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BIGAUSSIAN PELL AND PELL-LUCAS POLYNOMIALS

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Summary. In this paper, we define biGaussian Pell and Pell-Lucas Polynomials. We give Binet‘s formulas, generating functions, Catalan’s identities, Cassini’s identities for these polynomials. Matrix presentations of biGaussian Pell and Pell-Lucas polynomials are found. Also, NegabiGaussian Pell and Pell-Lucas Polynomials are defined. Finally, we give some properties for these polynomials.

1 INTRODUCTION

Number sequences have a very important place in the world of science. Mankind has enabled the definition and development of new number systems in line with some needs over time. It appears not only in mathematics but also in nature, electromagnetic waves, quantum mechanics, series analysis, stock market, aesthetics and many other fields.

The most well-known of these number sequences is Fibonacci sequences which first appeared in Leonardo Fibonacci's book with a rabbit problem. This sequence, which is obtained by adding the two numbers before it; It continues as 1,1,2,3,5,8,13,21,34,55,89,144, 233, 377,… Fibonacci sequences appear in many branches of mathematics. These include calculus, group theory, applied mathematics, linear algebra, etc. [2, 3, 7, 8, 11, 12]

Many sequences have been defined by generalizing these number sequences. In addition to these numbers, appeared similar number sequences as Lucas numbers, Pell numbers and Jacobsthal numbers. [1, 14, 20, 23]

Koshy gave important relations about Fibonacci numbers, Lucas numbers and Pell numbers. Also, he gave the relations among these numbers. [9, 10]

Polynomials of sequences of numbers have also been studied by scientists. [6, 13, 15-17, 19, 22, 24]

Horadam, and Mahon [6] defined Pell and Pell - Lucas polynomials and gave their properties. Yağmur [25] defined the Gaussian Pell-Lucas polynomials and gave some properties of these polynomials. Özkan and Taştan [18] described Gauss Fibonacci and Gauss Pell polynomials and show that there is a relation between Gauss Fibonacci and Gauss Lucas polynomials. Halıcı and Öz [5] defined the Gaussian Pell and Pell-Lucas sequences and obtained some important identities involving the Gaussian Pell and Pell-Lucas numbers. Saba and Baussayoud [21] gave the new generating functions for the products of (p, q)-modified Pell numbers with Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials, Gaussian Pell and Gaussian Pell Lucas polynomials.

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One of the latest works in this area is [4] where Gökbaş described a new type of Pell and Pell-Lucas numbers which are called biGaussian Pell and Pell-Lucas numbers.

In this work, we will introduce biGaussian Pell and Pell-Lucas Polynomials. In the following sections, biGaussian Pell and Pell-Lucas Polynomials will be defined. Some identities will be given for biGaussian Pell and Pell-Lucas Polynomials such as Binet‘s formulas, generating functions, Catalan's identities, Cassini's identities. Matrix presentations of biGaussian Pell and Pell-Lucas polynomials will be given. Also, NegabiGaussian Pell and Pell-Lucas Polynomials will be defined.

2 DEFINITIONS

2.1 Definition

The Pell Numbers P_n are definedby

$$P_n = 2P_{n-1} + P_{n-2}, n \geq 2$$

with $P_0 = 0$ and $P_1 = 1$.

2.2 Definition

The Pell-Lucas Numbers $Q_n$are givenby

$$Q_n = 2Q_{n-1} + Q_{n-2}, n \geq 2$$

with $Q_0 = 2$ and $Q_1 = 2$.

2.3 Definition

The Pell Polynomials$P_n(x)$are described by

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), n \geq 2$$

with $P_0(x) = 0$ and $P_1(x) = 1$.

2.4 Definition

The Pell-Lucas Polynomials $Q_n(x)$ are described by

$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x), n \geq 2$$

with $Q_0(x) = 2$ and $Q_1(x) = 2x$.

2.5 Definition

BiGaussian Pell Numbers BGP_n [4] are definedas follows

$$BGP_n = \{P_n + iP_{n-1} + jP_{n-2} + ijP_{n-3} |\, P_n, nth\ Pell\ number\}$$

$$BGP_n = 2BGP_{n-1} + BGP_{n-2}$$

where i and j satisfy the conditions $i^2 = -1,\ j^2 = -1,\ ij = ji$.

2.6 Definition

BiGaussian Pell-Lucas Numbers BGQ_n [4] are defined as follows

$$BGQ_n = \{Q_n + iQ_{n-1} + jQ_{n-2} + ijQ_{n-3} |\ Q_n, nth\ Pell - Lucas\ number\}$$

$$BGQ_n = 2BGQ_{n-1} + BGQ_{n-2}.$$

3MAIN RESULTS

3.1 Definition

BiGaussian Pell Polynomials $BGP_n(x)$ are defined as follows

$$BGP_n(x) = \{P_n(x) + iP_{n-1}(x) + jP_{n-2}(x) + ijP_{n-3}(x) |\ P_n(x), nth\ Pell\ polynomial\}$$

$$BGP_n(x) = 2xBGP_{n-1}(x) + BGP_{n-2}(x).$$

3.2 Definition

BiGaussian Pell-Lucas Polynomials $BGQ_n(x)$ are defined as follows:

$$BGQ_n(x) = \{Q_n(x) + iQ_{n-1}(x) + jQ_{n-2}(x) + ijQ_{n-3}(x) |\ Q_n(x), nth\ Pell\ polynomial\}$$

$$BGQ_n(x) = 2xBGQ_{n-1}(x) + BGQ_{n-2}(x).$$

3.3 Theorem

Binet formula for biGaussian Pell polynomials is given by

$$BGP_n(x) = \frac{\hat{\alpha}\alpha^{n-3}(x) - \hat{\beta}\beta^{n-3}(x)}{\alpha(x) - \beta(x)}$$

where $\hat{\alpha} = \alpha^3 + i\alpha^2 + j\alpha + ij$, $\alpha = x + \sqrt{x^2+1}$ and $\hat{\beta} = \beta^3 + i\beta^2 + j\beta + ij$, $\beta = x - \sqrt{x^2+1}$.

Proof.

$$BGP_n(x) = P_n(x) + iP_{n-1}(x) + jP_{n-2}(x) + ijP_{n-3}(x)$$

$$= \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + i\frac{\alpha^{n-1}(x) - \beta^{n-1}(x)}{\alpha(x) - \beta(x)} + j\frac{\alpha^{n-2}(x) - \beta^{n-2}(x)}{\alpha(x) - \beta(x)} + ij\frac{\alpha^{n-3}(x) - \beta^{n-3}(x)}{\alpha(x) - \beta(x)}$$

$$= \frac{\alpha^{n-3}(\alpha^3 + i\alpha^2 + j\alpha + ij) - \beta^{n-3}(\beta^3 + i\beta^2 + j\beta + ij)}{\alpha(x) - \beta(x)} = \frac{\hat{\alpha}\alpha^{n-3}(x) - \hat{\beta}\beta^{n-3}(x)}{\alpha(x) - \beta(x)}.$$

In addition, we can indicate a matrix generator for the biGaussian Pell polynomials:

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} BGP_2(x) & BGP_1(x) \\ BGP_1(x) & BGP_0(x) \end{bmatrix} = \begin{bmatrix} BGP_{n+2}(x) & BGP_{n+1}(x) \\ BGP_{n+1}(x) & BGP_n(x) \end{bmatrix}$$

for $n \geq 1$.

We give the first 5 terms of biGaussian Pell polynomials in Table 1.

	$BGP_n(x)$
$n=0$	$i-2xj+(4x^2+1)ij$
$n=1$	$1+j-2xij$
$n=2$	$2x+i+ij$
$n=3$	$4x^2+1+2xi+j$
$n=4$	$8x^3+4x+(4x^2+1)i+2xj+ij$

Table 1. Some terms of biGaussian Pell polynomials.

3.4 Theorem

Binet's formula for biGaussian Pell-Lucas polynomials is given by

$$BGQ_n(x)=\hat{\alpha}\alpha^{n-3}(x)+\hat{\beta}\beta^{n-3}(x).$$

Proof.

Binet's formula for biGaussian Pell-Lucas polynomials is obtained like the proof of Theorem 3.3.

Also, we can indicate a matrix generator for the biGaussian Pell-Lucas polynomials:

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} BGQ_2(x) & BGQ_1(x) \\ BGQ_1(x) & BGQ_0(x) \end{bmatrix} = \begin{bmatrix} BGQ_{n+2}(x) & BGQ_{n+1}(x) \\ BGQ_{n+1}(x) & BGQ_n(x) \end{bmatrix}$$

for $n \geq 1$.

We give some terms of biGaussian Pell-Lucas polynomials in Table 2.

	$BGQ_n(x)$
$n=0$	$2-2xi+(4x^2+2)j-(8x^3+6x)ij$
$n=1$	$2x+2i-2xj+(4x^2+2)ij$
$n=2$	$4x^2+2+2xi+2j-2xij$
$n=3$	$8x^3+6x+(4x^2+2)i+2xj+2ij$
$n=4$	$16x^4+16x^2+2+(8x^3+6x)i+(4x^2+2)j+2xij$
$n=5$	$32x^5+40x^3+10x+(16x^4+16x^2+2)i+(8x^3+6x)j+(4x^2+2)ij$

Table 2. Some terms of biGaussian Pell-Lucas polynomials.

3.5 Theorem

Catalan's identities for biGaussian Pell polynomials $BGP_n(x)$ and biGaussian Pell-Lucas polynomials $BGQ_n(x)$ are as follows, respectively.

i. $\left(BGP_n(x)\right)^2 - BGP_{n+r}(x)BGP_{n-r}(x) = 2(-1)^{n-r}[P_{r+2}(x) + P_{r-2}(x)]jP_r(x) + (-1)^{n-r}[P_{r-3}(x) - P_{r+3}(x) + P_{r+1}(x) - P_{r-1}(x)]ijP_r(x).$

ii. $\left(BGQ_n(x)\right)^2 - BGQ_{n+r}(x)BGQ_{n-r}(x) = 16(-1)^{n-r+1}[P_{r+2}(x) + P_{r-2}(x)]jP_r(x) + 8(-1)^{n-r}[P_{r+3}(x) - P_{r-3}(x) + P_{r-1}(x) - P_{r+1}(x)]ijP_r(x).$

Proof.

i. LHS= $\left(P_n(x) + iP_{n-1}(x) + jP_{n-2}(x) + ijP_{n-3}(x)\right)\left(P_n(x) + iP_{n-1}(x) + jP_{n-2}(x) + ijP_{n-3}(x)\right) - (P_{n+r}(x) + iP_{n+r-1}(x) + jP_{n+r-2}(x) + ijP_{n+r-3}(x))(P_{n-r}(x) + iP_{n-r-1}(x) + jP_{n-r-2}(x) + ijP_{n-r-3}(x))$

$$= 2(-1)^{n-r}[P_{r+2}(x) + P_{r-2}(x)]jP_r(x) + (-1)^{n-r}[P_{r-3}(x) - P_{r+3}(x) + P_{r+1}(x) - P_{r-1}(x)]ijP_r(x)$$

ii. LHS= $\left(Q_n(x) + iQ_{n-1}(x) + jQ_{n-2}(x) + ijQ_{n-3}(x)\right)\left(Q_n(x) + iQ_{n-1}(x) + jQ_{n-2}(x) + ijQ_{n-3}(x)\right) - (Q_{n+r}(x) + iQ_{n+r-1}(x) + jQ_{n+r-2}(x) + ijQ_{n+r-3}(x))(Q_{n-r}(x) + iQ_{n-r-1}(x) + jQ_{n-r-2}(x) + ijQ_{n-r-3}(x))$

$$= 16(-1)^{n-r+1}[P_{r+2}(x) + P_{r-2}(x)]jP_r(x) + 8(-1)^{n-r}[P_{r+3}(x) - P_{r-3}(x) + P_{r-1}(x) - P_{r+1}(x)]ijP_r(x)$$

3.6 Result

Cassini's identities for $BGP_n(x)$ and $BGQ_n(x)$ are as follows, respectively.

i. $BGP_{n+1}(x)BGP_{n-1}(x) - \left(BGP_n(x)\right)^2 = (-1)^n(8x^2 + 4)j - (-1)^n(8x^3 + 4x)ij$

ii. $BGQ_{n+1}(x)BGQ_{n-1}(x) - \left(BGQ_n(x)\right)^2 = (-1)^{n+1}(64x^2 + 32)j - (-1)^{n+1}(64x^3 + 32x)ij.$

Proof.
If it is taken $r = 1$ in the Catalan's identities, Cassini's identities are obtained.

3.7 Theorem

Generating functions for $BGP_n(x)$ and $BGQ_n(x)$ polynomials are as follows, respectively.

i. $h(t)=\frac{i-2xj+(4x^2+1)ij+[1-2i+(4x^2+1)j-(8x^3+4x)ij]t}{1-2xt-t^2}$

ii. $m(t)=\frac{2-2xi+(4x^2+2)j-(8x^3+6x)ij+[-2x+(4x^2+2)i-(8x^3+6x)j+(16x^4+16x^2+2)ij]t}{1-2xt-t^2}$.

Proof.

Let $h(t)$ be the generating function for biGaussian Pell polynomials as

$$h(t)=\sum_{n=0}^{\infty} BGP_n(x)\,t^n$$

Using $h(t)$, $2xth(t)$ and $t^2h(t)$, we get following equations

$$2xth(t)=\sum_{n=0}^{\infty} 2xBGP_n(x)\,t^{n-1},\ t^2h(t)=\sum_{n=0}^{\infty} BGP_n(x)\,t^{n+2}.$$

After needed calculations, the generating function for biGaussian Pell polynomials is obtained as

$$h(t)=\frac{BGP_0(x)+[BGP_1(x)-2xBGP_0(x)]t}{1-2xt-t^2}$$

$$h(t)=\frac{i-2xj+(4x^2+1)ij+[1-2i+(4x^2+1)j-(8x^3+4x)ij]t}{1-2xt-t^2}$$

Similarly, the generating function for biGaussian Pell-Lucas polynomials can be obtained.

3.8 Theorem

D'Ocagne's identities for $BGP_n(x)$ and $BGQ_n(x)$ polynomials are as follows, respectively.

i. $BGP_m(x)BGP_{n+1}(x)-BGP_{m+1}(x)BGP_n(x)=12(-1)^{n-1}iP_{m-n+2}(x)+6jP_{m-n}(x)[(-1)^{2m-n}+(-1)^n]+12(-1)^{n-1}ijP_{m-n+2}(x)$

ii. $BGQ_m(x)BGQ_{n+1}(x)-BGQ_{m+1}(x)BGQ_n(x)=12[(-1)^{2m-n}+(-1)^n]iP_{m-n}(x)-48j[(-1)^{2m-n}+(-1)^n]P_{m-n}(x)+96(-1)^n ijP_{m-n}(x)$.

Proof.

i. $BGP_m(x)BGP_{n+1}(x)-BGP_{m+1}(x)BGP_n(x)=\big(P_m(x)+iP_{m-1}(x)+jP_{m-2}(x)+ijP_{m-3}(x)\big)\big(P_{n+1}(x)+iP_n(x)+jP_{n-1}(x)+ijP_{n-2}(x)\big)-\big(P_{m+1}(x)+iP_m(x)+jP_{m-1}(x)+ijP_{m-2}(x)\big)(P_n(x)+iP_{n-1}(x)+jP_{n-2}(x)+ijP_{n-3}(x))$

$=12(-1)^{n-1}iP_{m-n+2}(x)+6jP_{m-n}(x)[(-1)^{2m-n}+(-1)^n]+12(-1)^{n-1}ijP_{m-n+2}(x)$

ii. $BGQ_m(x)BGQ_{n+1}(x) - BGQ_{m+1}(x)BGQ_n(x) = \big(Q_m(x) + iQ_{m-1}(x) + jQ_{m-2}(x) + ijQ_{m-3}(x)\big)\big(Q_{n+1}(x) + iQ_n(x) + jQ_{n-1}(x) + ijQ_{n-2}(x)\big) - \big(Q_{m+1}(x) + iQ_m(x) + jQ_{m-1}(x) + ijQ_{m-2}(x)\big)(Q_n(x) + iQ_{n-1}(x) + jQ_{n-2}(x) + ijQ_{n-3}(x))$

$$= 12[(-1)^{2m-n} + (-1)^n]iP_{m-n}(x) - 48j[(-1)^{2m-n} + (-1)^n]P_{m-n}(x) + 96(-1)^n ijP_{m-n}(x)$$

3.9 Definition

NegabiGaussian Pell$BGP_{-n}(x)$ and negabiGaussian Pell-Lucas Polynomials$BGQ_{-n}(x)$ are defined as follows, respectively.

$$BGP_{-n}(x) = \{P_{-n}(x) + iP_{-n-1}(x) + jP_{-n-2}(x) + ijP_{-n-3}(x) | P_{-n}(x), -nth\ Pell\ polynomial\}$$

$$BGQ_{-n}(x) = \{Q_{-n}(x) + iQ_{-n-1}(x) + jQ_{-n-2}(x) + ijQ_{-n-3}(x) |\ Q_{-n}(x), -nth\ Pell\ polynomial\}.$$

3.10 Theorem

The following relations are satisfied for $BGP_n(x)$ and $BGQ_n(x)$ polynomials

i. $2\big(xBGP_{n+1}(x) + BGP_n(x)\big) = BGQ_{n+1}(x)$

ii. $2\big(BGP_{n+1}(x) - xBGP_n(x)\big) = BGQ_n(x)$

iii. $BGP_{n+1}(x) + BGP_{n-1}(x) = BGQ_n(x)$

iv. $BGP_{n+1}(x) - BGP_{n-1}(x) = 2xBGP_n(x)$

v. $BGP_{n+2}(x) + BGP_{n-2}(x) = (4x^2 + 2)BGP_n(x)$

vi. $BGP_{n+2}(x) - BGP_{n-2}(x) = 2xBGQ_n(x)$

vii. $xBGQ_{n+1}(x) + BGQ_n(x) = (2x^2 + 2)BGP_{n+1}(x)$

viii. $BGQ_{n+1}(x) - xBGQ_n(x) = (2x^2 + 2)BGP_n(x)$

ix. $BGQ_{n+1}(x) + BGQ_{n-1}(x) = (4x^2 + 4)BGP_n(x)$

x. $BGQ_{n+1}(x) - BGQ_{n-1}(x) = 2xBGQ_n(x)$

xi. $BGQ_{n+2}(x) + BGQ_{n-2}(x) = (4x^2 + 2)BGQ_n(x)$

xii. $BGQ_{n+2}(x) - BGQ_{n-2}(x) = (8x^3 + 8x)BGP_n(x)$

Proof:

They can be easily proven using the necessary definitions.

4 CONCLUSIONS

In this paper, we defined biGaussian Pell and Pell-Lucas Polynomials. We gave some terms of biGaussian Pell and Pell-Lucas polynomials in a table. NegabiGaussian Pell and Pell-Lucas Polynomials are defined. We also gave the Binet‘s formula, generating function, Catalan’s identitiy, Casssini’s identity. Moreover, matrix presentations of biGaussian Pell and Pell-Lucas polynomials are found. Finally, we gave some properties of these polynomials.In the future, this topic can be applied to other number sequences. Also, their polynomials, some identities, and properties of the number sequences can be defined.

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