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SOME PROPERTIES OF k-GENERALIZED FIBONACCI NUMBERS

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Summary. In the present paper, we propose some properties of the new family k-generalized Fibonacci numbers which related to generalized Fibonacci numbers. Moreover, we give some identities involving binomial coefficients for k-generalized Fibonacci numbers.

1 INTRODUCTION

Fibonacci numbers have a great importance in mathematics. It is one of the most popular sequences that have a lot of applications in many branch of mathematics as in diverse sciences [1, 2, 6, 7, 10-13, 16-20]. The Fibonacci numbers F_n are given by the recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \qquad n \geq 1$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. Koshy [9] written one of the most popular books of Fibonacci and Lucas numbers, and gave numerous recurrence relations, generalizations and applications of Fibonacci and Lucas numbers. For $a, b \in \mathbb{R}$ and $n \geq 1$, the well-known generalized Fibonacci numbers are defined

$$G_{n+1} = G_n + G_{n-1}$$

where $G_0 = a$ and $G_1 = b$.

Falcon and Plaza [4] introduced general k-Fibonacci numbers and gave some properties of these numbers. Guleç et al. [5] presented some properties of generalized Fibonacci numbers with binomial coefficients.

El-Mikkawy and Sogabe [3] proposed a new family of k-Fibonacci numbers and gave the relationship between the k-Fibonacci numbers and Fibonacci numbers as follow:

$$F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r, \qquad n = mk + r.$$

In [14], Özkan et al. defined a new family of k-Lucas numbers and gave some identities of the new family of k-Fibonacci and k-Lucas numbers. Özkan et al. [15] introduced some identities of the new family of k-Fibonacci numbers.

In this study, we present some identities of the new family of k-generalized Fibonacci numbers. We give relationships between the new family of k-Fibonacci numbers and k-generalized Fibonacci numbers. Also, we introduce Cassini formulas of k-generalized Fibonacci numbers and some properties involving binomial coefficients. The rest of the paper is organized as follows: In Section 2 (Preliminaries), the fundamental definitions and theorems are given. Then main theorems and proofs are introduced in Section 3.

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2 PRELIMINIARIES

Definition 2.1. [21] For $n, k\ (k \neq 0) \in \mathbb{N}$, the new family of k-generalized Fibonacci numbers are defined by

$$G_n^{(k)} = \frac{1}{\left(\sqrt{5}\right)^k}\left([a+b\alpha]\alpha^{m+1} - [a+b\beta]\beta^{m+1}\right)^r\left([a+b\alpha]\alpha^m - [a+b\beta]\beta^m\right)^{k-r}$$

where $n = mk + r$, $0 \leq r < k$ and $m \in \mathbb{N}$.
It is clear that for $a = 0$ and $b = 1$, $G_n^{(k)} = F_n^{(k)}$ and for $k = 1$, $r = 0$ and $n = m$, $G_n^{(1)} = G_n$.
Then they gave the relationship of between the new family of k-generalized Fibonacci numbers and generalized Fibonacci numbers as follow:

$$G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r, \qquad n = mk + r. \tag{2.1}$$

Theorem 2.2. [9]

i. $G_{n+1}^3 - G_n^3 - G_{n-1}^3 = 3G_{n+1}G_nG_{n-1}$
ii. $\sum_{i=1}^{n} F_iG_{3i} = F_nF_{n+1}G_{2n+1}$
iii. $G_n^2 + G_{n-1}^2 = (3a-b)G_{2n-1} - (a^2+ab-b^2)F_{2n-1}$
iv. $F_{2n+1} = F_{n+1}^2 + F_n^2$
v. $G_{n-1}^6 + G_n^6 + G_{n+1}^6 = 2[2G_n^2 + (a^2+ab-b^2)(-1)^n]^3 + 3G_{n-1}^2G_n^2G_{n+1}^2$
vi. $G_{n+t}G_{n+t-2} - G_{n+t-1}^2 = (a^2+ab-b^2)(-1)^{n+t-1}F_k^2$

Theorem 2.3. [15]

$$\sum_{i=1}^{n} F_iF_{3i} = F_{2n+1}^{(2)}\left(F_{2n+3}^{(2)} - F_{2n-1}^{(2)}\right)$$

Theorem 2.4. [3]

i. $\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}F_{mk+i}^{(k)} = (-1)^{k-1}F_mF_{(m-1)(k-1)}^{(k-1)}$
ii. $\sum_{i=0}^{k-1}\binom{k-1}{i}F_{mk+i}^{(k)} = F_mF_{(m+2)(k-1)}^{(k-1)}$.

3 MAIN RESULTS

In this section, we present some properties of the new family k-generalized Fibonacci numbers.

Theorem 3.1. For $n \geq 1$, we have

$$G_{2n+2}^{(2)} + G_{2n}^{(2)} = 2\, G_{2n+1}^{(2)} + G_{2n-2}^{(2)}.$$

Proof. Using Theorem 2.2 (i), we have

$$G_{n+1}^3 - G_n^3 = G_{n-1}^3 + 3G_{n+1}G_nG_{n-1}$$
$$(G_{n+1} - G_n)(G_{n+1}^2 + G_{n+1}G_n + G_n^2) = G_{n-1}(G_{n-1}^2 + 3G_{n+1}G_n)$$
$$G_{n-1}\left(G_{2n+2}^{(2)} + G_{2n+1}^{(2)} + G_{2n}^{(2)}\right) = G_{n-1}\left(G_{2n-2}^{(2)} + 3G_{2n+1}^{(2)}\right)$$
$$G_{2n+2}^{(2)} + G_{2n+1}^{(2)} + G_{2n}^{(2)} = G_{2n-2}^{(2)} + 3G_{2n+1}^{(2)}$$
$$G_{2n+2}^{(2)} + G_{2n}^{(2)} = 2\, G_{2n+1}^{(2)} + G_{2n-2}^{(2)}\,.$$

Theorem 3.2. For $n \geq 1$, we have

$$(3a-b)\sum_{i=1}^{n} F_iG_{3i} = F_{2n+1}^{(2)}\left(G_{2n+3}^{(2)} - G_{2n-1}^{(2)}\right) + (a^2+ab-b^2)\sum_{i=1}^{n} F_iF_{3i}.$$

Proof. Using Theorem 2.2 (ii), (iii), (iv) and Theorem 2.3, we have

$$\begin{aligned}
(3a-b)\sum_{i=1}^{n} F_iG_{3i} &= (3a-b)F_nF_{n+1}G_{2n+1} \\
&= F_nF_{n+1}(G_n^2 + G_{n+1}^2 + (a^2+ab-b^2)F_{2n+1}) \\
&= F_nF_{n+1}\big(G_n(G_{n+1} - G_{n-1}) + G_{n+1}(G_{n+2} - G_n) \\
&\quad + (a^2+ab-b^2)(F_{n+1}^2 + F_n^2)\big) \\
&= F_nF_{n+1}\big(-G_nG_{n-1} + G_{n+1}G_{n+2} \\
&\quad + (a^2+ab-b^2)(F_{n+2}F_{n+1} - F_nF_{n-1})\big) \\
&= F_{2n+1}^{(2)}\left(G_{2n+3}^{(2)} - G_{2n-1}^{(2)} + (a^2+ab-b^2)\left(F_{2n+3}^{(2)} - F_{2n-1}^{(2)}\right)\right) \\
&= F_{2n+1}^{(2)}\left(G_{2n+3}^{(2)} - G_{2n-1}^{(2)}\right) + (a^2+ab-b^2)F_{2n+1}^{(2)}\left(F_{2n+3}^{(2)} - F_{2n-1}^{(2)}\right) \\
&= F_{2n+1}^{(2)}\left(G_{2n+3}^{(2)} - G_{2n-1}^{(2)}\right) + (a^2+ab-b^2)\sum_{i=1}^{n} F_iF_{3i}.
\end{aligned}$$

Theorem 3.3. For $n \geq 1$, we have

$$\left(G_{2n-2}^{(2)}\right)^3 + \left(G_{2n}^{(2)}\right)^3 + \left(G_{2n+2}^{(2)}\right)^3 = 2\left[2G_{2n}^{(2)} + (a^2+ab-b^2)(-1)^n\right]^3 + 3G_{2n-2}^{(2)}G_{2n}^{(2)}G_{2n+2}^{(2)}.$$

Proof. Using Theorem 2.2 (v), we get

$$\begin{aligned}
\left(G_{2n-2}^{(2)}\right)^3 + \left(G_{2n}^{(2)}\right)^3 + \left(G_{2n+2}^{(2)}\right)^3 &= (G_{n-1}^2)^3 + (G_n^2)^3 + (G_{n+1}^2)^3 \\
&= G_{n-1}^6 + G_n^6 + G_{n+1}^6 \\
&= 2[2G_n^2 + (a^2+ab-b^2)(-1)^n]^3 + 3G_{n-1}^2G_n^2G_{n+1}^2
\end{aligned}$$

$$= 2\left[2G_{2n}^{(2)} + (a^2 + ab - b^2)(-1)^n\right]^3 + 3G_{2n-2}^{(2)}G_{2n}^{(2)}G_{2n+2}^{(2)}\,.$$

Theorem 3.4. For $n \geq 1$, we have

$$G_{2n+2}^{(2)} - G_{2n}^{(2)} = G_{2n-2}^{(2)} + 2\,G_{2n-1}^{(2)}.$$

Proof. From equation (2.1) and recurrence relation of generalized Fibonacci numbers, we get

$$\begin{aligned}
G_{2n+2}^{(2)} - G_{2n}^{(2)} &= G_{n+1}^2 - G_n^2 \\
&= (G_{n+1} - G_n)(G_{n+1} + G_n) \\
&= G_{n-1}(G_{n+1} + G_n) \\
&= G_{n-1}G_{n+1} + G_{n-1}G_n \\
&= G_{n-1}(G_n + G_{n-1}) + G_{n-1}G_n \\
&= G_{n-1}^2 + 2G_{n-1}G_n \\
&= G_{2n-2}^{(2)} + 2\,G_{2n-1.}^{(2)}
\end{aligned}$$

Theorem 3.5. For $n \geq 1$, we have

$$G_{2n-2}^{(2)} + G_{2n-1}^{(2)} = G_{2n}^{(2)} + (a^2 + ab - b^2)(-1)^n\,.$$

Proof. Using Theorem 2.2 (vi), we have

$$\begin{aligned}
G_{2n-2}^{(2)} + G_{2n-1}^{(2)} &= G_{n-1}^2 + G_nG_{n-1} \\
&= G_{n-1}(G_{n-1} + G_n) \\
&= G_{n-1}G_{n+1} \\
&= G_n^2 + (a^2 + ab - b^2)(-1)^n \\
&= G_{2n}^{(2)} + (a^2 + ab - b^2)(-1)^n.
\end{aligned}$$

Theorem 3.6. For $n \geq 1$, we have

$$G_{4n+5}^{(4)} = \left(G_{2n}^{(2)}\right)^2 + G_{4n+1}^{(4)} + 2G_{4n-3}^{(4)} + \left(G_{2n-2}^{(2)}\right)^2 + 3G_{2n+3}^{(2)}G_{2n-1}^{(2)}\,.$$

Proof. Using Theorem 2.2 (i), we have

$$\begin{aligned}
G_{4n+5}^{(4)} &= (G_{n+1})^3G_{n+2} \\
&= (G_n^3 + G_{n-1}^3 + 3G_{n+1}G_nG_{n-1})G_{n+2} \\
&= G_n^3G_{n+2} + G_{n-1}^3G_{n+2} + 3G_{n+2}G_{n+1}G_nG_{n-1} \\
&= G_n^3(G_n + G_{n+1})G_{n-1}^3(2G_n + G_{n-1}) + 3G_{2n+3}^{(2)}G_{2n-1}^{(2)} \\
&= \left(G_{2n}^{(2)}\right)^2 + G_{4n+1}^{(4)} + 2G_{4n-3}^{(4)} + \left(G_{2n-2}^{(2)}\right)^2 + 3G_{2n+3}^{(2)}G_{2n-1.}^{(2)}
\end{aligned}$$

Theorem 3.7. For $k, n, t \geq 1$, we have

$$G_{kn+t}^{(k)} G_{kn+t-2}^{(k)} - \left(G_{kn+t-1}^{(k)}\right)^2 = \begin{cases} G_n^{2k-2}(-1)^n(a^2+ab-b^2), & t=1 \\ 0, & t \neq 1 \end{cases}.$$

Proof. For $t = 1$, we get

$$\begin{aligned} G_{kn+1}^{(k)} G_{kn-1}^{(k)} - \left(G_{kn}^{(k)}\right)^2 &= (G_n^{k-1} G_{n+1})(G_{n-1} G_n^{k-1}) - (G_n^k)^2 \\ &= G_{n-1} G_n^{2k-2} G_{n+1} - G_n^{2k} \\ &= G_n^{2k-2}[G_{n-1} G_{n+1} - G_n^2] \\ &= G_n^{2k-2}(-1)^n(a^2+ab-b^2). \end{aligned}$$

For $t \neq 1$, we get

$$\begin{aligned} & G_{kn+t}^{(k)} G_{kn+t-2}^{(k)} - \left(G_{kn+t-1}^{(k)}\right)^2 = (G_n^{k-t} G_{n+1}^t)(G_n^{k-t+2} G_{n+1}^{t-2}) - (G_n^{k-t+1} G_{n+1}^{t-1})^2 \\ & = G_n^{2k-2t+2} G_{n+1}^{2t-2} - G_n^{2k-2t-2} G_{n+1}^{2t-2} \\ & = 0. \end{aligned}$$

Theorem 3.8. For $n \geq 1$, we have

$$G_{2(n+s-1)}^{(2)} - G_{n+s} G_{n+s-2} = (-1)^{n+s}(a^2+ab-b^2).$$

Proof. From the equation (2.1) and Theorem 2.2. (vi), we acquire

$$\begin{aligned} G_{2(n+s-1)}^{(2)} - G_{n+s} G_{n+s-2} &= G_{n+s-1}^2 - G_{n+s} G_{n+s-2} \\ &= -(G_{n+s} G_{n+s-2} - G_{n+s-1}^2) \\ &= -\left((-1)^{n+s-1}(a^2+ab-b^2)\right) \\ &= (-1)^{n+s}(a^2+ab-b^2). \end{aligned}$$

Theorem 3.9. For $n \geq 1$, we have

$$\sum_{i=1}^{k-1} (-1)^i \binom{k-1}{i} G_{mk+i}^{(k)} = (-1)^{k-1} G_m G_{(m-1)(k-1)}^{(k-1)}.$$

Proof. By using the equation (2.1) and the well known binomial property, we obtain

$$\begin{aligned} \sum_{i=1}^{k-1} (-1)^i \binom{k-1}{i} G_{mk+i}^{(k)} &= (-1)^{k-1} \sum_{i=1}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} G_m^{k-i} G_{m+1}^i \\ &= (-1)^{k-1} G_m \sum_{i=1}^{k-1} \binom{k-1}{i} (-G_m)^{k-i-1} G_{m+1}^i \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{k-1} G_m (G_{m+1} - G_m)^{k-1} \\
&= (-1)^{k-1} G_m G_{m-1}^{k-1} \\
&= (-1)^{k-1} G_m G_{(m-1)(k-1)}^{(k-1)} .
\end{aligned}
$$

Theorem 3.10. For $n \geq 1$, we have

$$\sum_{i=1}^{k-1} \binom{k-1}{i} G_{mk+i}^{(k)} = G_m G_{(m+2)(k-1)}^{(k-1)} .$$

Proof. By taking account the equation (2.1) and the well known binomial property, we get

$$
\begin{aligned}
\sum_{i=1}^{k-1} \binom{k-1}{i} G_{mk+i}^{(k)} &= \sum_{i=1}^{k-1} \binom{k-1}{i} G_m^{k-i} G_{m+1}^{i} \\
&= G_m \sum_{i=1}^{k-1} \binom{k-1}{i} G_{m+1}^{i} (G_m)^{k-i-1} \\
&= G_m (G_{m+1} + G_m)^{k-1} \\
&= G_m G_{m+2}^{k-1} \\
&= G_m G_{(m+2)(k-1)}^{(k-1)} .
\end{aligned}
$$

4 CONCLUSIONS

In this study, we prove that some identities of the new family of k-generalized Fibonacci numbers. Then, we show that some properties of the new family of k-generalized Fibonacci numbers related to generalized Fibonacci numbers. Furthermore, we extend Cassini's formula to the new family of k-generalized Fibonacci numbers and present identities comprising binomial coefficients for the new family of k-generalized Fibonacci numbers.

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