

## NORMAL FAMILIES, THEOREMS OF LIOUVILLE AND PICARD AND BLOCH PRINCIPLE

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**Summary.** In the scientific literature there are several proofs for the Liouville theorem and the small Picard theorem for analytic and meromorphic functions. In this paper, new proofs are presented for the Liouville theorem and the small Picard theorem. These are obtained as direct (simple) consequences of the Väisälä theorem for the constantness of continuous functions and the Montel theorems on the normality of a family of functions. The exhibited material is analyzed in relation to the of Bloch and Zalcman principle.

### 1 INTRODUCTION

The French mathematician Paul Montel, a member of the French Academy of Sciences (1937), defended his doctoral thesis titled “Sur les suites infinies de fonctions” (Université de Paris) in 1907, in which he studied a normal family of functions. This laid the foundations of a new field of mathematics, known as the Theory of Normal Families of Functions. The name of a normal family of functions appears in his work [12], published in 1911.

The French mathematician Henri Milloux said the following about the importance of the theory of normal family functions: “*La theorie des familles normales de fonctions doit etre consideree comme l'une des decouvertes les plus belles, et les plus importantes par sa fecondite, de cette premiere moitie du siecle*”. (The theory of normal families of functions should be regarded as one of the most beautiful discoveries, and most importantly its fruitfulness in the first half of the 20th century) [21].

The truthfulness of Milloux's words is reiterated by the fact that the theory of normal families of functions has found wide applications in many mathematical disciplines: in the classical theory of a functions of a complex variable (in the proof of Riemann's theorem on conformal mapping of a disk, the proof of Picard and Julia's theorem on the boundary behavior of holomorphic functions in their singular points, and so on, see [1, 11, 14, 15, 16, 20, 21, 22]), holomorphic dynamics (the dynamics of the Möbius map, the dynamics of Riemann surfaces, in particular, the dynamics of hyperbolic surfaces, see [2, 11, 15]), the theory of boundary sets (asymptotic limit values of functions, see, e.g., [8, 9, 16, 15]), functional analysis (compactness theorems in metric function spaces, Arzela-Ascoli theorem, see [1,20]), the theory of iterations with rational functions, and so on.

We particularly emphasize the application of the theory of normal families of holomorphic (meromorphic) functions in obtaining results relating to the rapidly developing domain of the theory of complex variable functions associated with the Bloch principle. The Bloch principle is contained in the heuristic assumption: “*One guiding principle in their study has been the heuristic principle which says that a family of functions meromorphic (or holomorphic) in a*

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*domain and possessing a certain property is likely to be normal if there is no nonconstant function meromorphic (holomorphic) in the plane which has this property”.*

Originally, Bloch formulated this principle in the following manner: “*Nihil est in infinito quod non prius fuerit in finito.*” (see [3, 4]).

It should be noted that the Bloch Principles are used as a roadmap for formulating some mathematical hypotheses that must be further justified. There are many counterexamples which prove that the Bloch Principle does not always yield a correct mathematical statement (other articles [19]).

The generalization of the Bloch Principle is given by Robinson-Zalcman’s “Heuristic Principle”, which represents its mathematical formalization (see: [3, 21, 24]).

We emphasize that in addition to the Bloch Principle, the inverse of the Bloch Principle, which says: “*If any family of meromorphic functions satisfying a (suitable) property  $P$  in an arbitrary domain is necessarily normal, then a function that is meromorphic in  $\mathbb{C}$  and possesses  $P$  reduces to a constant*” (see: [16]) is studied. There are counterexamples for the inverse Bloch Principle (see: [10, 21]).

A detailed overview of the results related to the Bloch principle is given in [3].

Bloch's principle and it's inverse are illustrated by Montel's theorem on the normality of a family of analytic functions and the Liouville's theorems, as well as Montel's theorems on the normality of a family of analytic (meromorphic) functions which omit the fixed values and the first (small) Picard's theorem.

The theorem of Liouville concerning the constancy of entire analytic functions and the first Picard's theorem on the values of analytic functions on the complex plane have a special and significant place in classical theory of functions of a complex variable.

Many proofs of these theorems can be found in literature, from “elementary”, which use some inequalities from the theory of analytic functions, the Cauchy integral formula, the expansion of an analytic function to a Taylor series, to those which use the elliptic modular function, or the theorems mentioned above can be obtained as consequences of theorems in “complex” mathematical theories (see: [1, 4, 6, 7, 11, 17, 20, 22]).

For example, the "elementary" proof of Picard's small theorem without the use of an elliptic modular function was given by Borel in 1896 (see [6]). The "elementary" proof of Picard's small theorem was also given by Bloch in 1926 (see [4]). Namely, Picard's theorem can be easily obtained from the Landau theorem or from the Bloch theorem (see [17], pp. 258-260, or [18], pp. 233-235).

Encouraged by the concept of Bloch's principle, in this article, from Montel's theorem about the normality of a family of analytic and meromorphic functions (see [13, 21]) and Theorem 2 of Väisälä [23], we will simply (directly) prove the Liouville theorems and the small Picard theorem.

This evidence differs from the obtaining of the Liouville theorem and the small Picard theorem from 4.1.6 in [21], which represents the mathematical formalization of the inverse Bloch principle, which is in fact obtained as a consequence of the Marti's theorem about the normal families of functions and conditions i) and ii) from Definitions 4.1.1 from [21].

It is interesting to highlight that Liouville's theorem was proved by O. Cauchy in 1844, and that in that same year, Liouville proved only a special case of that assertion. (see [22]).

## 2 PRELIMINARY NOTATIONS, DEFINITIONS AND RESULTS

We will denote the set of real numbers with  $\mathbb{R}$ , the set of complex numbers with  $\mathbb{C} = \{z | z = x + iy, x, y \in \mathbb{R}\}$ , and the Riemannian sphere with  $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . We will also set  $G(\mathbb{C}) = \{az + b | a, b \in \mathbb{C}, a \neq 0\}$  to be the Mobius automorphism group of the conformal complex plane  $\mathbb{C}$ .

The convergence on  $\mathbb{C}$  and  $\overline{\mathbb{C}}$  are given in Euclidean and spherical metrics. These metrics are equivalent on compact sets in  $\mathbb{C}$ .

A family  $\mathfrak{F} = \{f | f : O \rightarrow \overline{\mathbb{C}}\}$  of functions on a domain  $O \subset \overline{\mathbb{C}}$  is normal in  $O$  if every sequence of functions  $(f_n) \subset \mathfrak{F}$  contains either a subsequence which converges to a limit function  $f \neq \infty$  uniformly on each compact subset of  $O$ , or a subsequence which converges uniformly to  $\infty$  on each compact subset. A family  $\mathfrak{F}$  of functions is normal at a point  $z \in O$  if it is a normal family in some neighborhood of  $z$ . If  $\infty \in O$ , then the family functions  $\mathfrak{F}$  normal at the point  $\infty$  if the family  $\mathfrak{F}' = \{f(z^{-1}) | f \in \mathfrak{F}\}$  is a normal family in  $0$ . It is known that a family of functions  $\mathfrak{F}$  is normal in a domain  $O$  if and only if  $\mathfrak{F}$  is normal at each point of  $O$  (see ([13, 21])).

Here we formulate propositions on the normality of a family of functions that are necessary for the proof of the Liouville's theorem and the Picard's theorem.

**Theorem A** (theorem 2, [23] p. 17). *If  $f(z)$  is continuous function on  $\mathbb{C}$  and  $\mathfrak{F} = \{(f \circ g)(z) | g \in G(\mathbb{C})\}$  is a normal family on  $\mathbb{C}$ , then  $f(z)$  is constant.*

**Theorem B** (Montel 's Theorem, [12]; [16], p. 35). *If  $\mathfrak{F}$  is a locally bounded family of analytic functions on a domain  $O$ , then  $\mathfrak{F}$  is a normal family in  $O$ .*

**Theorem C** (Montel 's Theorem, [13], p. 58 ; Fundamental Normality Test [21], p. 54). *Let  $\mathfrak{F}$  be the family of analytic functions on a domain  $O$  which omit two fixed values  $a$  and  $b$  in  $\mathbb{C}$ . Then  $\mathfrak{F}$  is normal in  $O$ .*

**Theorem D** (Montel 's Theorem, [13], p. 108 ; Fundamental Normality Test [21], p. 74). *Let  $\mathfrak{F}$  be the family of meromorphic functions on a domain  $O$  which omit three distinct values  $a, b, c$  in  $\overline{\mathbb{C}}$ . Then  $\mathfrak{F}$  is normal in  $O$ .*

**Remark 1.** Using Schottky's theorem, which is obtained by using the Bloch theorem (see [17], p. 261), Theorem C can be obtained from Theorem B without the use of an elliptic modular function (see [21], p. 59).

**Remark 2.** Theorem D can be easily obtained from Theorem C (see [13], p. 106, or, p. [21], p. 74).

### 3 PROOF OF THEOREMS OF LIOUVILLE AND PICARD

**Theorem 1** (*Liouville's Theorem*). *A bounded function  $f$  which is defined and analytic everywhere on  $\mathbb{C}$  must be constant.*

*Proof.* From the boundedness of the analytic function  $f$  it follows that the  $\mathfrak{F}_1 = \{(f \circ g)(z) \mid g \in G(\mathbb{C})\}$  is a bounded family of analytic functions on  $\mathbb{C}$ . Then it follows from theorem B that  $\mathfrak{F}_1$  is a normal family of functions on  $\mathbb{C}$ . Now the assertion of the theorem follows directly from Theorem A.  $\square$

**Theorem 2** (small Picard's Theorem). *Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a non-constant analytic function. Then there exists at most one  $z \in \mathbb{C}$  which does not lie in the image of  $f$ .*

*Proof.* Suppose that Theorem 2 is false, that is, that there exists an analytic function  $f : \mathbb{C} \rightarrow \mathbb{C}$  which is not a constant, which takes all values except two or more fixed values from the set  $\mathbb{C}$ . Consider the family  $\mathfrak{F}_1 = \{(f \circ g)(z) \mid g \in G(\mathbb{C})\}$ . The functions of the family  $\mathfrak{F}_1$ , of course, do not take two fixed values in  $\mathbb{C}$ . Therefore, it follows from theorem C that the family is a normal family of analytic functions on  $\mathbb{C}$ . Then theorem A implies that  $f$  is a constant in  $\mathbb{C}$ , which contradicts our assumption. The above contradiction completes the proof of theorem 2.  $\square$

**Theorem 3** (small Picard's Theorem for meromorphic functions). *Let  $f : \mathbb{C} \rightarrow \overline{\mathbb{C}}$  be a non-constant meromorphic function. Then there exist at most two values with  $\overline{\mathbb{C}}$  not lying in the image of  $f$ .*

Theorem 3 is proved in the same way as theorem 2 with the distinction that theorem D is used instead of theorem C in the proof.

**Question.** Can the theorems of Liouville and Picard for harmonic functions be obtained be obtained in the same way?

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