MATHEMATICA MONTISNIGRI
Vol XLIV (2019)

$\varphi-$ BIPROJECTIVE AND (φ,ψ) - AMENABLE BANACH ALGEBRAS

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DOI: 10.20948/mathmon-2019-44-1

Summary. In this paper we introduce and study the concept of a $\varphi-$ biprojective and $(\varphi,\psi)-$amenable Banach algebra A, where φ is a continuous homomorphism on A and $\psi \in \Phi_A$. We show that A is $(\varphi,\psi)-$amenable if and only there exists a bounded net $(m_\alpha) \subset A$ such that $\|\varphi(a)m_\alpha - m_\alpha\varphi(a)\| \to 0$ and $\psi \circ \varphi(m_\alpha) = 1$ for all α.

1 INTRODUCTION

Amenable Banach algebra were introduced by Johnson in [10]. He showed that A is amenable Banach algebra if and only if A has a approximate diagonal, that is a bounded net (m_α) in $(A \hat{\otimes} A)$ such that $m_\alpha a - a m_\alpha \to 0$ and $\pi(m_\alpha)a \to a$ for every $a \in A$. The notion of a biflat and biprojective Banach algebra was introduced by Helemskii [8, 9]. Indeed, A is called biprojective, if here exists a bounded A-bimodule map $\theta : A \to A \hat{\otimes} A$ such that $\pi \circ \theta = id_A$.

He considered a Banach algebra A is amenable if A biflat and has a bounded approximate identity[7, 9]. In fact, A is called biflat if there exists a bounded A-bimodule map $\theta : (A \hat{\otimes} A)^* \to A^*$ such that $\theta \circ \pi^* = id_{A^*}$.

Given a continuous homomorphism φ from A into A, authers in [13, 14] defined and studied φ-derivations and φ-amenability.

Recall that a character on A is a non-zero homomorphism from A into the scalar field. The set of all characters on A the character space of A, is denoted by Φ_A.

Motivated by these considerations, author with M. Lashkarizadeh bami introduced some generalizations of Helemskiis concepts like φ-biflatness and φ-biprojectivity, where φ is a continuous homomorphism from A into A [5, 6]. The auther states that Banach algebra A is φ- biflat(φ- biprojective) if there exists a bounded A-bimodule map $\theta : A \to (A \hat{\otimes} A)^{**}$ ($\theta : A \to (A \hat{\otimes} A)$) such that $\pi^{**} \circ \theta \circ \varphi = \kappa$ is the canonical embedding of A into A^{**} ($\pi \circ \theta \circ \varphi = id_A$).

2010 Mathematics Subject Classification: 43A20, 46H25, 47B47.
Key words and Phrases: Banach algebra, φ -biprojective, φ -biflat.

In this paper, we define $(\varphi,\psi)-$amenability Banach algebra A, where φ is a continuous homomorphism on A and $\psi\in\Phi_A$. We show that A is $(\varphi,\psi)-$amenable if and only if there exists a bounded net $(m_\alpha)\subset A$ such that $\|\varphi(a)m_\alpha-m_\alpha\varphi(a)\|\to 0$ and $\psi\circ\varphi(m_\alpha)=1$ for all α.

2 THE RESULTS

Let A be a Banach algebra and X,Y be a Banach A-bimodules, then A-bimodule morphism from X to Y is a morphism $\varphi: X\to Y$ with

$$\varphi(a\cdot x)=a\cdot\varphi(x),\quad \varphi(x\cdot a)=\varphi(x)\cdot a\ \ (a\in A, x\in X)$$

In the next result $\varphi: A\to A$ is a homomorphism and I is a closed ideal of A. We define the map $\widetilde{\varphi}: A/I\to A/I$ by $\widetilde{\varphi}(a+I)=\varphi(a)+I$.

Theorem 2.1 *Suppose that A is a $\varphi-$ biprojective Banach algebra. If I is a closed ideal of A, then A/I is $\widetilde{\varphi}-$biprojective.*

Proof. Assume that $\theta: A\to(A\hat{\otimes}A)$ is a continuous A-bimodule map such that $\pi\circ\theta\circ\varphi=id_A$. Let $q: A\to A/I$ be the quotient map. Define the map $\widetilde{\theta}: A/I\to(A/I\hat{\otimes}A/I)$ by $a+I\mapsto(q\hat{\otimes}q)\circ\theta(a)\ (a\in A)$. We prove that $\widetilde{\theta}$ is an A/I - bimodul map. To do, take $a,b,c\in A$, then we have

$$\begin{aligned}\widetilde{\theta}((a+I)(b+I)(c+I))&=\widetilde{\theta}(abc+I)\\&=(\hat{q}\otimes q)\circ\theta(abc)\\&=(\hat{q}\otimes q)(a\cdot\theta(b)\cdot c)\\&=a\cdot(\hat{q}\otimes q)(\theta(b)\cdot c\\&=(a+I)\cdot\widetilde{\theta}(b+I)\cdot(c+I).\end{aligned}$$

and also we have

$$\begin{aligned}\pi_{A/I}\circ\widetilde{\theta}\circ\widetilde{\varphi}(a+I)&=\pi_{A/I}\circ\widetilde{\theta}(\varphi(a)+I)\\&=\pi_{A/I}\circ(\hat{q}\otimes q)\circ\theta(\varphi(a))\\&=q\circ\pi_A\circ\theta\circ\varphi(a)=q(a)=a+I.\end{aligned}$$

That is, A/I is $\widetilde{\varphi}-$biprojective.

Theorem 2.2 *Suppose that A is a $\varphi-$ biprojective Banach algebra. If I is a closed deal of A with one sided bounded approximate identity and $\varphi(I)\subset I$. Then I is $\varphi|_I-$ biprojective.*

Proof. Assume that $\theta : A \to (A \hat{\otimes} A)$ is a continuous A-bimodule map such that $\pi \circ \theta \circ \varphi = id_A$. Let $\iota : I \to A$ be the inclusion map. Then $\theta|_I = \theta \circ \iota : I \to (A \hat{\otimes} A)$ is I-bimodule homomorphisms. If I^3 denotes span $\{abc : a,b,c \in I\}^-$, then $I^3 = I$ because I has a one sided bounded approximate identity, and

$$\begin{aligned} \theta|_I &= \theta(I) \\ &= \theta_(I^3) \\ &\subseteq span\{a \cdot \theta(b) \cdot c\}^- \\ &\subseteq span\{a \cdot m \cdot c : a,c \in I, m \in \hat{A} \otimes A\}^- \subseteq \hat{I} \otimes I. \end{aligned}$$

Therefore for every $a \in I$,

$$\begin{aligned} \pi \circ \theta|_I \circ \varphi(a) &= \pi(\theta(\varphi(a))) \\ &= \varphi(a). \end{aligned}$$

Proposition 2.3 *Let A be a unital Banach algebra, and B be a Banach algebra cont-aining a non-zero idempotent b_0. If $A \hat{\otimes} B$ is $\varphi \otimes \psi-$ biprojective. Then A is $\varphi-$ bipr-ojective.*

Proof. There exists an $A \hat{\otimes} B$-bimodule $\theta : A \hat{\otimes} B \to (A \hat{\otimes} B) \hat{\otimes} (A \hat{\otimes} B)$ with $\pi_{A \hat{\otimes} B} \circ \theta \circ (\varphi \otimes \psi) = id_{A \hat{\otimes} B}$. We consider $A \hat{\otimes} B$ as an $A-$bimodule with the actions given by

$$a_1 \cdot (a_2 \otimes b) = a_1 a_2 \otimes b, \ and \ (a_2 \otimes b) \cdot a_1 = a_2 a_1 \otimes b \quad (a_1, a_2 \in A, b \in B)$$

Thus for every $a_1, a_2 \in A$ we have

$$\begin{aligned} \theta(a_1 a_2 \otimes b_0) &= \theta((a_1 \otimes b_0)(a_2 \otimes b_0)) \\ &= (a_1 \otimes b_0) \cdot \theta((a_2 \otimes b_0)) \\ &= a_1 \cdot (e_A \otimes b_0) \cdot \theta((a_2 \otimes b_0)) \\ &= a_1 \cdot \theta(a_2 \otimes b_0). \end{aligned}$$

Similarly we can show a right-module version of this equation. Hence we have

$$\theta(a_1 a_2 \otimes b_0) = a_1 \cdot \theta(a_2 \otimes b_0) = \theta(a_1 \otimes b_0) \cdot a_2 \quad (a_1, a_2 \in A)$$

Take $f \in \Phi_A$ with $f(b_0) = 1$ and define

$$\rho : (A \hat{\otimes} B) \hat{\otimes} (A \hat{\otimes} B) \to (A \hat{\otimes} A), \ (a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \mapsto f(b_1 b_2) a_1 \otimes a_2$$

Then ρ is an A-bimodule morphism.

We now define $\widetilde{\theta} : A \to (A \hat{\otimes} A)$ by

$$\widetilde{\theta}(a) = \rho \circ \theta(a \otimes \psi(b0) \quad (a \in A).$$

Then $\widetilde{\theta}$ is an A-bimodule morphism and

$$\pi_A \circ \rho = (id_A \otimes f) \circ \pi_{A \hat{\otimes} B}.$$

Therefore

$$\begin{aligned}\pi_A \circ \widetilde{\theta} \circ \varphi(a) &= \pi_A \circ \rho \circ \theta(\varphi(a) \otimes \psi(b0)) \\ &= (id_A \otimes f) \circ \pi_{\hat{A} \otimes B} \circ \theta(\varphi(a) \otimes \psi(b0)) \\ &= a.\end{aligned}$$

That is, A is $\varphi-$biprojective.

Proposition 2.4 *Let A be a unital Banach algebra, and B be a Banach algebra containing a non-zero idempotent b_0. If $A \hat{\otimes} B$ is $\varphi \otimes \psi-$ bifllat. Then A is $\varphi-$biflt.*

Proof. There exists an $\hat{A} \otimes B$-bimodule $\theta : A \hat{\otimes} B \to (A \hat{\otimes} B) \hat{\otimes} (A \hat{\otimes} B)^{**}$ with $\pi^{**}_{A \hat{\otimes} B} \circ \theta \circ (\varphi \otimes \psi) = \kappa_{A \hat{\otimes} B}$.

The following proof is similar to that of Proposition 2.3 We now define $\widetilde{\theta} : A \to (A \hat{\otimes} A)^{**}$ by

$$\widetilde{\theta}(a) = \rho^{**} \circ \theta(a \otimes \psi(b0)) \quad (a \in A).$$

Then $\widetilde{\theta}$ is an A-bimodule morphism and

$$\pi^{**}_A \circ \widetilde{\theta} \circ \varphi = \kappa_A.$$

That is, A is $\varphi-$biflat.

Suppose that A is a Banach algebra, and let Λ is a non-empty set. We denote by $M_\Lambda(A)$ the set of $\Lambda \times \Lambda$ matrices $(a_{ij})_{i,j \in \Lambda}$ with entries in A such that

$$\| (a_{ij}) \| = \sum_{i,j} \| a_{ij} \|_A < \infty.$$

$M_\Lambda(A)$ is a Banach algebra with matrix multiplication. The matrix units in $M_\Lambda(\mathbb{C})$ are denoted by $e_{i,j}$, so that

$$e_{i,j} e_{k,l} = \delta_{j,k} e_{i,l} \quad (i,j,k,l \in \Lambda),$$

where $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq K$. The map

$$\theta : M_\Lambda(A) \to (A \hat{\otimes} M_\Lambda(\mathbb{C})) \ by \ (a_{ij}) \mapsto \sum_{i,j} a_{ij} \otimes e_{i,j},$$

is an isometric algebra isomorphism.

Corollary 2.5 *Let A be a unital Banach algebra, and let Λ be a non-empty set and $\varphi=\varphi_0\otimes\varphi_1,\varphi_0\in Hom(A)$ and $\varphi_1\in Hom(M_\Lambda(\mathbb{C}))$. Then $M_\Lambda(A)$ $\varphi-$ biprojectiv ($\varphi-$ bflat) if and only if A is φ_0- biprojective($\varphi-$bflat).*

Proof. Let $M_\Lambda(A)$ is $\varphi-$ biprojectiv($\varphi-$bflat). Since $M_\Lambda(A)=A\hat{\otimes}M_\Lambda(\mathbb{C})$, the result follows from proposition 2.3(2.4).

Conversely, fix $k_0\in\Lambda$, and define $\theta: M_\Lambda(\mathbb{C})\to(M_\Lambda(\mathbb{C})\hat{\otimes}M_\Lambda(\mathbb{C}))$ by

$$\theta(a)=\sum_{i,j\in\Lambda}a_{ij}e_{i,k_0}\otimes e_{k_0,j}\quad (a=(a_{i,j})\in M_\Lambda(\mathbb{C})).$$

The sum converges since $\sum_{i,j}|a_{ij}|<\infty$. So that

$$\begin{aligned}\pi_{M_\Lambda(\mathbb{C})}\circ\theta\circ\varphi_1(a)&=\pi_{M_\Lambda(\mathbb{C})}\circ\sum_{i,j\in\Lambda}\varphi_1(a_{ij})e_{i,k_0}\otimes e_{k_0,j}\\&=\varphi_1(a_{ij})=\varphi_1(a).\end{aligned}$$

That is $M_\Lambda(\mathbb{C})$ is $\varphi-$biprojective. Therefore by [6,Proposition 2.12], $M_\Lambda(A)$ $\varphi-$ biprojectiv($\varphi-$bflat).

We quote the following result from [13].

Lemma 2.6 *Let A be a Banach algebra . Then there exists an A-bimodule homomorphism $\gamma:(A\hat{\otimes}A)^*\to(A^{**}\hat{\otimes}A^{**})^*$ such that for any functional $f\in(A\hat{\otimes}A)^*$, elements $\varphi,\psi\in A^{**}$ and nets $(a_\alpha),(b_\beta)$ in A with $w^*-lim_\alpha a_\alpha=\varphi$ and $w^*-lim_\beta b_\beta=\psi$ we have*

$$\gamma(f)(\varphi\otimes\psi)=lim_\alpha lim_\beta f(a_\alpha\otimes b_\beta).$$

Theorem 2.7 *Suppose that A is a Banach algebra and $\varphi\in Hom(A)$. If A^{**} is $\varphi^{**}-$biprojective. Then A is $\varphi-$biflat.*

Proof. Let $\kappa: A\to A^{**}$, $\kappa_1: A^*\to A^{***}$ and $\kappa_*: A^{**}\to A^{****}$ denote the natural inclusions, π (${}^{**}\pi$, respectively) the product maps on A (A^{**}, respectively) and let γ be defined as in Lemma 2.6. Then for each $a^*\in A^*$, elements $a_1^{**},a_2^{**}\in A^{**}$ and nets $(a_\alpha),(b_\beta)\subset A$ with $w^*-\lim_\alpha a_\alpha=a_1^{**},w^*-\lim_\beta b_\beta=a_2^{**}$, we have

$$\begin{aligned}(\gamma(\pi^*(a^*)))(a_1^{**}\otimes a_2^{**})&=\lim_\alpha\lim_\beta\pi^*(a^*)(a_\alpha\otimes b_\beta)\\&=\lim_\alpha\lim_\beta a^*(a_\alpha b_\beta)\end{aligned}$$

$$
\begin{aligned}
&= w^* - \lim_{\alpha} w^* - \lim_{\beta} \kappa(a_\alpha b_\beta)(a^*) \\
&= \kappa_1(a^*)(a_1^{**} a_2^{**}) \\
&= \kappa_1(a^*)(^{**}\pi(a_1^{**} \otimes a_2^{**})) \\
&= (^{**}\pi^*(\kappa_1(a^*)))(a_1^{**} \otimes a_2^{**}).
\end{aligned}
$$

Thus $\gamma \circ \pi^* = ^{**}\pi^* \circ \kappa_1$. Hence $\pi^{**} \circ \gamma^* = \kappa_1^* \circ ^{**}\pi^{**}$. Since A^{**} is $\varphi^{**}-$ biprojective, there is an A-bimodule map $\theta_0 : A^{**} \to (A^{**} \hat{\otimes} A^{**})$, such that $\pi \circ \theta_0 \circ \varphi^{**} = id_{A^{**}}$. Putting $\theta := \gamma^* \circ \theta_0 \circ \kappa$, then for each $a \in A$ we have

$$
\begin{aligned}
\pi^{**} \circ \theta \circ \varphi(a) &= \pi^{**} \circ \gamma^* \circ \theta_0 \circ \kappa \circ \varphi(a) \\
&= \kappa_1^* \circ \pi^{**} \circ \theta_0 \circ \kappa \circ \varphi(a) \\
&= \kappa_1^* \circ \pi^{**} \circ \theta_0 \circ \varphi^{**}(a) \\
&= \kappa_1^*(a) = \kappa(a).
\end{aligned}
$$

That is, A is $\varphi-$biflat.

Recall that A is $\varphi-$ approximate biflat if there is a net $\theta_\alpha : A \to (A \hat{\otimes} A)^{**}$ $(\alpha \in I)$ of bounded A-bimodule morphisms such that $\pi^{**} \circ \theta_\alpha \circ \varphi(a) \to \varphi(a)$.

Theorem 2.8 *Suppose that A is a $\varphi-$ approximate biflat Banach algebra with one sided bounded approximate identity. If I is a closed deal of A. Then A/I is $\tilde{\varphi}-$approximately biflat.*

Proof. Let $\theta_\alpha : A \to (A \hat{\otimes} A)^{**}$ be a bounded A-bimodule map such that $\lim_\alpha \pi^{**} \circ \theta_\alpha \circ \varphi(a) = \varphi(a)$. Let $q : A \to A/I$ be the quotient map. Define the map $\tilde{\theta}_\alpha : A/I \to (A/I \hat{\otimes} A/I)^{**}$ by $a + I \mapsto (q \hat{\otimes} q)^{**} \circ \theta_\alpha(a)$ $(a \in A)$. If (e_β) is a bounded left approximate identity for A (the right case is analogous), then

$$
\begin{aligned}
\| (\hat{q} \otimes q)^{**}(\theta_\alpha(a)) \| &= \lim_{\beta} \| (\hat{q} \otimes q)^{**}(\theta_\alpha(e_\beta a)) \| \\
&= \lim_{\beta} \| q(a)(\hat{q} \otimes q)^{**}(\theta_\alpha(e_\beta)) \| \\
&\le \| q \|^2 \| \theta_\alpha \| \sup_{\beta} \| e_\beta \| \|q(a) \|.
\end{aligned}
$$

And $\tilde{\theta}_\alpha$ are well-defined. We prove that $\tilde{\theta}_\alpha$ is an A/I - bimodul map. To do this, choose $a,b,c \in A$, then we have

$$
\tilde{\theta}_\alpha((a + I)(b + I)(c + I)) = \tilde{\theta}_\alpha(abc + I)
$$

$$
\begin{aligned}
&= (\hat{q} \otimes q)^{**} \circ \theta_\alpha(abc) \\
&= (\hat{q} \otimes q)^{**}(a \cdot \theta_\alpha(b) \cdot c) \\
&= a \cdot (\hat{q} \otimes q)^{**}(\theta_\alpha(b) \cdot c \\
&= (a + I) \cdot \widetilde{\theta}_\alpha(b + I) \cdot (c + I).
\end{aligned}
$$

We also have

$$
\begin{aligned}
\lim_\alpha \pi_{A/I}^{**} \circ \widetilde{\theta}_\alpha \circ \widetilde{\varphi}(a + I) &= \lim_\alpha \pi_{A/I}^{**} \circ \widetilde{\theta}_\alpha(\varphi(a) + I) \\
&= \lim_\alpha \pi_{A/I}^{**} \circ (\hat{q} \otimes q)^{**} \circ \theta_\alpha(\varphi(a)) \\
&= \lim_\alpha q^{**} \circ \pi_A^{**} \circ \theta_\alpha \circ \varphi(a) \to q(a) = a + I.
\end{aligned}
$$

That is, A/I is $\widetilde{\varphi}-$approximately biflat.

The proof of the following result is similar to that of Theorem 2.8 and we omit it.

Theorem 2.9 *Suppose that A is a Banach algebra and $\varphi \in Hom(A)$. If A^{**} is $\varphi^{**}-$ approximate biflat. Then A is $\varphi-$approximate biflat.*

Example 2.10 *Let $A = \{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \}$ under the standard operator norm, we see that A has no identity and right approximate identity. Therfore A is not $\varphi-$approximate amenable Banach algebra. Put*

$$
f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; g = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
$$

we define

$$
\theta(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}) = a(f \otimes f) + b(f \otimes g).
$$

Then for $a \in A$ and $\varphi \in Hom(A)$, $\pi \circ \theta \circ \varphi(a) = \varphi(a)$. Thus A is $\varphi-$ biprojective Banach algebra, but A is not $\varphi-$approximate biflat.

3 $(\varphi, \psi)-$AMENABLE BANACH ALGEBRAS

We start this section by introducing the following:

Let $\psi \in \Phi_A$. Then ψ has a unique extension on A^{**} and defined by $\widetilde{\psi}(F) = F(\psi)$ for every $F \in A^{**}$.

Definition 3.1 *Let A be a Banach algebra and $\varphi \in Hom(A), \psi \in \Phi_A$. Then A is called $(\varphi,\psi)-$amenable if there exists $M \in A^{**}$, such that $M(\psi \circ \varphi)=1$ and $M(\varphi(a)\cdot f)=M(f \cdot \varphi(a))$ for all $a \in A, f \in A^*$.*

Theorem 3.2 *Let A be a Banach algebra and $\varphi \in Hom(A), \psi \in \Phi_A$. Then A is $(\varphi,\psi)-$amenable if and only there exists a bounded net $(m_\alpha) \subset A$ such that $\| \varphi(a)m_\alpha - m_\alpha\varphi(a) \| \to 0$ and $\psi \circ \varphi(m_\alpha)=1$ for all α.*

Proof. There exists $M \in A^{**}$ such that $M(\psi \circ \varphi)=1, M(\varphi(a)\cdot f)=M(f \cdot \varphi(a))$ for $a \in A, f \in A^*$. Choose a net a_α in A with $a_\alpha \to M$ in the w^*-topology of A^{**} and $\| a_\alpha \| \le \| M \|$ for all α. Since $< \psi \circ \varphi, a_\alpha > \to < \psi \circ \varphi, M >= 1$, passing to a subnet and replacing (a_α) by $(1/\psi \circ \varphi(a_\alpha))a_\alpha$, we may assume that $\psi \circ \varphi(a_\alpha)=1$ and $\| a_\alpha \| \le \| M \| + 1$ for all α. Consider the product space A^A endowed with the product of norm toppoogical. Define a linear map $T: A \to A^A$ by $T(b)=\varphi(a)b - b\varphi(a)$, for all $b \in A$. Let

$$B=\{b \in A : \| b \| \le \| M \| + 1 \text{ and } \psi \circ \varphi(ba)=1\}$$

Clearly, B is convex and so $T(B)$ is a convex subset of A^A. For every $f \in A^*$, we have

$$\begin{aligned} < f, \varphi(a)a_\alpha - a_\alpha\varphi(a) > &= < f, \varphi(a)a_\alpha > - < f, a_\alpha\varphi(a) > \\ &= < f \cdot \varphi(a), a_\alpha > - < \varphi(a)\cdot f, a_\alpha > \\ &\to < M, f \cdot \varphi(a) > - < M, \varphi(a)\cdot f > = 0. \end{aligned}$$

By Goldestin's Theorem we can replace weak* convergence in equations by weak convergence. Applying Mazur's Theorem, then we have obtain a net m_α in A such that $\| \varphi(a)m_\alpha - m_\alpha\varphi(a) \| \to 0$ and $\psi \circ \varphi(m_\alpha)=1$.

Conversely, assume that a net (m_α) exists. Let M be a w^*-cluster point of the net (m_α) in A^{**}. Then, $< M, \psi \circ \varphi >= \lim_\alpha < \psi \circ \varphi, m_\alpha >= 1$. For every $a \in A$ and $f \in A^*$, we get

$$\begin{aligned} < M, f \cdot \varphi(a) > &= \lim_\alpha < f \cdot \varphi(a), a_\alpha > = \lim_\alpha < f, \varphi(a)a_\alpha > \\ &= \lim_\alpha < f, \varphi(a)a_\alpha - a_\alpha\varphi(a) > + \lim_\alpha < f, a_\alpha\varphi(a) > \\ &= \lim_\alpha < \varphi(a)\cdot f, a_\alpha > = < M, \varphi(a)\cdot f > . \end{aligned}$$

Definition 3.3 *Let A be a Banach algebra and $\varphi \in Hom(A), \psi \in \Phi_A$. An element M of $(A \hat{\otimes} A)^{**}$ is a $(\varphi,\psi)-$ virtual diagonal for A if*

i) $\varphi(a)\cdot M = M \cdot \varphi(a) \qquad (a \in A)$

ii) $\tilde{\psi} \circ \pi^{**}(M)\cdot\varphi(a)=\varphi(a) \qquad (a \in A)$.

Proposition 3.4 *Let A be a Banach algebra and $\varphi \in Hom(A), \psi \in \Phi_A$. The A Banach algebra A has a $(\varphi,\psi)-$ virtual diagonal if and only if there exists a bounded net (m_α) in $(A \hat{\otimes} A)$ such that $m_\alpha \cdot \varphi(a) - \varphi(a) \cdot m_\alpha \to 0$ and $\psi \circ \pi(m_\alpha) \cdot \varphi(a) \to \varphi(a)$ for every $a \in A$.*

Proof. Let M be a $\varphi-$ virtual diagonal for A, and let (m_α) be a net in $(A \hat{\otimes} A)$ such that $M = w^* - \lim_\alpha m_\alpha$. Then, a routine verification shows that for the net (m_α), $m_\alpha \cdot \varphi(a) - \varphi(a) \cdot m_\alpha \to 0$ and $\psi \circ \pi(m_\alpha) \cdot \varphi(a) \to \varphi(a)$ for every $a \in A$, holds in the $weak^* -$ topology. Following the argument given in the proof of [6, Lemma 2.9.64] we can show that there exists a net (m_β) of convex combinations of (m_α)'s satisfying both conditions.

Conversely, let $(m_\alpha) \subset (A \hat{\otimes} A)$ be a bounded net such that $m_\alpha \cdot \varphi(a) - \varphi(a) \cdot m_\alpha \to 0$ and $\psi \circ \pi(m_\alpha) \cdot \varphi(a) \to \varphi(a)$ for every $a \in A$. After passing to a sunnet if necessary, let $M \in (A \hat{\otimes} A)^{**}$ be a w^*-cluster point of the net (m_α). Since $w^* - \lim m_\alpha \cdot \varphi(a) - \varphi(a) \cdot m_\alpha = 0$, it can easily show that $\varphi(a) \cdot M = M \cdot \varphi(a)$, for every $a \in A$. Also the w^*-continuity of π^{**}, implies that $\tilde{\psi} \circ \pi^{**}(M) \cdot \varphi(a) = \varphi(a)$ and the proof is complete.

We close the section with the following result.

Theorem 3.5 *Let A be a Banach algebra and $\varphi \in Hom(A), \psi \in \Phi_A$. If A is $(\varphi,\psi)-$amenable then A has a $(\varphi,\psi)-$ virtual diagonal.*

Proof. Suppose that A is $(\varphi,\psi)-$amenable then by Theorem (3.2) there exists a bounded net $(m_\alpha) \subset A$ such that $\| \varphi(a) m_\alpha - m_\alpha \varphi(a) \| \to 0$ and $\psi \circ \varphi(m_\alpha) = 1$ for all α. Define $a_\alpha = \varphi(m_\alpha) \otimes \varphi(m_\alpha)$, therefore

$$\begin{aligned} \psi \circ \pi(a_\alpha) \cdot \varphi(a) &= \psi \circ \pi(\varphi(m_\alpha) \otimes \varphi(m_\alpha)) \cdot \varphi(a) \\ &= \psi \circ \varphi(m_\alpha) \psi \circ \varphi(m_\alpha) \cdot \varphi(a) \\ &= \varphi(a). \end{aligned}$$

Then by proposition (3.4), A has a $(\varphi,\psi)-$ virtual diagonal .

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Received January 20, 2019