MATHEMATICA MONTISNIGRI
Vol XLIII (2018)

ENVELOPE SOLITARY WAVES AT A WATER–ICE INTERFACE: THE CASE OF POSITIVE INITIAL TENSION

A. T. IL'ICHEV

Steklov Mathematical Institute of the Russian Academy of Sciences
Gubkina Street 8, 119991 Moscow, Russia
e-mail: ilichev@mi.ras.ru

Summary. In this paper, we focus our attention on physical parameters of so-called envelope solitary waves beneath an ice cover. The form and propagation of waves in water basins under the ice cover are described by the two-dimensional Euler equations. The ice cover is modeled by an elastic Kirchhoff–Love plate and is assumed to be of considerable thickness so that the inertia of the plate is taken into account in the formulation of the model. The Euler equations involve the additional pressure from the plate that is freely floating at the surface of the fluid. We consider the self-focusing case, when envelope solitary waves exist, for which the envelope speed (group speed) is equal to the speed of filling (phase speed). The indicated families of envelope solitary waves are parameterized by the speed of the waves, and their existence is proved earlier for speeds lying in some neighborhood of the critical value corresponding to the quiescent state. The envelope solitary waves, in turn, bifurcate from the quiescent state and lie in some neighborhood of it. Analyzing the form of the envelope solitary wave and the critical parameters for it we determine characteristic values of length and speed of the wave for the initially pre-stressed ice cover. These physical parameters can be compared with possible observations detecting such waves in practice.

1 INTRODUCTION

The physical processes arising in plane parallel water flows are of great interest (see, e.g., [1]). In this paper we focus our attention on physical parameters of so-called envelope solitary waves beneath an ice cover. Both envelope solitary waves and dark solitons correspond to solutions of traveling wave type of the full two-dimensional (2D) Euler equations of an ideal incompressible fluid in the presence of the ice cover. The ice cover is usually modeled either by an elastic Kirchhoff–Love plate [2–4] or by a Cosserát shell [5], though a form of surface traveling wave patterns is qualitatively the same in these two cases. The description of envelope solitary waves either of small or of finite amplitudes and their properties (including stability analysis using numerical methods) can be found, for example, in [6–9].

The form and propagation of waves in such basins are described by the full 2D Euler equations. The Euler equations involve the additional pressure from the plate that is freely floating at the surface of the fluid. The ice cover is modeled by an elastic Kirchhoff–Love plate and is assumed to be of considerable thickness so that the inertia of the plate is taken into account in

2010 Mathematics Subject Classification: 76B15.
Key words and phrases: ice cover, surface waves, envelope solitary waves, centre manifold.

the formulation of the model. We consider the self-focusing case, when envelope solitary waves exist, for which the envelope speed (group speed) is equal to the speed of filling (phase speed). The indicated families of envelope solitary waves are parameterized by the speed of the waves, and their existence is proved earlier for speeds lying in some neighborhood of the critical value corresponding to the quiescent state. The envelope solitary waves, in turn, bifurcate from the quiescent state and lie in some neighborhood of it.

It is known [6] that for each value of ice thickness there exist some critical values of water depth and speed where envelope solitary waves cease to exist, and they are replaced by dark solitary waves. Hence, there exist a finite range of values of water depth (for fixed value of the ice cover thickness) where a family of envelope solitary waves locally exist (this family may be continued with respect to the wave amplitude [10]). Analyzing the form of the envelope solitary wave from this range and the critical parameters for it we determine characteristic values of length and speed of the wave. For each value of the ice cover thickness and the water depth we find the corresponding values of wave length, speed and frequency.

We first, give the brief formulation of the problem, then consider the methodology, results and give our conclusion and discussion.

2 FORMULATION

2.1 Model of the ice cover

The equation of the balance of forces acting from the plate to the fluid is reduced to the following form [4]:

$$p = p_0 + \frac{\sigma_0 h}{R_m} - \partial^2_{\mathrm{xx}} M + \rho_s h \frac{\partial^2 \eta}{\partial t^2}, \quad M = \frac{J}{R_m}, \quad J = \frac{Eh^3}{12(1-\nu^2)}. \tag{1}$$

The curvature of a middle surface which is identified with the plate after averaging procedure with respect to plate thickness is given by [2] as

$$\frac{1}{R_m} = -\frac{\partial_{\mathrm{xx}}\eta}{(1+(\partial_x \eta)^2)^{3/2} - h\partial_{\mathrm{xx}}\eta/2}.$$

The notations for the dimensional physical values and fundamental constants are given in table 1.

2.2 Equations of the fluid–ice system

We study plane potential motions of the ideal incompressible fluid of finite depth with a horizontal bottom.

Symbol	Value	Dimension
p	pressure in the fluid	$[\mathrm{ML^{-1}T^{-2}}]$
p_0	atmospheric pressure	$[\mathrm{ML^{-1}T^{-2}}]$
φ	velocity potential	$[\mathrm{L^2T^{-1}}]$
η	surface deviation of the ice–fluid interface	$[\mathrm{L}]$
R_m	radius of the middle surface of the plate	$[\mathrm{L}]$
x	horizontal coordinate	$[\mathrm{L}]$
z	vertical coordinate	$[\mathrm{L}]$
h	ice thickness	$[\mathrm{L}]$
H	fluid depth	$[\mathrm{L}]$
g	gravity acceleration	$[\mathrm{LT^{-2}}]$
k	wave number	$[\mathrm{L^{-1}}]$
ω	frequency	$[\mathrm{T^{-1}}]$
σ_0	pre-stress in the plate	$[\mathrm{ML^{-1}T^{-2}}]$
E	Young module	$[\mathrm{ML^{-1}T^{-2}}]$
ν	Poisson coefficient	$[\ ]$
ρ	fluid density	$[\mathrm{ML^{-3}}]$
ρ_s	ice density	$[\mathrm{ML^{-3}}]$

Table 1: Notations of the dimensional physical values and fundamental constants.

The fluid occupies the domain

$$D = \{x \in \mathbb{R};\ 0 < z < H + \eta(x)\},$$

with the boundary

$$\partial D = \partial D^+ \cup \partial D^- = \{x \in \mathbb{R}; z = H + \eta(x) \cup z = 0\}.$$

The interface between the fluid and ice is given by the equation $z = H + \eta(x)$, $x \in \mathbb{R}$.

From (1) it follows that the Euler equations of the ideal incompressible fluid of finite depth with a horizontal bottom in presence of the mentioned surface effects has the form [4]

$$\begin{gathered}
\varphi_{\mathrm{xx}} + \varphi_{\mathrm{zz}} = 0, \quad (x,z) \in D, \\
\varphi_z = 0, \quad (x,z) \in \partial D^-, \\
\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_z^2) + g\eta - \hat{b}\hat{\kappa}_1 + \frac{J}{\rho}\hat{\kappa}_2 + \hat{c}\eta_{\mathrm{tt}} = 0, \quad (x,z) \in \partial D^+, \\
\eta_t + \eta_x\varphi_x = \varphi_z, \quad (x,z) \in \partial D^+,
\end{gathered} \tag{2}$$

where

$$\hat{b} = \frac{h\sigma_0}{\rho}, \quad \hat{c} = \rho_s h/\rho, \quad \hat{\kappa}_1 = \frac{\eta_{\rm xx}}{(1+\eta_x^2)^{3/2} - h\eta_{\rm xx}}, \quad \hat{\kappa}_2 = \partial_{\rm xx}^2 \frac{\eta_{\rm xx}}{(1+\eta_x^2)^{3/2} - h\eta_{\rm xx}}.$$

The letter subscripts denote differentiation with respect to corresponding variables.

3 TRAVELING WAVES

Consider a traveling wave which propagates to the left with the speed V along the x-axis. In the coordinates moving with the speed V the components of the velocity vector of particles $\mathbf{v} = (u,v)^\top$ satisfy the following asymptotic conditions $u \to V$, $v \to 0$, $x \to \infty$.

Make, further, the following scaling transformations:

$$(x,z) \to \left(\frac{x}{H}, \frac{z}{H}\right), \quad \eta \to \frac{\eta}{H}, \quad \mathbf{v} \to \frac{\mathbf{v}}{V}.$$

For these transformations $D \to \Omega$, $\partial D^\pm \to \partial\Omega^\pm$,

$$\Omega = \{x \in \mathbb{R};\ 0 < z < 1+\eta(x)\}, \quad \partial\Omega = \partial\Omega^+ \cup \partial\Omega^- = \{x \in \mathbb{R}; z = 1+\eta(x) \cup z = 0\}.$$

In new dimensionless variables (for them we use the previous notations), the Euler equation (2.2) for traveling waves have the form

$$\begin{aligned}
\operatorname{rot}\mathbf{v} = \mathbf{0}, \quad \operatorname{div}\mathbf{v} = 0, \quad (x,z) \in \Omega;\\
\frac{1}{2}|\mathbf{v}|^2 + \lambda\eta - b\kappa_1 + \gamma\kappa_2 + c\partial_{\rm xx}\eta = \text{const}, \quad (x,z) \in \partial\Omega^+;\\
\partial_x\eta\, u - v = 0, \quad (x,z) \in \partial\Omega^+;\\
v = 0, \qquad (x,z) \in \partial\Omega^-.
\end{aligned} \tag{3}$$

The constants λ, b, γ, c and κ are determined by

$$\lambda = gH/V^2, \quad b = \frac{\hat{b}}{HV^2} - c, \quad \gamma = \frac{J}{\rho V^2 H^3}, \quad c = \frac{\hat{c}}{H}, \quad \kappa = \frac{h}{H}.$$

The functions κ_j, $j = 1,2$, are given by

$$\kappa_1 = \frac{\partial_{\rm xx}\eta}{(1+(\partial_x\eta)^2)^{3/2} - \kappa\partial_{\rm xx}\eta}, \quad \kappa_2 = \partial_{\rm xx}^2 \frac{\partial_{\rm xx}\eta}{(1+(\partial_x\eta)^2)^{3/2} - \kappa\partial_{\rm xx}\eta}.$$

It can be shown that equations (3) can be written locally in the operator form

$$\partial_x \mathbf{u} = \mathscr{A}(\boldsymbol{\lambda})\mathbf{u} + \mathbf{G}(\boldsymbol{\lambda}, c, \mathbf{u}), \tag{4}$$

where $\boldsymbol{\lambda} = (\lambda, b, \gamma)^\top$, vector function $\mathbf{u}$ is the unknown, $\mathscr{A}(\boldsymbol{\lambda})$ is the linear operator acting in certain functional spaces, $\mathbf{G}(\boldsymbol{\lambda}, c, \mathbf{u})$ is the nonlinearity, μ is a small parameter [4]. It can be also shown that due to properties of the operator $\mathscr{A}$ in the neighborhood of the the surface in the parameter (λ, b, γ) space, parameterized by $\{q, \gamma_0\}$, where q is the dimensionless wave number $q = kH$ (wave length $l = 2\pi/k$), as follows

$$\begin{aligned} \lambda_0 &= \gamma_0 q^4 + \frac{q \coth q}{2} + \frac{q^2 \sinh^{-2} q}{2}, \\ b_0 &= -2\gamma_0 q^2 + \frac{\coth q}{2q} - \frac{\mathrm{sinhh}^{-2} q}{2}. \end{aligned} \tag{5}$$

It can be shown [4] that (4) can be rewritten as

$$\dot{\mathbf{u}} = \mathscr{A}\mathbf{u} + \mathbf{F}(\mu, \mathbf{u}), \quad \mathscr{A} = \mathscr{A}(\boldsymbol{\lambda}_0), \quad \mathbf{F}(0, \mathbf{0}) = 0, \quad \partial_{\mathbf{u}} \mathbf{F}(0, \mathbf{0}) = 0. \tag{6}$$

4 CENTRE MANIFOLD REDUCTION AND NORMAL FORM APPROXIMATION

In the case in question the following results are valid (see [4] and references therein).

Theorem 1 (about centre manifold, [11]). *The manifold*

$$M_\mu = \big\{ \big(\mathbf{u}_0, \mathbf{h}(\mu, \mathbf{u}_0)\big) \in X, \quad ||\mathbf{u}_0|| < \varepsilon \big\}, \quad \varepsilon \ll 1, \quad \mathbf{u}_0 \in X_0,$$

($||\cdot||$*denotes a norm in* X) *is the invariant manifold of the dynamic system* (6), *contains all small bounded solutions of this system and is called the centre manifold. The dimension of the space* X_0 *is finite, that is equivalent to finiteness of number of the imaginary eigenvalues (with their multiplicity) of the operator* $\mathscr{A}$. *The system of equations* (6) *after projection on the space* X_0 *and its complementary one in* X *(denoted further as* X_1*) takes the form*

$$\begin{aligned} \dot{\mathbf{u}}_0 &= \mathscr{A}_0 \mathbf{u}_0 + \mathbf{F}_0(\mu, \mathbf{u}_0 + \mathbf{u}_1), \\ \dot{\mathbf{u}}_1 &= \mathscr{A}_1 \mathbf{u}_1 + \mathbf{F}_1(\mu, \mathbf{u}_0 + \mathbf{u}_1), \end{aligned} \tag{7}$$

where

$$\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1)^\top \in X = X_0 \times X_1, \quad \mathscr{A}_0 = \mathscr{A}|_{X_0}, \quad \mathscr{A}_1 = \mathscr{A}|_{X_1 \cap D(\mathscr{A})},$$

and $\mathbf{F}_j,\ j=0,1$ *are the projections of* $\mathbf{F}$ *on* X_0 *and* X_1*, correspondingly. Besides,*

• $\mathbf{h}(0,\mathbf{0}) = \partial_{\mathbf{u}_0}\mathbf{h}(0,\mathbf{0}) = \mathbf{0}$;

• *if* $\mathscr{R} = \mathscr{R}_0 \oplus \mathscr{R}_1$, $\mathscr{R}_0 : X_0 \to X_0$, $\mathscr{R}_1 : X_1 \to X_1$ *are linear isometries, such that* $\mathbf{F}_j(\mu, \mathscr{R}_0\mathbf{u}_0, \mathscr{R}_1\mathbf{u}_1) = -\mathscr{R}_j\mathbf{F}_j(\mu,\mathbf{u}_0,\mathbf{u}_1)$, $\mathscr{A}_j\mathscr{R}_j = -\mathscr{R}_j\mathscr{A}_j$, $j=0,1$, *then* $\mathbf{h}(\mu,\mathscr{R}_0\mathbf{u}_0) = \mathscr{R}_1\mathbf{h}(\mu,\mathbf{u}_0)$.

The assertions of Theorem 1 mean that until the inequality $||\mathbf{u}_0|| < \varepsilon$ is valid, the solution $\mathbf{u} = (\mathbf{u}_0,\mathbf{u}_1)^\top$ of (4) belongs to M_μ, i.e., $\mathbf{u}_1 = \mathbf{h}(\mu,\mathbf{u}_0)$. Consequently, the set of all small bounded solutions obeys the finite dimensional dynamic system of equations:

$$\dot{\mathbf{u}}_0 = \mathscr{A}_0\mathbf{u}_0 + \mathbf{f}_0(\mu,\mathbf{u}_0), \quad \mathbf{f}_0(\mu,\mathbf{u}_0) = \mathbf{F}_0(\mu,\mathbf{u}_0 + \mathbf{h}(\mu,\mathbf{u}_0)). \tag{8}$$

The equations (8) are called the reduced equations.

It can be shown, that for the case in question $\mathbf{u}_0 = \left(A,B,A^*,B^*\right)^\top$ [12], and the following theorem is valid.

Theorem 2 [13]. *The reduced equations* (8) *are approximated by the system in the normal form*

$$\partial_x A = iqA + B + iAR\Big(AA^*, \frac{i}{2}(AB^* - A^*B)\Big),$$
$$\partial_x B = iqB + AQ\Big(AA^*, \frac{i}{2}(AB^* - A^*B)\Big) + iBR\Big(AA^*, \frac{i}{2}(AB^* - A^*B)\Big)$$

up to arbitrary algebraic order with respect to μ*. Here* R *and* Q *are polynomials with real coefficients:*

$$R(\mu,u,K_0) = p_1\mu + p_2u + p_3K_0 + O\big((|\mu| + |u| + |K_0|)^2\big),$$
$$Q(\mu,u,K_0) = q_1\mu - q_2u + q_3K_0 + O\big((|\mu| + |u| + |K_0|)^2\big).$$

Under the action of the isometry $\mathscr{R}_0$*,* $A \to A^*$*,* $B \to -B^*$*. The system of equations* (4) *has two first integrals*

$$K_0 = \frac{i}{2}(AB^* - A^*B),\ \ H_0 = |B|^2 - S(\mu,|A|^2,K_0), \quad S = \int\limits_0^{|A|^2} Q(\mu,u,K_0)\,du,$$

and, consequently, appears to be the integrable one.

5 RESULTS

It can be shown with the help of (4) [12] that the reduced system (8) has solitary wave solutions with equal phase and group speeds; the interface deviation is given by

$$\eta = 1 \pm \frac{2\,\mathrm{tanhh}\, q}{q^2}\sqrt{\frac{2\mu q_1}{q_2}}\cosh^{-1}\sqrt{\mu q_1}x\cos qx + O(|\mu|^{3/2}),$$

where $q_1 > 0$, $q_2 > 0$ are some constants given in [4], $0 < \mu \ll 1$ is chosen via the equality $\lambda = \lambda_0 + \mu$. From the definition of dimensionless parameters λ, b and γ for $\mu = 0$ we have

$$\frac{\lambda_0}{\gamma_0} = \frac{H^4\rho g}{J}, \quad V = \sqrt[8]{g^3\frac{J}{\rho\lambda_0^3\gamma_0}}, \quad \sigma_0 = (b_0 + c)\frac{H}{h}\sqrt[4]{\frac{g^3 J\rho^3}{\gamma_0\lambda_0^3}}. \tag{9}$$

Further, it is more convenient for us to fix b_0 in (3) and consider γ_0 and λ_0 as functions of q:

$$\begin{aligned} \lambda_0 &= -\frac{b_0 q^2}{2} + \frac{3}{4}q\coth q + \frac{q^2\sinh^{-2} q}{4}, \\ \gamma_0 &= -\frac{b_0}{2q^2} + \frac{\coth q}{4q^3} - \frac{\sinh^{-2} q}{4q^2}. \end{aligned} \tag{10}$$

From (5) it follows that

$$\lambda_0 = 1 + \left(\frac{1}{6} - \frac{b_0}{2}\right)q^2 + O(q^3), \quad \gamma_0 = \left(\frac{1}{6} - \frac{b_0}{2}\right)q^{-2} - \frac{1}{45} + O(q^2), \quad q \to 0,$$

and

$$\lambda_0 \to -\frac{b_0}{2}q^2 + \frac{3}{4}q, \quad \gamma \to -\frac{b_0}{2q^2} + \frac{1}{4q^3}, \quad q \to \infty.$$

We here examine the case when $b_0 > 0$, i.e., when the ice plate is subjected to extension, though stable compression is also possible [14]. For small enough positive b_0 the parameters γ_0 and λ_0 are still positive. For example, from (9) it follows, that for $H = 100$ m, $h = 0.5$ m, ($E = 5\times 10^9$ N/m^2, $\nu = 0.3$, $g = 10$ m/s^2, $\rho = 1000$ kg/m^3), $b_0 = 1/20$ we have $q = kH \approx 8.7$, wave length $l \approx 72$ m, the wave speed $V \approx 12.4$ m s^{-1} and the initial pre-stress in the ice cover $\sigma_0 \approx 1.3\times 10^4$ N m^{-2}.

6 CONCLUSIONS

The result of the present research is an attempt to make the correspondence between theoretical predictions and results of measurements of waves characteristics beneath the ice, which are made, in particular, at the Institute of Arctic and Antarctic in Saint-Petersburg, Russia. For theoretical description we use the full 2D Euler equations. The Euler equations involve the additional pressure from the plate that is freely floating at the surface of the fluid. The ice cover is modeled by an elastic Kirchhoff–Love plate and is assumed to be of considerable thickness so that the inertia of the plate is taken into account in the formulation of the model. We consider the self-focusing case, when envelope solitary waves exist, for which the envelope speed (group speed) is equal to the speed of filling (phase speed). The self-focusing takes place for $q \in I$ (the lower boundary of I is zero, and the upper boundary is q_c depending on h and H). When $q > q_c$ the envelope solitary wave is replaced by the dark soliton, which is the indicator of modulation stability [6]. Considering the case $q \in I$ and analyzing envelope solitary waves for different values of parameters, according to (9), (5) we are able to find theoretical values for corresponding wave length, wave speed and initial extension force in the ice cover.

Calculations show that even for large pre-stresses corresponding to the initial extension of the ice plate, the wave parameters weakly depend on b_0. This justifies the consideration of wave propagation beneath the ice cover without initial pre-stress as in [7, 15].

Acknowledgments: The work is supported by the Russian Science Foundation (grant No. 14-50-00005). The paper is based on the proceedings of the XXXIII International Conference on Equations of State for Matter, which was held in Elbrus and Tegenekli settlements, in the Kabardino-Balkar Republic of the Russian Federation, from March 1 to 6, 2018.

REFERENCES

[1] S. V. Polyakov, Yu. N. Karamzin, T. A. Kudryasov, and N. I. Tarasov, "Mathematical modelling of water treatment processes", *Mathematica Montisnigri* **40**, 110–126 (2017).

[2] L. K. Forbes, "Surface waves of large amplitude beneath an elastic sheet. Part 1. High-order series solution", *J. Fluid Mech.* **169**, 409–428 (1986).

[3] L. K. Forbes, "Surface waves of large amplitude beneath an elastic sheet. Part 2. Galerkin solution", *J. Fluid Mech.* **188**, 491–508 (1988).

[4] A. T. Il'ichev, "Soliton-like structures on a water–ice interface", *Russ. Math. Surveys* **70**(6), 1051–1103 (2015).

[5] P. I. Plotnikov and J. F. Toland, "Modelling nonlinear hydroelastic waves", *Philos. Trans. R. Soc., A* **369**(1947), 2942–2956 (2011).

[6] A. T. Il'ichev, "Envelope solitary waves and dark solitons at a water–ice interface", *Proc. Steklov Inst. Math.* **289**(1), 152–166 (2015).

[7] P. A. Milewski, J. M. Vanden-Broeck, and Z. Wang, "Hydroelastic solitary waves in deep water", *J. Fluid Mech.* **679**, 628–640 (2011).

[8] P. Guyenne and E. I. Părău, "Computations of fully non-linear hydroelastic solitary wave on deep water", *J. Fluid Mech.* **713**, 307–329 (2012).

[9] P. Guyenne and E. I. Părău, "Finite-depth effects on solitary waves in a floating ice sheet", *Journal of Fluids Structures* **49**, 242–262 (2012).

[10] J. M. Vanden-Broeck and E. I. Părău, "Two-dimensional generalized solitary waves and periodic waves under an ice sheet", *Philos. Trans. R. Soc., A* **369**, 2957–2972 (2011).

[11] A. Mielke, "Reduction of quasilinear elliptic equations in cylindrical domains with applications", *Math. Meth. Appl. Sci.* **10**, 51–66 (1988).

[12] G. Iooss and M. Adelmeyer, *Topics in Bifurcation Theory and Applications*, Singapore: World Scientific, (1992).
[13] G. Iooss and M. C. Perouéme, "Perturbed homoclinic solutions in reversible 1 : 1 resonance vector fields", *J. Diff. Eqns* **102**, 62–88 (1993).
[14] A. T. Il'ichev, "Solitary wave packets beneath a compressed ice cover", *Fluid Dynamics* **51**, 327–337 (2016).
[15] E. Părău and F. Dias, "Nonlinear effects in the response of a floating ice plate to a moving load", *J. Fluid Mech.* **460**, 281–305 (2002).

Received July 12, 2018