of smoothing steps is $\nu = 2p-1$.

The numerical results are characterized by the number m of multigrid iterations, CPU time, and the convergence rate $\rho = \| r_m \| / \| r_{m-1} \|$ where r_{m-1} and r_m are the residuals. The convergence is achieved if $\| r_m \| < tol \cdot \| r_0 \|$ with prescribed tolerance $tol = 10^{-7}$. The solution of the Neumann problem with smoother $Cheb(p)$ at $p = 2$ takes $m = 9$ multigrid iteration steps, regardless of the grid size $N = 32^3$, 64^3, 128^3, 512^3. The average convergence rate is $\rho = 0.167$, which are higher than the theoretical convergence rate $\rho = 0.20$ of the method in the two-grid representation with two pointwise relaxation steps [13].

Let us present a typical situation which simulates ifluence of uncertainces in determination of λ^*_{min} on convergence characteristics of the multigrid algorithm. Consider three values of λ^*_{min}: $\lambda_{max}/20$, $\lambda_{max}/6$, $\lambda_{max}/1.5$. Table 1 presents the numerical results for the Neumann problem on the grid of 512^3 nodes with indicated values λ^*_{min}.

	Chebyshev, $p = 3$			LIM, $\nu = 3$		
λ^*_{min}	m	ρ	$time, s$	m	ρ	$time, s$
$\lambda_{max}/6$	8	0.126	188	9	0.155	234
$\lambda_{max}/20$	9	0.157	212	9	0.155	234
$\lambda_{max}/1.5$	12	0.243	280	9	0.155	234

Table 1. Convergence characteristics of the multigrid for various λ^*_{min}

We compare the smoothing procedures Cheb(3) and LIM(2) (with the number of smoothing steps given by $\nu = 2p - 1 = 3$). The time costs of smoothing in these two procedures are identical. For `this problem, the exact value of λ^*_{min} is known: $\lambda^*_{min} = \lambda_{max}/6$. In this case, the Chebyshev smoother ensures a higher rate of convergence than LIM. For other two cases $\lambda_{max}/20$, $\lambda_{max}/1.5$ it can be seen that, for the Chebyshev smoother, the inaccuracy slows down the convergence rate. In contrast to the Chebyshev smoother, the quality of LIM is independent on the given range of λ^*_{min}.

For a problem with strong anisotropy both smoothing procedures require some estimations of λ^*_{min} or implementation of the adaptive technique described below. This technique is based on analysis of multigrid convergence history, and is implemented by monitoring the convergence of the smoothing procedure and the convergence of the overall multigrid solver on each multigid iteration as well as.

3.6 Adaptation of smoothers to the spectrum of operators

Adaptation is performed by improving the lower bound λ^*_{min} of the high-frequency interval $[\lambda^*_{min};\ \lambda_{max}]$ for each grid level. This improvement can allow us to achieve automatically better smoothing properties and the best possible rate of multigrid convergence.

Suppose that, in the course of few multigrid iterations, after p smoothing steps with a given tolerance ε_{smooth}, the given bounds λ^*_{min} and λ_{max} corresponding to the current grid level, the relation $\delta = \| r_p \| / \| r_0 \|$ is obtained for the initial r_0 and final r_p residuals. If $\delta < 1$, for the Chebyshev smoother, inverting the formulas of form (7), we find the new lower bound

$$\lambda^*_{min} = \left(\frac{\rho_1 - 1}{\rho_1 + 1} \right)^2 \cdot \lambda_{max}, \qquad \rho_1 = \left(\delta^{-1} + \sqrt{\delta^{-2} - 1} \right)^{1/p}. \tag{20}$$

For the LIM smoother, using (15) and (18), the value of λ^*_{min} is updated using the formula

$$\lambda^*_{min} = \frac{\pi^2}{16 p^2} \cdot (\delta^{-1} - 1) \cdot \lambda_{max}. \tag{21}$$

Based on (20) and (21) we elabarated the rather sophisticated procedure which provides updating the bounds of the high frequency spectrum. We compute the optimal λ^*_{min} not only by monitoring the convergence of the smoothing procedure, but also the convergence of the overall multigrid solver on each multigid iteration. This technique provides to achieve automatically the multigrid convergence rate $\rho = \varepsilon^2_{smooth}$. This relation is expected on the basis of the analysis of two-grid representation (5).

Anisotropy test case. Consider the equation (1) in a unit cube with diffusion coefficients k_1, k_2, k_3, included strong anisotropy $k_1 = 10000$, $k_2 = 100$, $k_3 = 1$, see Table 2. The right hand side is

$$f(x, y, z) = \begin{cases} 100, & (x, y, z) \text{ in cube } [0.2;\ 06]^3, \\ 0\ , & \text{out}\ . \end{cases}$$

Dirichlet boundary conditions are imposed: $u = 0$ on $\partial\Omega$.

Five grid levels are used. The numerical results are characterized by the number m of multigrid iterations, total number Σp smoothing steps on the fine grid and the average convergence rate $\rho = (\| r_m \| / \| r_0 \|)^{1/m}$, where r_0 and r_m are the initial and final residuals. The convergence is achieved if $\| r_m \| < tol \cdot \| r_0 \|$ with the prescribed tolerance $tol = 10^{-6}$. Instead of λ^*_{min} we characterize spectrum separation by relative spectral bound $\eta = \lambda^*_{min} / \lambda_{max}$.

Table 2 presents the numerical results for computations on the grid of 128^3 nodes. We compare the smoothing procedures Cheb and LIM with and without adaptation procedures. The goal of an adaptation procedure is to adjust λ^*_{min} in order to achieve automatically better smoothing properties and possible to guarantee the prescribed rate of multigrid convergence.

For different choice of the diffusion cotfficients k_1, k_2, k_3 we consider three variants of smoothing strategy:

- Var0: without adaptation, a given relative bound $\eta = 1/6$,
- Var1: without adaptation, a given relative bound $\eta = \eta_{exact}$,
- Var2: with adaptation, an initial relative bound $\eta = 1/6$.

Here η_{exact} is given by (9) and for four cases k_1, k_2, k_3 (see Table 2) this value η_{exact} is equal $0.17,\ 5.0e-03,\ 2.6e-03,\ 1.7e-04$ respectively. In process of adaption we compute the relative bound η_{adjust} which is close to the best bound η_{exact}, see Table 2.

k_1, k_2, k_3	Smoothing strategy	Chebyshev	LIM
		m; Σp; ρ	m; Σp; ρ
1; 1; 1	Var0 Var2	8; 32; 0.15 7; 38; 0.12	8; 48; 0.15 8; 48; 0.15
100; 1; 1	Var0 Var1 Var2 $\eta_{exact} = 5.0e-03$	240; 960; 0.94 10; 200; 0.22 13; 194; 0.32 $\eta_{adjust} = 5.5e-03$	202; 1212; 0.93 7; 294; 0.12 13; 338; 0.32 $\eta_{adjust} = 6.3e-03$
100; 100; 1	Var0 Var1 Var2 $\eta_{exact} = 2.6e-03$	431; 1724; 0.97 9; 234; 0.19 13; 240; 0.34 $\eta_{adjust} = 3.0e-03$	362; 2172; 0.96 6; 348; 0.09 13; 430; 0.32 $\eta_{adjust} = 4.0e-03$
10000; 100; 1	Var0 Var1 Var2 $\eta_{exact} = 1.7e-04$	stagnation 8; 816; 0.17 15; 854; 0.39 $\eta_{adjust} = 2.0e-04$	stagnation 4; 936; 0.03 15; 1570; 0.40 $\eta_{adjust} = 3.0e-04$

Table 2. Convergence characteristics of the multigrid for the coefficients k_1, k_2, k_3.

Adaptation can increase or reduce the number of smoothing steps at every grid level. The lower bound of the operator on the coarsest grid can be improved in a similar manner.

In the computations, the adaptation of the smoothers improves the convergence rate of the multigrid algorithm, but the adaptation procedure has a few tuning parameters and further investigation and experimental verification is required to enhance robustness, which goes beyond the scope of this paper.

4. CONCLUSION

We proposed an adaptive technique based on analysis of convergence history which provides efficient design of smoothers with optimal smoothing factors. This technique allows us to achieve automatically the prescribed convergence rate of the multigrid solver. The computations confirm that the adaptive technique is a useful tool in designing efficient multigrid algorithms improves the efficiency of the multigrid method even in the presence of a strong anisotropy in the problems. Here, this technique is applied on structured Cartesian grids but we believe that such approach will be useful for unstructured grids and for algebraic multigrid as well. The proposed multigrid algorithm is implemented successfully on parallel computers.

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