**Multidimensional, Self-similar, strongly-interacting, Consistent (MuSIC) Riemann Solvers – Applications to Divergence-Free MHD, Maxwell & ALE Schemes**

By

Dinshaw S. Balsara

(dbalsara@nd.edu) Univ. of Notre Dame

http://physics.nd.edu/people/faculty/dinshaw-balsara

http://www.nd.edu/~dbalsara/Numerical-PDE-Course
What is the Multidimensional Riemann Problem?
You have always chosen to ignore it!
But it should have been there in your code all along!

How Is It Solved?

**MuSIC Riemann solver** -- **Multidimensional, Self-similar Riemann Solver**, based on a strongly-interacting state that is **Consistent with the governing hyperbolic law**.
**MuSIC Riemann solver** -- **Multidimensional, Self-similar Riemann Solver**, based on a strongly-interacting state that is **Consistent** with the governing hyperbolic law. **Four 1D Riemann Problems interact to produce Strongly-Interacting State.**

Note some important properties about the **Strongly-Interacting State:**

1) Strongly-Interacting state **evolves self-similarly in space-time.**

2) For any **1D RS**, you always build a **1D wave model**. The strongly-interacting state is bounded by a **multidimensional wave model**. The multidimensional wave model also evolves self-similarly in space-time.

3) **Strongly-Interacting state propagates into the four one-dimensional Riemann problems.** I.e., it literally engulfs the fluid in the four one-dimensional Riemann problems that are on all four sides of it. The fluid from those four one-dimensional Riemann problems goes to make up the strongly-interacting state.

4) “In-the-small” the multidimensional RP **always exists in any code at the vertices of the mesh.** Ignoring multid. effects reduces the timestep.
Overview:-
1) Motivating the need for Multidimensional Riemann Solvers using the MHD Equations.
Utility also to ALE Schemes

2) Divergence-Free Reconstruction of Magnetic Fields for MHD and AMR-MHD

3) Overview of Multidimensional Riemann Solvers – 2D and 3D!

4) Formulating the Multidimensional Riemann Solver in Self-Similar Variables

5) Approximating the Multidimensional Riemann Problem with just ONE call to the Multidimensional Riemann Solver!

6) Extension to Maxwell Equations

7) Results and Applications

8) Conclusions
Multidimensional HLL Riemann solvers:-


Inclusion of Substructure (Lowers Dissipation):-


Extension to Unstructured Meshes:-


For more information, please see Appendix A from the following website:-
http://www.nd.edu/~dbalsara/Numerical-PDE-Course
Applications to ALE:-

Applications to MHD & RMHD:-
• D.S. Balsara, Divergence-free reconstruction of magnetic fields and WENO schemes for magnetohydrodynamics, Journal of Computational Physics, 228 (2009) 5040-5056

Applications to Electromagnetism:-

For more information, please see Appendix A from the following website:-
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Advances over Previous Work:-
ANY self-similar 1D Riemann solver can be used as a building block in the Multidimensional Riemann Solver!

Formulation in Similarity Variables is easier to understand; especially 3D Formulation in Similarity Variables is entirely equivalent to previous Space-Time Formulation.

Enables seamless inclusion of sub-structure in the Strongly Interacting State of the Multidimensional Riemann Problem. The sub-structure can naturally pick out any orientation w.r.t. mesh – isotropic propagation.

Galerkin formulation has a plug-and-chug flavor. Closed form expressions suitable for applications. States and Fluxes are uniquely defined by Integral Constraints.

Can approximate the Multidimensional Riemann Problem with just ONE call to the Multidimensional Riemann Solver!
I.a) MHD Equations and Motivation:

\[
\begin{bmatrix}
\rho \\
\rho v_x \\
\rho v_y \\
\rho v_z \\
\varepsilon \\
B_x \\
B_y \\
B_z
\end{bmatrix}
+ \frac{\partial}{\partial t}
\begin{bmatrix}
\rho v_x \\
\rho v_x v_y - B_x B_y / 4\pi \\
\rho v_y \\
\rho v_y v_z - B_y B_z / 4\pi \\
\rho v_z \\
\rho v_x v_y - B_x B_y / 4\pi \\
\rho v_y v_z - B_y B_z / 4\pi \\
\rho v_z^2 + P + B_z^2 / 8\pi - B_y^2 / 4\pi
\end{bmatrix}
+ \frac{\partial}{\partial y}
\begin{bmatrix}
\rho v_y \\
\rho v_x v_y - B_x B_y / 4\pi \\
\rho v_y v_z - B_y B_z / 4\pi \\
-\left( v_x B_y - v_y B_x \right) \\
\left( v_y B_z - v_z B_y \right)
\end{bmatrix}
+ \frac{\partial}{\partial z}
\begin{bmatrix}
\rho v_z \\
\rho v_x v_z - B_x B_z / 4\pi \\
\rho v_y v_z - B_y B_z / 4\pi \\
\rho v_z^2 + P + B_z^2 / 8\pi - B_y^2 / 4\pi
\end{bmatrix}
\begin{bmatrix}
\varepsilon + P + B^2 / 8\pi \\
\varepsilon + P + B^2 / 8\pi \\
\varepsilon + P + B^2 / 8\pi \\
\varepsilon + P + B^2 / 8\pi
\end{bmatrix}
\begin{bmatrix}
B_x \\
B_y \\
B_z \\
B_x v_x - B_y v_y / 4\pi
\end{bmatrix}
= 0
\]

\[ E \equiv -v \times B \]

Hydro + Lorenz Force

\[ \mathbf{E_z} \]
\[ \mathbf{E_y} \]

Notice the dualism between the flux terms and the electric field
MHD Is Different: \[ \frac{\partial \mathbf{B}}{\partial t} + \frac{c}{\gamma} \nabla \times \mathbf{E} = 0 \]; \( \mathbf{E} \equiv -\frac{1}{c} \mathbf{v} \times \mathbf{B} \); constraint \( \nabla \cdot \mathbf{B} = 0 \)

Necessitates use of Yee-type mesh. Fluid variables still zone-centered. Magnetic field components at zone faces; electric fields at zone edges.

Since \( \mathbf{B} \) is defined on faces, we need a reconstruction of \( \mathbf{B} \) over the zone that respects the constraint:

\[ \nabla \cdot \mathbf{B} = 0! \]

Electric fields at edges require genuinely multi-d treatment – Need for at least genuinely 2D Riemann Solvers
There is a dualism between the fluxes of the conservation law and the electric field (Balsara & Spicer 1999). That can be exploited to obtain the electric field.

For this concept to truly work, the electric field should be treated truly multidimensionally.

All prior work had tried combinations of 1D RP to introduce multidimensionality into the electric field evaluation.

There was even an attempt to stabilize the electric field by doubling the dissipation (Gardiner&Stone 2005). Why double dissipation all the time?

Any doubling of dissipation proves to be completely unnecessary when the genuinely Multidimensional Riemann solvers are used to obtain the edge-centered electric field (Balsara 2010, 2012, 2014).

**Essential ideas:**

**Div-Free Reconstruction +**

**Dualism of fluxes and E-field** + **true multidimensional RS**
Motivation for Divergence-Free Reconstruction of Magnetic Fields

Why have Divergence-Free MHD Reconstruction?

1) **Divergence-cleaning** strategies (Brackbill & Barnes) run into problems. There are often modes that the divergence-cleaning routines do not remove (Balsara & Kim 2004).

2) **Powell (1994) fix** destroys the conservation form of the MHD system. Requires very substantial modification of the Riemann solvers. Also results in accumulation of divergence at stagnation points.

3) **GLM** formulation by Dedner et al (2002) require an a priori evaluation of the extremal speed. In most real-world applications, such a speed is not available. Try any magnetospheric problem, any astrophysics problem or any fusion problem!
   -- Low dissipation HLL, HLLC, HLLD RS are hard to design when the extremal speeds belong to the lagrange multiplier field.
   -- Linearized RS not very successful for MHD.

4) For closed/periodic geometries, the divergence never goes away.
Motivation for Multidimensional Riemann Solvers

Why have multidimensional Riemann solvers?

1) For divergence-free MHD, the motivation is especially compelling: There is no unique mesh-oriented direction for the edge-centered electric field. Electric field evaluation is fundamentally multidimensional.

2) Better representation of the physics. Astrophysical, space science, AME and fusion calculations are carried out on resolution-starved meshes. Multidimensional Riemann solvers give more isotropic propagation of small-scale flow features.

3) Multidimensional effects in fluxes \(\Rightarrow\) larger timesteps; larger CFL.

4) Multidimensional Riemann solvers are cost-competitive with 1d Riemann solver technology.

5) Extended now to unstructured meshes and ALE formulations.
II) Divergence-Free Methods in MHD & AMR-MHD

**MHD is different, Reason**: The magnetic field evolves according to Faraday’s law, i.e. a Stokes-law type update equation.

\[
\frac{\partial \mathbf{B}}{\partial t} + c \nabla \times \mathbf{E} = 0; \quad \mathbf{E} = -\frac{1}{c} \mathbf{v} \times \mathbf{B}
\]

Which satisfies the constraint: \( \nabla \cdot \mathbf{B} = 0 \)

_Violating this constraint_ results in _unphysical forces_ along the magnetic field.

Numerical methods for satisfying the constraint exist and rely on a _staggered mesh formulation_. Yee (1966), Brecht et al (1981)

The _magnetic field components_ are defined at the _face centers_.

The _electric field components_ are defined at the _edge centers_.
Notice: Face-centered magnetic fields are the fundamental quantity!

\[
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{c} \cdot \nabla \times \mathbf{E} = 0; \quad \nabla \cdot \mathbf{B} = 0
\]

\[
\Rightarrow B_{x, i+1/2, j, k}^{n+1} = B_{x, i+1/2, j, k}^n - \frac{c \Delta t}{\Delta y \Delta z} \left( \Delta z E_{z, i+1/2, j+1/2, k}^{n+1/2} - \Delta z E_{z, i+1/2, j-1/2, k}^{n+1/2} + \Delta y E_{y, i+1/2, j, k+1/2}^{n+1} - \Delta y E_{y, i+1/2, j, k-1/2}^{n+1} \right)
\]

**Important Questions:**

1) How do we reconstruct the magnetic field if it resides on the boundaries?

2) How do we obtain upwinded electric fields?
Divergence-Free Prolongation / Reconstruction of B-Field

Start with magnetic fields at faces in 2D – This is only first order accurate.

This will only give us first order accuracy. We wish to have at least second order, so we endow the fields with linear variation.

Count: Boundaries have 3 independent pieces of information!

Divergence-free Constraint

\[
(B_x^+ - B_x^-) \Delta y + (B_y^+ - B_y^-) \Delta x = 0
\]
Divergence-Free Prolongation / Reconstruction of B field

Fields defined at mesh faces are endowed with linear profiles for 2nd order accuracy. (Just like fluid slabs having linear profiles.)

**Reconstruction** ➞ Find the Divergence-free polynomial in the *interior* of the (coarse) zone so that it matches the linear profiles at the *boundaries*:

Count: Boundaries have 7 independent pieces of information!

Divergence-free Constraint ➞

\[
(B_x^+ - B_x^-) \Delta y + (B_y^+ - B_y^-) \Delta x = 0
\]
How do we obtain the facial variation in the field? Focus on 2nd order, piecewise linear reconstruction. For a structured mesh, this is easy. For an unstructured mesh, it is only a little more difficult.

Limit in vertical direction to obtain slope in y-direction for the x-magnetic field.

Limit in the horizontal direction to obtain slope in the x-direction for the y-magnetic field.
B_y profile = field + slope

\[ B(x,y) + B(x,y) = 0 \]
\[ \partial_x B(x,y) + \partial_y B(y,x) = 0 \]
\[ \Rightarrow a_x + b_y = 0 \]

First try: Use only piecewise linear profiles for B.

Clearly, we need more terms in the polynomial....
By profile = field + slope

\[ \partial_x B_x (x,y) + \partial_y B_y (x,y) = 0 \]

\[ \Rightarrow a_x + b_y = 0 ; \]
\[ 2a_{xx} + b_{xy} = 0 ; \]
\[ a_{xy} + 2b_{yy} = 0 \]

10 polynomial coefficients; contain all needed 2nd order terms + more (underlined)

Count!: 3 (fields) + 4 (slopes) = 7 = 10 (coefficients) – 3 (constraints)

While demonstrated for second order on rectangles, this process can be carried out for all orders for cubes and tetrahedra.
Divergence-free reconstruction in 3d is a little more intricate. But it has been done in Balsara (2001), Balsara (2004).

Higher order reconstruction, up to fourth order, leading to higher order divergence-free MHD schemes have been done in Balsara (2009).

Divergence-free AMR-MHD schemes also presented in Balsara (2001).

Extension to unstructured meshes has also been done recently by Balsara & Dumbser (2015) and Xu, Balsara & Du (2016).
III) Overview of Multidimensional Riemann Solvers

All Riemann problems (RP) are self-similar solutions of a hyperbolic conservation law. Often shown as a space-time diagram, see below.

A one-dimensional RP arises in computer codes when two constant states come together at a zone boundary.

Example for Euler equations shown below: 1– left state; 6– right state; 2– left-going rarefaction; 5– right-going rarefaction; 3&4 – states that separate the contact discontinuity.

Its job → produce a flux F*!
Many of the details produced by a Riemann solver (RS) are never used in a computer code.

Motivates need for an approximate Riemann solver – topic of this talk.

See fig. below. The approximate RS has to satisfy some requirements:

1) A **self-similar wave model** in space-time.
2) **Consistency** with the conservation law, $\partial_t U + \partial_x F = 0$. Gives $U^*$ & $F^*$!
3) **Entropy enforcement**. Provide dissipation at rarefaction fans.
4) V. Desirable but not essential: Preservation of *internal sub-structures*.

**Exact Riemann Solver**

**Approximate HLL Riemann Solver**
Previous slides only described the 1d situation. Obtaining the strongly-interacting subsonic state $U^*$, and associated flux $F^*$, was of interest there.

We will only see **multidimensional effects** at the **vertices** of a mesh.

It is desirable to introduce **multi-dimensional effects** in order to get more consistency with the physics. When 4 states come together at a **vertex**; we have a **multi-dimensional RP**.

Two space + 1 time dimension.

**1D HLL RS**: 1 space + 1 time

**2D HLL RS**: 2 space + 1 time
Wave model changes: - Inverted triangle $\rightarrow$ Inverted Pyramid –
Contains the Strongly-Interacting state: -
(Two-dimensional SI state is formed by eating into the 1D RP solutions.)
1) Strongly-Interacting state evolves self-similarly. Reduces to 1D RP in limit where flow is one-dimensional – notice the 1D RP in side panels.
2) $U^*, F^*, G^*$ are obtained from 2D conservation law $\partial_t U + \partial_x F + \partial_y G = 0$ via consistency.
3) If inverted pyramid is wide enough, we get entropy enforcement in 2D!
The 1D and 2D HLL Riemann solvers, shown previously, average over important internal sub-structures in the RP. Specifically, the contact discontinuity is smeared.

The HLLC/HLLD/linearized Riemann solvers are approximate RS that restore the sub-structure back into the Riemann problem. See below.
Restoring the contact discontinuity in multi-dimensions is highly desirable. It permits flow structures to propagate isotropically in all directions relative to the mesh. CD moves at any angle w.r.t. mesh.

Restoring the contact discontinuity has been done in Balsara (2012), BDA14.

Supersonic cases are easy.
Restoring the sub-structure in multi-dimensions is highly desirable. It permits flow features to propagate isotropically in all directions relative to the mesh. Sub-structure can be at any angle w.r.t. mesh.

Restoring the general sub-structure has been done in Balsara (2014).

Supersonic cases are easy.
When flow features/shocks are mesh-aligned (on logically rectangular meshes) the 2D Riemann Solver reduces exactly to the 1D RS!

The flat panels in the construction of the RS ensure that this happens. It explains why the wave model was chosen to have flat panels on its sides.

The 2D HLLC reduces to 1D HLLC; the self-similar 2D RS reduces to 1D HLLI!
From Space-Time to Self-Similar Variables: The space-time approach is very difficult to extend to three dimensions. That is why we invented the self-similarity variables. It reduces the dimensionality of the problem by casting the variables in terms of speed of propagation.

Space-Time Picture

Variables in 2D: $x, y, t$

Picture in Self-Similar Variables

Variables in 2D: $\xi \equiv \frac{x}{t}, \quad \psi \equiv \frac{y}{t}$
Formulation in similarity variables equivalent to space-time formulation!

The constant state $U^*$ now becomes the **region of strong interaction $U^*$**.

Because of **self-similarity**, the constant state $U^*$ forms an inverted pyramid with polygonal base in a 3D space-time. Seen from the top, the pyramid is a rectangle.

The introduction of the subsonic constant state $U^*$, whether in 1D or 2D, provides the requisite **dissipation** as well as **entropy enforcement**.
Extension to 3D: The eight initial states and the three-dimensional wave propagation model. The positive parts of the x and z-axes are shown. The positive y-axis goes into the page.
The three-dimensional strongly-interacting state. The strongly-interacting two-dimensional states are shown with darker solid shading. The one-dimensional resolved states from the one-dimensional Riemann solvers are shown by the hachured regions.
IV) Formulating the Multid. Riemann Solver in Similarity Variables

Two-Rimensional Riemann Problems have been explored (using 1D RS technology) by Shulz-Rinne et al (1993).

They arise when four constant states come together at a corner. See the four states $U_{RU}$, $U_{LU}$, $U_{LD}$ and $U_{RD}$ and their evolution in time-sequence:
The Strongly-Interacting State (S.I.State) is formed by the interaction of four 1D Riemann problems. Notice the Self-Similarity of S.I. State! The S.I. State is bounded by the four 1D Riemann problems. This gives us the Wave Model. 1D RPs help us pick out multid. Wave model! The 1D RP’s lie within the boundaries of the Wave Model which circumscribes the S.I. State.
Notice the Self-Similarity of the Strongly-Interacting State!
Observe that as the strongly-interacting state moves outward, it engulfs the one-dimensional Riemann problems! Importance of Lagrangian flux! In that sense, the one-dimensional Riemann problems literally provide the boundary conditions for the multidimensional Riemann problem.
Game Plan:

1) Use 1D RPs to identify the boundaries of the Multidimensional Wave Model.

2) Assert self-similar evolution of the Strongly Interacting State within the Multidimensional Wave Model.

3) Recast the Conservation Law in similarity variables.

4) Solve for the Strongly Interacting State using 1D RP as boundary conditions for the Multidimensional Wave Model.

The Result:

MuSIC RS == Multidimensional, Self-similar, strongly-Interacting, Consistent Riemann Solver
Use similarity variables!

Consider \( \frac{\partial U(x,t)}{\partial t} + \frac{\partial F(x,t)}{\partial x} = 0 \) and replace \textit{two} coordinates, \((x,t)\), with \textit{one} similarity variable \( \xi = \frac{x}{t} \)

\[
\frac{\partial U}{\partial t} =
\]

\[
\frac{\partial F}{\partial x} =
\]

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} =
\]
Consider \( \frac{\partial U(x,t)}{\partial t} + \frac{\partial F(x,t)}{\partial x} = 0 \) and replace \textit{two} coordinates, \((x,t)\),

with \textit{one} similarity variable \( \xi = \frac{x}{t} \)

\[
\frac{\partial U}{\partial t} = \frac{\partial \xi}{\partial t} \frac{\partial U}{\partial \xi} = -x \frac{\partial U}{t^2 \partial \xi} = -\frac{1}{t} \xi \frac{\partial U}{\partial \xi}
\]

\[
\frac{\partial F}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial F}{\partial \xi} = \frac{1}{t} \frac{\partial F}{\partial \xi}
\]

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = -\frac{1}{t} \xi \frac{\partial U}{\partial \xi} + \frac{1}{t} \frac{\partial F}{\partial \xi} = 0
\]

\[\iff \quad \frac{\partial F}{\partial \xi} - \xi \frac{\partial U}{\partial \xi} = 0 \iff \quad \frac{\partial (F - \xi U)}{\partial \xi} + U = 0\]
How to Work With Similarity Variables in Multidimensions?

Insert $\tilde{\xi} = \frac{x}{t}$ ; $\tilde{\psi} = \frac{y}{t}$ in $\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0$  
Mnemonic: $x, y \rightarrow ksi, psi$

In Similarity variables:-

Replace *three* coordinates $(x, y, t)$ with *two* similarity variables $(\tilde{\xi}, \tilde{\psi})$: -

$U(x, y, t) \rightarrow \tilde{U}(\tilde{\xi}, \tilde{\psi})$ ; $F(x, y, t) \rightarrow \tilde{F}(\tilde{\xi}, \tilde{\psi})$ ; $G(x, y, t) \rightarrow \tilde{G}(\tilde{\xi}, \tilde{\psi})$

We Get:

\[
\frac{\partial (\tilde{F} - \tilde{\xi} \tilde{U})}{\partial \tilde{\xi}} + \frac{\partial (\tilde{G} - \tilde{\psi} \tilde{U})}{\partial \tilde{\psi}} + 2\tilde{U} = 0
\]

With a slight shift and rescaling of variables, we get:

\[
\frac{1}{\Delta \xi} \frac{\partial}{\partial \xi} \left[ \tilde{F} - (\xi_c + \xi \Delta \xi) \tilde{U} \right] + \frac{1}{\Delta \psi} \frac{\partial}{\partial \psi} \left[ \tilde{G} - (\psi_c + \psi \Delta \psi) \tilde{U} \right] + 2\tilde{U} = 0
\]

This is the Master Equation!
**Solution methodology:** Assert Self-Similar solution using self-similar trial/basis functions:

\[ \tilde{U}(\xi, \psi) = \tilde{U} + U_{\xi} \xi + U_{\psi} \psi + U_{\xi\xi} \xi^2 - \frac{1}{12} + U_{\psi\psi} \psi^2 - \frac{1}{12} + U_{\xi\psi} \xi \psi \]  

(Similarly for fluxes!)

Using test functions \( \phi(\xi, \psi) \), in Galerkin sense, integrate Master Equation over S.I. State:

\[
\frac{1}{\Delta \xi} \frac{\partial}{\partial \xi} \left\{ \phi(\xi, \psi) \left[ \tilde{F} - (\xi_c + \xi \Delta \xi) \tilde{U} \right] \right\} + \frac{1}{\Delta \psi} \frac{\partial}{\partial \psi} \left\{ \phi(\xi, \psi) \left[ \tilde{G} - (\psi_c + \psi \Delta \psi) \tilde{U} \right] \right\} \\
- \frac{1}{\Delta \xi} \left[ \tilde{F} - (\xi_c + \xi \Delta \xi) \tilde{U} \right] \frac{\partial \phi(\xi, \psi)}{\partial \xi} - \frac{1}{\Delta \psi} \left[ \tilde{G} - (\psi_c + \psi \Delta \psi) \tilde{U} \right] \frac{\partial \phi(\xi, \psi)}{\partial \psi} + 2 \phi(\xi, \psi) \tilde{U} = 0
\]
If there is no sub-structure (i.e., $\tilde{U}(\xi, \nu) = \bar{U}$ etc.) we get **3 V.Imp. Eqns.**:

$$2 \ A_{ABCD} \ \bar{U} = -\frac{1}{\Delta S} \int_{A\rightarrow B\rightarrow C\rightarrow D\rightarrow A} \left( F_\eta (\ell) - S_\eta U(\ell) \right) d\ell \ \leftarrow \text{Gives State}$$

$$- \ \frac{A_{ABCD}}{\Delta S} \ \bar{F} = -\frac{\xi_c}{\Delta S} \ A_{ABCD} \ \bar{U} - \frac{1}{\Delta S} \int_{A\rightarrow B\rightarrow C\rightarrow D\rightarrow A} \xi \left( F_\eta (\ell) - S_\eta U(\ell) \right) d\ell$$

$$- \ \frac{A_{ABCD}}{\Delta S} \ \bar{G} = -\frac{\psi_c}{\Delta S} \ A_{ABCD} \ \bar{U} - \frac{1}{\Delta S} \int_{A\rightarrow B\rightarrow C\rightarrow D\rightarrow A} \psi \left( F_\eta (\ell) - S_\eta U(\ell) \right) d\ell \ \leftarrow \text{Gives x- and y-fluxes}$$

Use Gauss Law, Integrate over S.I. State

Interpret above equations physically – they depend on **Lagrangian flux**! Reason: The Wave Model is a moving boundary! Multid. Wave model eats into 1d RP!

State and fluxes are **uniquely defined** by values in the 1D Riemann problems that lie in the boundary of the S.I. State.

These Integral Constraints **MUST** be respected!
Supersonic cases are easily treated. Illustrated here for the 1D HLLC RS when flow is supersonic to the right. S.I. State does not overlie vertex.
Introducing **Sub-Structure in the Strongly-Interacting State** is easy – just retain more moments and the Galerkin formulation does the rest!

Start with: \( \tilde{U}(\xi, \psi) = \bar{U} + U_\xi \xi + U_\psi \psi \); \( \tilde{F}(\xi, \psi) = \bar{F} + F_\xi \xi + F_\psi \psi \); \( \tilde{G}(\xi, \psi) = \bar{G} + G_\xi \xi + G_\psi \psi \)

For example, Integrating the test function \( \phi(\xi, \psi) = \xi \) over the wave model, we get:

\[
\frac{1}{4} U_\xi - \frac{1}{\Delta \xi} \bar{F} = - \frac{\xi_c}{\Delta \xi} \bar{U} - \left[ \frac{1}{2 \Delta \xi} \int_{-1/2}^{1/2} \left( F(1/2, \psi) - S_R U(1/2, \psi) \right) d\psi + \frac{1}{2 \Delta \xi} \int_{-1/2}^{1/2} \left( F(-1/2, \psi) - S_L U(-1/2, \psi) \right) d\psi 
+ \frac{1}{\Delta \psi} \int_{-1/2}^{1/2} \xi \left( G(\xi, 1/2) - S_U U(\xi, 1/2) \right) d\xi - \frac{1}{\Delta \psi} \int_{-1/2}^{1/2} \xi \left( G(\xi, -1/2) - S_D U(\xi, -1/2) \right) d\xi \right]
\]

For example, Integrating the test function \( \phi(\xi, \psi) = \psi \) over the wave model, we get:

\[
\frac{1}{4} U_\psi - \frac{1}{\Delta \psi} \bar{G} = - \frac{\psi_c}{\Delta \psi} \bar{U} - \left[ \frac{1}{2 \Delta \xi} \int_{-1/2}^{1/2} \psi \left( F(1/2, \psi) - S_R U(1/2, \psi) \right) d\psi - \frac{1}{2 \Delta \xi} \int_{-1/2}^{1/2} \psi \left( F(-1/2, \psi) - S_L U(-1/2, \psi) \right) d\psi 
+ \frac{1}{2 \Delta \psi} \int_{-1/2}^{1/2} \left( G(\xi, 1/2) - S_U U(\xi, 1/2) \right) d\xi + \frac{1}{2 \Delta \psi} \int_{-1/2}^{1/2} \left( G(\xi, -1/2) - S_D U(\xi, -1/2) \right) d\xi \right]
\]
Further equations detailed in papers.

This process has been explicitly carried out for linear variations in Balsara (2014), Balsara & Dumbser (2015), Balsara et al. (2016a,b).

Methods to treat linearly degenerate waves (which need to have their profiles steepened) and genuinely nonlinear waves (which do not need steepening) are also documented in these papers.
V) Approximating the Multidimensional Riemann Problem with just ONE call to the Multidimensional Riemann Solver!

\( \tilde{U}(\xi, \psi) = \bar{U} + U_{\xi} \xi + U_{\psi} \psi + U_{\xi\xi} \left[ \xi^2 - \frac{1}{12} \right] + U_{\psi\psi} \left[ \psi^2 - \frac{1}{12} \right] + U_{\xi\psi} \xi \psi \); 

Linearize fluxes: (using \( \bar{A} = \partial F(\bar{U})/\partial \bar{U} \) and \( \bar{B} = \partial G(\bar{U})/\partial \bar{U} \))

\( \bar{F}(\xi, \psi) = \bar{F}(\bar{U}) + \bar{A} \left( \tilde{U}(\xi, \psi) - \bar{U} \right) \); \( \bar{G}(\xi, \psi) = \bar{G}(\bar{U}) + \bar{B} \left( \tilde{U}(\xi, \psi) - \bar{U} \right) \)

\[
2 \bar{U} = - \left[ \frac{1}{\Delta \xi} \int_{-1/2}^{1/2} \left( \mathbf{F}(1/2, \psi) - S_R \mathbf{U}(1/2, \psi) \right) d\psi - \frac{1}{\Delta \xi} \int_{-1/2}^{1/2} \left( \mathbf{F}(-1/2, \psi) - S_L \mathbf{U}(-1/2, \psi) \right) d\psi \right. \\
+ \frac{1}{\Delta \psi} \int_{-1/2}^{1/2} \left( \mathbf{G}(\xi, 1/2) - S_U \mathbf{U}(\xi, 1/2) \right) d\xi - \frac{1}{\Delta \psi} \int_{-1/2}^{1/2} \left( \mathbf{G}(\xi, -1/2) - S_D \mathbf{U}(\xi, -1/2) \right) d\xi \right]
\]

\[
\frac{1}{4} U_{\xi} = \frac{1}{\Delta \xi} \left( \bar{F} - \xi_c \bar{U} \right)
\]

\[
- \left[ \frac{1}{2 \Delta \xi} \int_{-1/2}^{1/2} \left( \mathbf{F}(1/2, \psi) - S_R \mathbf{U}(1/2, \psi) \right) d\psi + \frac{1}{2 \Delta \xi} \int_{-1/2}^{1/2} \left( \mathbf{F}(-1/2, \psi) - S_L \mathbf{U}(-1/2, \psi) \right) d\psi \right. \\
+ \frac{1}{\Delta \psi} \int_{-1/2}^{1/2} \xi \left( \mathbf{G}(\xi, 1/2) - S_U \mathbf{U}(\xi, 1/2) \right) d\xi - \frac{1}{\Delta \psi} \int_{-1/2}^{1/2} \xi \left( \mathbf{G}(\xi, -1/2) - S_D \mathbf{U}(\xi, -1/2) \right) d\xi \right]
\]
\[
\frac{1}{4} U_\psi = \frac{1}{\Delta \psi} \left( \bar{G} - \psi_c \bar{U} \right)
\]

\[
= \left[ \frac{1}{\Delta \xi} \int_{-1/2}^{1/2} \psi \left( F(1/2, \psi) - S_R U(1/2, \psi) \right) d\psi - \frac{1}{\Delta \xi} \int_{-1/2}^{1/2} \psi \left( F(-1/2, \psi) - S_L U(-1/2, \psi) \right) d\psi \right]
\]

\[
+ \frac{1}{2 \Delta \psi} \int_{-1/2}^{1/2} \left( G(\xi, 1/2) - S_U U(\xi, 1/2) \right) d\xi + \frac{1}{2 \Delta \psi} \int_{-1/2}^{1/2} \left( G(\xi, -1/2) - S_D U(\xi, -1/2) \right) d\xi
\]

Further equations detailed in paper.

While this process has been demonstrated here for linear variations, quadratic, cubic and quartic variations have also been documented in Balsara (2014).

Allows us to construct a series solution for the multidimensional Riemann problem.
V) Approximating the Multidimensional Riemann Problem with just ONE call to the Multidimensional Riemann Solver!
VI) Multid RS & Constraint-Preserving Reconstruction for Maxwell’s Equations

Issues similar to MHD also prevail for Maxwell’s equations:

\[
\frac{\partial}{\partial t} \mathbf{B} = -\nabla \times \mathbf{E} \quad \leftarrow \text{Faraday's law} \quad ; \quad \frac{\partial}{\partial t} \mathbf{E} = \nabla \times \mathbf{B} - 4\pi \mathbf{j} \quad \leftarrow \text{Ampere's law}
\]

\[
\nabla \cdot \mathbf{E} = 4\pi \rho_c \quad \leftarrow \text{Gauss' law} \quad ; \quad \nabla \cdot \mathbf{B} = 0 \quad \leftarrow \text{no monopoles}
\]

\[
\frac{\partial}{\partial t} \rho_c + \nabla \cdot \mathbf{j} = 0
\]

Written in conservation law form:

\[
\frac{\partial}{\partial t} \begin{pmatrix} E^x \\ E^y \\ E^z \\ B^x \\ B^y \\ B^z \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} 0 & cB^z & \vdots & \vdots & \vdots & \vdots \\ cB^z & 0 & \vdots & \vdots & \vdots & \vdots \\ -cB^y & -cB^x & 0 & -cE^z & -cE^y & -cE^x \\ 0 & cB^x & -cB^y & 0 & -cE^x & -cE^y \\ -cE^z & cE^x & cE^y & 0 & 0 & 0 \\ cE^y & -cE^x & -cE^z & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -cB^z \\ -cB^x \\ -cB^y \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4\pi j_x \\ -4\pi j_y \\ -4\pi j_z \end{pmatrix}
\]
Take facial electric and magnetic field components as primary variables.
From MuSIC Riemann solver we get **edge-centered** Electric & Magnetic Fields:

\[
E^z_{\star} = \frac{1}{4} \left( E^z_{RU} + E^z_{RD} + E^z_{LU} + E^z_{LD} \right) + \frac{1}{2} \left( B^y_{R} - B^y_{L} \right) - \frac{1}{2} \left( B^x_{U} - B^x_{D} \right)
\]

**Centered electric field**

\[
B^z_{\star} = \frac{1}{4} \left( B^z_{RU} + B^z_{RD} + B^z_{LU} + B^z_{LD} \right) - \frac{1}{2} \left( E^y_{R} - E^y_{L} \right) + \frac{1}{2} \left( E^x_{U} - E^x_{D} \right)
\]

**Centered magnetic field**
Briefly Recall First Order Upwind Scheme

Realize that information always flows from the *upwind direction* in the advection equation. We try to build that intuition into our scheme in the simplest way.

For $a > 0$ we have:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a \left( \frac{u_j^n - u_{j-1}^n}{\Delta x} \right)$$

The scheme is also called the *donor cell scheme*.

It is stable because we can write:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a \left( \frac{u_j^n - u_{j-1}^n}{\Delta x} \right)$$

as centered + parabolic terms:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a \left( \frac{\left( u_{j+1}^n + u_j^n \right) - \left( u_j^n + u_{j-1}^n \right)}{2 \Delta x} \right) + a \left( \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2 \Delta x} \right)$$

The parabolic term enables the scheme to be stable!
Learn by Analogy: First order upwind scheme in 1D can be written as a central scheme + parabolic dissipation terms.

The electric and magnetic fields from the previous page are multidimensionally upwinded \( \Rightarrow \)

For a first order evolution of Faraday's law, the equations must return the centered scheme + parabolic dissipation:

\[
\frac{\partial B^x_{i+1/2,j,k}}{\partial t} = -\frac{c}{\Delta y} \left( \overline{E}^{z*}_{i+1/2,j+1/2,k} - \overline{E}^{z*}_{i+1/2,j-1/2,k} \right) + \frac{c}{\Delta z} \left( \overline{E}^{y*}_{i+1/2,j,k+1/2} - \overline{E}^{y*}_{i+1/2,j,k-1/2} \right) \\
+ \frac{c}{2\Delta x} \left( B^x_{i+3/2,j,k} - 2B^x_{i+1/2,j,k} + B^x_{i-1/2,j,k} \right) + \frac{c}{2\Delta y} \left( B^x_{i+1/2,j+1,k} - 2B^x_{i+1/2,j,k} + B^x_{i+1/2,j-1,k} \right) \\
+ \frac{c}{2\Delta z} \left( B^x_{i+1/2,j,k+1} - 2B^x_{i+1/2,j,k} + B^x_{i+1/2,j,k-1} \right)
\]

The multidimensional Riemann solver retrieves the appropriate stabilization!

At higher order, the contribution from the parabolic terms will be reduced!
The magnetic field is still divergence-free. However, the electric field satisfies a different constraint associated with Gauss' law:
\[ \nabla \cdot \mathbf{E} = 4\pi \rho_c \]

At second order, we can illustrate the issue in 2D. The reconstructed charge density becomes
\[ \rho_c(x, y) = q_0 + q_x x + q_y y \]
The reconstructed electric field becomes
\[ E^x(x, y) = a_0 + a_x x + a_y y + a_{xx} (x^2 - 1/12) + a_{xy} xy \]
\[ E^y(x, y) = b_0 + b_x x + b_y y + b_{xy} xy + b_{yy} (y^2 - 1/12) \]
Enforcing constraints via Gauss' Law gives:
\[ a_x + b_y = 4\pi q_0 \quad ; \quad 2a_{xx} + b_{xy} = 4\pi q_x \quad ; \quad a_{xy} + 2b_{yy} = 4\pi q_y \]
This gives a constraint-preserving reconstruction for the Electric field!
VII) Results & Applications


ADER is used in the predictor step with WENO reconstruction. Multid. RS provides the corrector step.

Test problems which emphasize advantages of multi-d approach are also presented. We find vastly reduced mesh imprinting.

CFL numbers that are higher than those in conventional 2\textsuperscript{nd} order Godunov schemes are used.

Doubling dissipation for MHD is completely unnecessary.

Hydro, MHD & RMHD tests presented.

ANY self-similar 1D Riemann solver can be used in the Multid RS!\textsuperscript{56}
Euler Flow: Spherical 3D Hydrodynamical Blast problem.
With sufficient resolution, all test problems will pin this one well. However, real applications are resolution-starved. $64^3$ zones in 3D.

On coarser meshes, mesh imprinting shows up; latter is more isotropic.

Conventional 2\textsuperscript{nd} order: (CFL 0.3)

With 2D HLLC RS: (CFL 0.6)
**Euler Flow: Isentropic Vortex – Accuracy Analysis on Unstructured Mesh**

Run with **CFL of 0.95**. Accuracy shown with increasing mesh resolution. ADER-WENO Schemes used.

<table>
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<tr>
<th>#of elements, 1d</th>
<th>L₁ Error</th>
<th>L₁ Order</th>
<th>L₂ Error</th>
<th>L₂ Order</th>
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Euler Flow: Double Mach Reflection Problem on Structured Mesh

Run with CFL of 0.8. Usually requires a 1920x480 zone mesh to see the KH instability at the Mach stem even with 4th order DG methods. Results at 2400x600 zones with 2nd order shown here

Multid RS technology at 2nd order seems to catch up with high order schemes with conventional RS technology.

CFL # vastly larger than that for 4th order DG schemes!
MHD Flow: Long term decay of Alfven Waves
Alfven waves propagating very obliquely to mesh.

The decay over long times is shown. Of all the choices shown, 2D HLLD RS with some amount of WENO technology shows the least dissipation.

Effect of second, third and fourth order of accuracy also shown.

Decay of $V_z$

Decay of $B_z$
MHD Flow: Magnetic Field Loop Advection

Run with CFL of 0.9. Magnetic loop advected diagonally on a rectangular domain.

Conventional scheme doubles dissipation of the electric field (Gardiner & Stone 2005). The scheme with Multid. RS does not double dissipation.

The propagation of the field loop is much more isotropic for Multid. RS

Conventional 2\textsuperscript{nd} order (Gardiner & Stone 2005): (CFL 0.45)

With 2D HLLC RS Balsara (2010, 2012): (CFL 0.9)
MHD Rotor Problem on ALE Mesh

CFL 0.9; 80,000 elements ; ALE mesh
Accuracy for the MHD Vortex problem on an ALE Mesh

Accuracy demonstrated from 1\textsuperscript{st} to 5\textsuperscript{th} order on 2D ALE Mesh.

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MHD Flow: 3D MHD Blast with very low Plasma Beta
Run with a CFL of 0.6. Near-infinite blast wave propagating through a magnetic plasma with $\beta=0.001$.

Accurate div-free propagation of B-field also gives better pressure positivity. ($\log_{10}$ of density and pressure shown.)
RMHD Rotor Problem in 2D: A Case Study with Increasing Lorenz Factor

\( \gamma = 10 \)

\( \gamma = 30 \)

With J. Kim
RMHD Blast Problem in 3D

Nearly-Infinite Strength RMHD Blast problem in low plasma-β medium.

Density | Pressure | Lorenz Factor

With J. Kim
## Accuracy of Electromagnetic Wave Propagation

Table 2

Shows the accuracy analysis for the third order scheme for the propagation of an electromagnetic wave in vacuum. A CFL of 0.45 was used. The errors and accuracy in the $y$-component of the electric and magnetic fields are shown.

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<th>$B_y L_{inf}$ accuracy</th>
</tr>
</thead>
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<td>3.8375E-2</td>
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<tr>
<td>$32^3$</td>
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<td>2.93</td>
<td>5.2845E-3</td>
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<td>$64^3$</td>
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<td>3.00</td>
<td>6.5957E-4</td>
<td>3.01</td>
</tr>
<tr>
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<td>3.01</td>
<td>8.1875E-5</td>
<td>3.00</td>
</tr>
</tbody>
</table>

Table 3

Shows the accuracy analysis for the fourth order scheme for the propagation of an electromagnetic wave in vacuum. A CFL of 0.45 was used. The errors and accuracy in the $y$-component of the electric and magnetic fields are shown.

<table>
<thead>
<tr>
<th>Zones</th>
<th>$E_y L_1$ error</th>
<th>$E_y L_1$ accuracy</th>
<th>$E_y L_{inf}$ error</th>
<th>$E_y L_{inf}$ accuracy</th>
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<tr>
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<td>9.6550E-6</td>
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<tr>
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<td>4.07</td>
<td>5.8010E-7</td>
<td>4.05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Zones</th>
<th>$B_y L_1$ error</th>
<th>$B_y L_1$ accuracy</th>
<th>$B_y L_{inf}$ error</th>
<th>$B_y L_{inf}$ accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16^3$</td>
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<td>2.2960E-3</td>
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<tr>
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<td>9.7013E-5</td>
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<td>4.07</td>
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<td>4.05</td>
</tr>
</tbody>
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VIII) Conclusions

1) Genuinely Multidimensional (MuSIC) Riemann Solvers presented. **Input**: multiple states in 2D. **Output**: 1 resolved state + 2 numerical fluxes. Any 1D RS can be used as a building block.

2) Addressed all the issues with introducing self-similar sub-structure in the strongly-interacting state. It can propagate at any orientation relative to the mesh and provide reduced dissipation.

3) All the self-similar equations needed for the formulation are presented as explicit, computer-implementable, closed-form formulae. This makes the present MuSIC RS technology very accessible.

4) The process of obtaining the numerical fluxes explicitly is presented.

5) Larger CFL numbers possible compared to conventional RS-based technology.

6) Predictor-Corrector like ADER-WENO formulation for any order, cost-competitive implementation is presented.
7) Much more isotropic propagation of flow features demonstrated for hydro, MHD and RMHD flows.

8) There is no need to double dissipation when evaluating electric fields in MHD calculations.

9) The 2D MuSIC RS also helps out with retaining pressure positivity in MHD and RMHD problems with very low plasma-β.

10) Demonstrated value of the MuSIC Riemann solver for ALE meshes.

11) It is very satisfying that the MuSIC RS approximates the multid. RP quite well with only a few terms in the series expansion.

More At: http://physics.nd.edu/people/faculty/dinshaw-balsara
Please also see the website for my book:
   http://www.nd.edu/~dbalsara/Numerical-PDE-Course

Thanks for your attention!