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NEW PROPERTIES OF AN ARITHMETIC FUNCTION

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Summary. Recently the author and Derbal introduced and studied some elementary properties of arithmetic functions related the greatest common divisor. New properties of them are given in this paper.

1. INTRODUCTION

Throughout this paper, we let $\mathbb{N}^*$ denote the set $\mathbb{N} \setminus \{0\}$ of positive integers and we let (m, n) and $[m, n]$ denote, respectively, the greatest common divisor and the least common multiple of any two integers m and n. A sequence of positive integers $(a_n)_{n\geq 1}$ is simply denoted by $\mathfrak{a}$. Let the prime factorization of the positive integer $n > 1$ be

$$n = \prod_{i=1}^{r} p_i^{e_i}$$

where $r, e_1, e_2, \ldots, e_r$ are positive integers and $p_1, p_2, \ldots, p_r$ are different primes.

In number theory, an arithmetic function, is a function whose domain is the positive integers and whose range is a subset of the complex numbers. Their various properties were studied by several authors (see e.g., [1-4]) and they still represent an important research topic up to now. Recently, the author and Derbal [5] introduced and studied some elementary properties of the following arithmetic function, for a positive integer α:

$$\begin{cases} f_\alpha(1) = 1, \\ f_\alpha(n) = \prod_{i=1}^{r} p_i^{(e_i,\alpha)}, \end{cases}$$

which can be considered a generalization of the radical function, since

$$f_1(n) = \prod_{i=1}^{r} p_i = \mathrm{rad}(n) \text{ for all } n.$$

In the present paper, we will discuss other properties of the functions f_α and will define new integer sequences related to them.

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2. MAIN RESULTS

It can be easily seen that:

$$f_\alpha(mn) = f_\alpha(m)f_\alpha(n) \text{ whenever } (m,n) = 1 \tag{2.1}$$

which means that f_α is a multiplicative function, for all α. It is not completely multiplicative, since for a prime number p:

$$f_\alpha(p) = \begin{cases} p, & \text{if } \alpha \text{ is odd;} \\ p^2, & \text{if } \alpha \text{ is even.} \end{cases}$$

While

$$f_\alpha(p)f_\alpha(p) = p^2 \text{ for all } \alpha.$$

The next theorem gives a condition for m and n (which are not necessarily co-prime) to be satisfied the equation (2.1), for all even positive integers α.

Theorem 2.1 Let α be an even positive integers. Then

$$f_\alpha(mn) = f_\alpha(m)f_\alpha(n)$$

for all square-free positive integers m and n.

Proof. Let m and n be square-free positive integers. Then it follows that:

$$f_\alpha(mn) = \prod_{p|m, p\nmid n} p \prod_{q\nmid m, q|n} q \prod_{r|m, r|n} r^{(2,\alpha)}, \tag{2.2}$$

where p, q, and r are prime numbers. Also, we have

$$f_\alpha(m)f_\alpha(n) = mn = \prod_{p|m, p\nmid n} p \prod_{q\nmid m, q|n} q \prod_{r|m, r|n} r^2. \tag{2.3}$$

The right-hand sides of (2.2) and (2.3) are equal only if $(2,\alpha) = 2$, i.e., only if α is even, as claimed. The proof is complete.

Let α and k be positive integers. It is clear that if $(k,\alpha) = 1$, then $f_\alpha(n^k) = f_\alpha(n)$ for all $n \in \mathbb{N}^*$. In the general case we have the following theorem.

Theorem 2.2 Let the prime factorization of the positive integer $n > 1$ be $\prod_{i=1}^r p_i^{e_i}$. Let α and k be positive integers with $(k,\alpha) = d$. Then

$$f_\alpha(n^k) = \prod_{i=1}^r p_i^{\gamma_i d},$$

where $\gamma_i = \left(\frac{\alpha}{d}, e_i\right)$. In particular, if $(\alpha, e_i) = e$ $(1 \le i \le r)$, then

$$f_\alpha(n^k) = f_\alpha(n)^l, \text{ with } l = \left(\frac{\alpha}{e}, k\right).$$

Proof. According to [6, Ex. 24, p. 22] we have the following multiplicative property of the greatest common divisor. If a, b, h, and k be positive integers, then

$$(ah, bk) = (a, b)(h, k)\left(\frac{a}{(a, b)}, \frac{k}{(h, k)}\right)\left(\frac{b}{(a, b)}, \frac{h}{(h, k)}\right).$$

By taking $h = 1$ we obtain

$$(a, bk) = (a, b)\left(\frac{a}{(a, b)}, k\right).$$

It follows by using this fact that:

$$\begin{aligned} f_\alpha(n^k) &= f_\alpha\left(\prod_{i=1}^{r} p_i^{ke_i}\right) \\ &= p_1^{(ke_1,\alpha)} p_2^{(ke_2,\alpha)} \dots p_r^{(ke_r,\alpha)} \\ &= p_1^{(k,\alpha)\left(\frac{\alpha}{(\alpha,k)}, e_1\right)} p_2^{(k,\alpha)\left(\frac{\alpha}{(\alpha,k)}, e_2\right)} \dots p_r^{(k,\alpha)\left(\frac{\alpha}{(\alpha,k)}, e_r\right)} \\ &= \prod_{i=1}^{r} p_i^{\gamma_i d}, \end{aligned}$$

where $\gamma_i = \left(\frac{\alpha}{d}, e_i\right)$, as claimed. The proof is finished.

Theorem 2.3 Let α be a positive integer. Then the following two systems of inequalities:

$$\begin{cases} m < n \\ f_\alpha(m) < f_\alpha(n) \end{cases} \text{ and } \begin{cases} m < n \\ f_\alpha(m) > f_\alpha(n) \end{cases}$$

hold for infinitely many positive integers m and n.

Proof. Let us distinguish the following two cases:

Case 1: If $\alpha = 1$. For the first system, let m be a positive integer and let n be any square-free number such that $m < n$. Then

$$f_\alpha(m) \le m < n = f_\alpha(n),$$

which confirms the first system. For the second one, let p and q be prime numbers with $p < q$. Let

$$S \coloneqq \{a \in \mathbb{N}^*;\ p^a > q\}.$$

Clearly $S \neq \emptyset$ and it contains infinitely many elements. We put $m = q$ and $n = p^s$, where $s \in S$. From which $m < n$ and

$$f_\alpha(q) = \mathrm{q} > p = f_\alpha(p^s) \ \Rightarrow\ f_\alpha(m) > f_\alpha(n),$$

this confirms the second system.

Case 2: If $\alpha > 1$. Let M be the set of the positive multiples of α and let $k \in M$. So $(k, \alpha) = \alpha$, $k - 1 \notin M$ since $\alpha > 1$, and $(k - 1, \alpha) < \alpha$. Now we put $m = \varphi(2^k) = 2^{k-1}$, where φ is the totient's Euler function (see e.g., [6, Chapter 2]) and $n = 2^k$. Clearly, $m < n$ and we have

$$
\begin{aligned}
f_\alpha\left(\varphi(2^k)\right) &= f_\alpha(2^{k-1}) \\
&= 2^{(k-1,\alpha)} \\
&< 2^\alpha \ \text{(since } (k-1,\alpha) < \alpha) \\
&= 2^{(k,\alpha)} \ \text{(since } k \in M) \\
&= f_\alpha(2^k) \\
&\Rightarrow f_\alpha(m) < f_\alpha(n),
\end{aligned}
$$

from which the validity of the first system follows. Next, let us choose an integer k such that $k-1 \in M$. So $k \notin M$, $(k-1,\alpha) = \alpha$ and $(k,\alpha) < \alpha$. If we put $m = \varphi(p^k)$ and $n = p^k$, where p is an odd prime, then we have $m < n$ and

$$
\begin{aligned}
f_\alpha\left(\varphi(p^k)\right) &= f_\alpha((p-1)p^{k-1}) \\
&= f_\alpha(p-1)f_\alpha(p^{k-1}) \ \text{(since } (p-1,p^{k-1}) = 1) \\
&= f_\alpha(p-1)\, p^{(k-1,\alpha)} \\
&> p^{(k-1,\alpha)} \ \text{(since } f_\alpha(p-1) > 1) \\
&> 2^{(k,\alpha)} \ \text{(since } (k-1,\alpha) = \alpha > (k,\alpha)) \\
&= f_\alpha(p^k) \\
&\Rightarrow f_\alpha(m) > f_\alpha(n),
\end{aligned}
$$

which confirms the second system and completes this proof.

Theorem 2.4 Let $n > 1$ be an integer and let d be a proper positive divisor of n. Then we have

1. If n is a square-free number, then $f_\alpha(d)|f_\alpha(n)$ $(\forall\alpha \in \mathbb{N}^*)$.
2. If d is a square-free number, then $f_\alpha(d)|f_\alpha(n)$ $(\forall\alpha \in \mathbb{N}^*)$.
3. If n and d are not square-free number, then there are infinitely many positive integers α such that $f_\alpha(d)|f_\alpha(n)$.

Proof. Let the prime factorization of the positive integer $n > 1$ be $\prod_{i=1}^{r} p_i^{e_i}$. It is well known that the positive divisors of n are all integers of the form $\prod_{i=1}^{r} p_i^{h_i}$ with $0 \le h_i \le e_i (1 \le i \le r)$.

1. If n is a square-free integer, then d must itself to be a square-free. According to [5, Theorem 2.1] we can write
$$f_\alpha(d) = d|n = f_\alpha(n) \ (\forall\alpha \in \mathbb{N}^*),$$
which proves the first property.
2. Suppose that d is a square-free number, i.e., $h_i \le 1$ $(1 \le i \le r)$. Then
$$(h_i,\alpha) = 1 \le (e_i,\alpha) \ (1 \le i \le r \text{ and } \forall\alpha \in \mathbb{N}^*).$$
Therefore $f_\alpha(d) = d|f_\alpha(n)$ $(\forall\alpha \in \mathbb{N}^*)$, as required.

3. We let e denote the least common multiple of $e_1, e_2, \dots, e_r$. Then it is well known [1, Theorem 2.3] that f_α is e-periodic function as a function of α, in other words $f_{\alpha+e}(n) = f_\alpha(n)$, for all α. It follows by taking $\alpha = ke$, $(k \in \mathbb{N}^*)$ that:
$$(h_i, \alpha) \leq h_i \leq e_i = (e_i, \alpha) \ (1 \leq i \leq r),$$
from which the validity of the last property follows. The proof of the theorem is complete.

A sequence of positive integers $\mathfrak{a} = (a_n)_{n\geq 1}$ is said to be a divisibility sequence if it satisfies, for all $n, m \geq 1$, the property:
$$n|m \Rightarrow a_n|a_m.$$
It is said to be a strong divisibility sequence if it satisfies, for $n, m \geq 1$, the stronger property:
$$(a_m, a_n) = a_{(m,n)}.$$
For any positive integer α we define $\mathfrak{f}_\alpha$ to be the sequence of integer such that $\mathfrak{f}_\alpha = \{f_\alpha(n); n \in \mathbb{N}^*\}$. For examples:
$$\mathfrak{f}_1 = \{1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 2, 17, 6, 19, 10, \dots\},$$
which is is the sequence A007947 in the *On-Line Encyclopedia of Integer Sequences* (OEIS, see [7]).
$$\mathfrak{f}_2 = \{1, 2, 3, 4, 5, 6, 7, 2, 9, 10, 11, 12, 13, 14, 15, 4, 17, 18, 19, 20, \dots\},$$
which is the sequence A066990 in the (OEIS).
$$\mathfrak{f}_3 = \{1, 2, 3, 2, 5, 6, 7, 8, 3, 10, 11, 6, 13, 14, 15, 2, 17, 6, 19, 10, \dots\},$$
which is the sequence A331737 in the (OEIS).
$$\mathfrak{f}_4 = \{1, 2, 3, 4, 5, 6, 7, 2, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, \dots\}.$$

Theorem 2.5 Let $\mathfrak{F} := \{\mathfrak{f}_\alpha, \ \alpha \in \mathbb{N}^*\}$. Then $\mathfrak{f}_1$ is the only strong divisibility sequence in $\mathfrak{F}$.

Proof. It easy to show that $\mathfrak{f}_1$ is a strong divisibility sequence, since
$$(f_1(m), f_1(n)) = (m, n) = f_1\big((m, n)\big),$$
for all $m, n \in \mathbb{N}^*$. Now we assume that $\alpha \geq 2$ and we wish to prove that $\mathfrak{f}_\alpha$ is not a strong divisibility sequence. To do so, it suffices to prove that $\mathfrak{f}_\alpha$ is not a divisibility sequence, since every strong divisibility sequence is divisibility sequence. For a prime number p we have $p|p^{\alpha+1}$ but
$$f_\alpha(p^\alpha) = p^\alpha \nmid p = f_\alpha(p^{\alpha+1}),$$
which means that $\mathfrak{f}_\alpha$ is not a divisibility sequence. The proof is finished.

Definition 2.6 Let $A \subset \mathbb{N}$ be an infinite set. Then $\mathfrak{a}$ is said to be a A-strong divisibility sequence if it satisfies

$$(a_m, a_n) = a_{(m,n)} \text{ whenever } m, n \in A.$$

Theorem 2.7 Let $\alpha \geq 2$ be an integer. Then

1. $\mathfrak{f}_\alpha$ is a $\mathfrak{f}_1$-strong divisibility sequence.
2. $\mathfrak{f}_\alpha$ is a $\mathbb{P}$-strong divisibility sequence, where $\mathbb{P}$ is the set of all primes.

Proof.

1. The first item follows at once from [5, Theorem 2.1] which states that the square-free positive integers are the only integers satisfying $f_\alpha(n) = n$ for all positive integers α, so
$$f_\alpha((m,n)) = (m,n) = \big(f_\alpha(m), f_\alpha(n)\big) \ (\forall m, n \in \mathfrak{f}_1 \text{ and } \forall \alpha \geq 2).$$
2. The second item follows from the following elementary property:
$$(m,n) = 1 \Rightarrow \big(f_\alpha(m), f_\alpha(n)\big) = 1.$$
This completes the proof of the theorem.

Remark 2.8 The second item of Theorem 2.7 remains true even when we replace $\mathbb{P}$ with any infinite subset of $\mathbb{N}$ which elements are pairwise co-prime.

We close the paper with the following theorem which proves existing, under some conditions on m and n, of a positive integer α such that $\big(f_\alpha(m), f_\alpha(n)\big) \neq f_\alpha((m,n))$.

Theorem 2.9 Let m and n be positive integer such that $p^r \parallel m$ and $p^s \parallel n$ for a prime p and positive integers r and s with $2 \leq r < s$ and $r \nmid s$. Then

$$\big(f_r(m), f_r(n)\big) \neq f_r((m,n))$$

Proof. On one hand

$$\begin{cases} p^r \parallel m \\ p^s \parallel n \end{cases} \Rightarrow \begin{cases} p^{(r,r)} = p^r \parallel f_r(m) \\ p^{(s,r)} = p^{sr/[s,r]} \parallel f_r(n) \end{cases}$$

$$\Rightarrow p^{\min(r, sr/[s,r])} \parallel \big(f_r(m), f_r(n)\big).$$

We have $s < [s,r]$, since $r \nmid s$, so $\frac{sr}{[s,r]} < r$. Thus

$$p^{sr/[s,r]} \parallel \big(f_r(m), f_r(n)\big). \tag{2.4}$$

On the other hand,

$$p^r \parallel m \Rightarrow p^r \parallel (m,n) \ \ (\text{since } r < s)$$

$$\Rightarrow p^r = p^{(r,r)} \parallel f_r((m,n)). \tag{2.5}$$

Consequently, (2.4) and (2.5) show that $\big(f_r(m), f_r(n)\big) \neq f_r((m,n))$, which concludes this proof.

3. CONCLUSION

In this paper, we give some new interesting properties of the arithmetic functions f_α. For example we prove (Theorem 2.3) that the sequence $f_\alpha(n)$ $(n \in \mathbb{N}^*)$ can not be monotonically increasing or decreasing for all α. Also, we show (Theorem 2.5) that the sequence $\mathfrak{f}_1$ (see A007947 in the OEIS) is a strong divisibility sequence. On the other hand, the purpose of our next papers will be the study their relationship with some other well-studied arithmetic functions. Also, we will try to find asymptotic formulas for the Dirichlet series associated with f_α and other sums related to them.

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