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FRAGMENTS OF DYNAMIC OF MÖEBIOUS MAPPINGS AND SOME APPLICATIONS. PART II

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Summary. Using dynamic and geometry of Möebious mappings we prove Lindelöf type theorems for much larger class of functions on the unit disk than previously considered class of meromor-phic functions.

1. INTRODUCTION

In classical theory of boundary behaviour of functions of one complex variable and in the theory of boundary sets the special important place is for the Lindelöf theorem and the Fatou theorem (we refer to [3, 12]) on radial and nontangential boundary values of holomorphicc functions. The first one concerns the local property of functions, i.e., it is about the existence of nontanegntial boundary value in a single point in the domain of a holomorphic function, the second one is about global boundary behaviour, i.e., it concerns the almost everywhere existence of radial boundary values of a holomorphic function. Nowadays there exist many proofs of these theorems but all of them use classical results of analytic theory of functions (see [3, 12, 13, 23]). Generalizations of Lindelöf theorems and Fatou theorems goes in many directions. One direction is for analytic functions by proving „stronger“ results, i.e., by proving the existence of nontangentail boundary values under weaker conditions then those in the Lindelof theorem (see [17-19]). The second direction is to consider similar theorems for broader class of functions: meromorphic functions, endomorphic mappings, holomorphic mappings of several complex variables, quasiconformal mappings in n, Rn≥ 2, harmonic functions and similar [22, 24, 25].

In this paper we prove how one can efficasely use the geometry or dynamic of Möebious mappings in order to derive the results on asymptotical behavior of holomorphic functions. Namely, we prove theorems that give necessary and sufficient conditions and criteria in order that a meromorphic function on the unit disk has tangential and nontangential boundary values.

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These theorems show that the conditions in the classical Lindelöf theorem and in the theorem of Lehto and Virtanen, Bagemihl and Seidel, Gavrilov and Burkova on angular boundary values of meromorphic functions may be relaxed. In the proofs of these theorems we use the Main Lemma 1 and the Main Lemma 2 in the Section 5 (see [18]). These results give the necessary and sufficient condition on a function defined on the unit disk in the complex plane, to has a boundary set consisted of one point, along the set which is obtain applying cyclic semi-group produced by an element in the hyperbolic or parabolic Moebius group on the unit disk. More on the topic on boundary asymptotic properties of functions one may found in [13, 17-19, 22-24].

2. PRELIMINARY NOTATIONS, DEFINITIONS AND RESULTS

By *D* we denote the open unit disk $\{z \mid |z|<1\}$ in the complex plane C, and with Γ we denote the boundary of *D*, and $D^+ = D\cap\{z \mid \operatorname{Im} z>0\}$, $D^- = D\cap\{z \mid \operatorname{Im} z<0\}$, and $D_r=\{z \mid |z|<r\}$, 0<r<1 is the disk with radius r. By P_θ i p_θ we denote the diameter and the radius of *D* with one endpoint in $e^{i\theta}$. Further, we denote by $d(z_1,z_2)=|z_1-z_2|$, $z_1,z_2\in\mathbb{C}$ the Euclidean distance on $\mathbb{C}$, $d_{ph}(z,w)=\left|\frac{z-w}{1-z\overline{w}}\right|$ and $d_h(z,w)=\frac{1}{2}\log\frac{1+d_{ph}(z,w)}{1-d_{ph}(z,w)}$, $z,w\in D$, stand for the pseudohyperbolic and hyperbolic distance between z and w in the dsik D, respecitvely, and

$$d_s(z,w)=\begin{cases}\dfrac{2|z-w|}{\sqrt{1+|z|}\cdot\sqrt{1+|w|}}, & z,w\in C;\\[2ex] \dfrac{1}{\sqrt{1+|z|^2}}, & z\in C,\ w=\infty\end{cases}$$

is the spherical distance on the Rimanian sphere $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$.

It is well known that d_h is the metric in the Poencare model of the hyperbolic geometry on the disk *D* introduced by Lobachevsy.

All convergencies in this paper are with respect to the distances introduces above.

The set $D_{ph}(w,r')=\{z \mid z\in D, d_{ph}(z,w)<r'\}$, $w\in D$, $0<r'<1$, is the pseudohyperbolic disk, and $D_h(w,r)=\{z \mid z\in D, d_h(z,w)<r\}$, $w\in D$, $r>0$, is the disk with respect to the hyperbolic distance.

Lemma 1. *We have* $D_h(w,r)=D_{ph}(w,r')$, *where* $r=\frac{1}{2}\ln\frac{1+r'}{1-r'}$ $\left(r'=\frac{e^{2r}-1}{e^{2r}+1}=thr\right)$.

The pseudohyperbolic disk $D_{ph}(w,r)$ is the Euclidean disk $D(c,R)=\{z\in D \,\|z-c|<R\}$ for $c=\dfrac{1-r^2}{1-r^2|w|^2}$, and $R=\dfrac{1-|w|^2}{1-r^2|w|^2}r.$

Therefore, the boundaries of hyperbolic and pseudohyperbolic disks are the ordinary cycles. The cycle which lies in *D* and with Γ has one common point is the *oricycle D*. The radius of *D* and arcs in *D* and in intersection with Γ have two points are *hypercycles* in *D*.

An arbitrary hypercycle will be denoted by H, an arbitrary oricycle will be denoted by O. We denote by $H^\theta, \theta\in[0,\pi)$, the hypercycle which connects the points $-e^{i\theta}$ and $e^{i\theta}$; we denote by $O^\theta,\ \theta\in[0,2\pi)$, the oricycle which is tangent to Γ in $e^{i\theta}$, and $O_0{}^\theta=\left\{\dfrac{u}{u-i}e^{i\theta}\,\middle|\, u\in(-\infty,\infty)\right\}$ is the oricycle $\left\{z \,\middle|\, \left|z-\dfrac{1}{2}e^{i\theta}\right|=\dfrac{1}{2}\right\}$.

We will also consider the family of all hypercycles with two common points in Γ.

The hyperbolic distance between a point z, $z\in D$, to the curve $\gamma,\ \gamma\subset D$, is $d_h(z,\gamma)=\inf\limits_{w\in\gamma} d_h(z,w)$.

For $\gamma=H^\theta$, one can prove that $d_h(z,H^\theta)=\min\limits_{w\in H^\theta} d_h(z,w)$ and that $d_h(z,H^\theta)$ does not depend on z if $z\in H$, where H is a hyper-cycle from the family of all hypercycles which is defined by the hypercycle H^θ (see [10]). Also one can prove (see [10]) that there exists unique point w_0 in H^θ such that

$$d_h(z,H^\theta)=\min_{w\in H^\theta} d_h(z,w)=d_h(z,w_0). \qquad (1)$$

From above, by “symmetric thinking”, it follows that for $w\in H^\theta$ there exists unique point z_0 in *H* such that

$$d_h(w,H)=\min_{z\in H} d_h(w,z)=d_h(w,z_0) \qquad (2)$$

And this distance does not depend on $w\in H^\theta$.

From (1) and (2) it follows that for any $w\in H^\theta$ and $z\in H$ there exists unique points $w_0\in H^\theta$ and $z_0\in H$ such that

$$d_h(w,H)=d_h(z,H^\theta)=d_h(w_0,z_0). \qquad (3)$$

Having in mind all the preceding, the equality (3) define the hyperbolic distance between hypercycles H^{θ} *and H. Notation:* $d_h\left(H^{\theta},H\right)$.

From the all given above we have:

Lemma 2 *(*see [10]*)**.** *The set of points in D such the hyperbolic distance between the hypercycle H is the hypercycle which belongs to the family of all hypercycles defined by the hypercycle H.*

From Lemma 2 we obtain:

Lemma 3 (see [8, 14]). *The set* $\Delta_H\left(\theta,r\right)=\bigcup\limits_{a\in(-1,1)} D_h\left(ae^{i\theta},r\right)$, $r\in\left(0,+\infty\right)$, *is a domain in the disk D bounded by two hyper-cycles* $H^{\theta}\left(r\right)$ *and* $H^{\theta}\left(-r\right)$ *such that their hyperbolic distance to the radius* P_{θ} *is equal to r, and which contains the points* $-th\ re^{i\left(\frac{\pi}{2}+\theta\right)}$ *and* $th\ re^{i\left(\frac{\pi}{2}+\theta\right)}$ *and contains the points* $-e^{i\theta}$ *and* $e^{i\theta}$ (see the Figure 1).

Lemma 4 (see [10])**.** *Let r be the hyperbolic distance of hyper-cycle* H^{θ} *from the diameter* P_{θ} *of the disk D. The angle* α *between* H^{θ} *and* P_{θ} *is equal to* $\alpha=\frac{\pi}{2}-arctg\ e^{\pm 2r\sqrt{\pi}}$.

If $h(\theta,\alpha_1)$ and $h(\theta,\alpha_2)$, $-\frac{\pi}{2}<\alpha_1<\alpha_2<\frac{\pi}{2}$, are arcs in *D* that with the radius p_{θ} of the disk *D* with endpoint in point $e^{i\theta}$ make angles α_1 and α_2, then the domain in *D* which is bounded by these arcs and by the circle $D_r=\left\{z\,|\left|z-e^{i\theta}\right|=r\right\}$ is the Stolz angle with vertex at $e^{i\theta}$. This domain is denoted by $\Delta\left(\theta,\alpha_1,\alpha_2\right)$. By $\Delta\left(\theta,\alpha\right)$ we denote the Stolz angle with boundary $h(\theta,\alpha)$ and $h(\theta,-\alpha)$, $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$. We denote it by $\Delta\left(\theta,\alpha_1,\alpha_2\right)$. With $\Delta\left(\theta,\alpha\right)$ we denote the Stolz angle with boundary $h(\theta,\alpha)$ and $h(\theta,-\alpha)$, $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, i.e.,

$$\Delta\left(\theta,\alpha\right)=\left\{z\,\middle|\ \mathrm{z}\in\mathrm{D},\ \left|\arg\left(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{z}\right)\right|<\alpha,\ 0<\alpha<\frac{\pi}{2}\right\}.$$

Threfore, the Stolz angle is the domain which is an usualty geometic object (see Figure 1).

From Lemma 4 we obtain:

Lemma 5. *For every* α**,** $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, *there exist* r, $r\in\left(0,+\infty\right)$, r_1, $r_1\in\left(0,1\right)$, *such that* $\left\{z\middle|\ \left|y-e^{i\theta}\right|<r_1\right\}\cap\Delta\left(\theta,\alpha\right)\subset\left\{z\middle|\ \left|y-e^{i\theta}\right|<r_1\right\}\cap\Delta_H\left(\theta,r\right)$. *For every* r, $r\in\left(0,+\infty\right)$, *there exists* α**,** $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, *such that* $\Delta_H\left(\theta,r\right)\subset\Delta\left(\theta,\alpha\right)$.

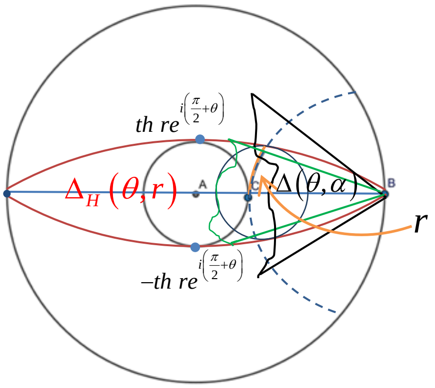

Figure 1

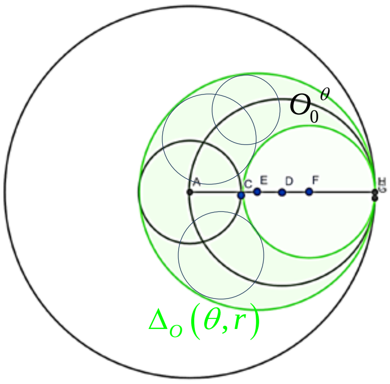

Figure 2

If $d_h(z,\gamma)=\inf_{w\in\gamma} d_h(z,w)$ and if $\gamma=O^{\theta}$, then $d_h(z,O^{\theta})=\min_{w\in O^{\theta}} d_h(z,w)$ and $d_h(z,H^{\theta})$ do not depend on $z\in O$, where O is the oricycle from the family of all oricycles generated by O^{θ}. One can also prove that (see[10]) there exists only one point w_0 in O^{θ} such that $d_h(z,O^{\theta})=\min_{w\in O^{\theta}} d_h(z,w)=d_h(z,w_0)$. Analogy one may define the distance between two ori-cycles from the same family of ori-cycles in the following way $d_h(H^{\theta},H)=d_h(w,H)=d_h(z,H^{\theta})$. It may be shown that there exist unique points $w_0\in O^{\theta}$ and $z_0\in O$ such that $d_h(H^{\theta},H)=d_h(w,H)=d_h(z,H^{\theta})=d_h(w_0,z_0)$.

Now, we have the following statements:

Lemma 6 (see [10])**.** *The set of points in D for which the hyperbolic distance is constant from the oricycle O is the orycycle in the family of all orycycles defined by the oricycle O.*

From lemma 6, we have:

Lemma 7 (see [14]). *The set* $\Delta_O(\theta,r)=\bigcup_{u\in(-\infty,\infty)} D_h\left(\frac{u}{u+i}e^{i\theta},r\right)$, $r\in(0,+\infty)$, *is a domain in the disk D which is bounded by two ori-cycles* $O^{\theta}(-r)$ *and* $O^{\theta}(r)$ *such that the hyperbolic distance between them and the oricycle* $O_0^{\theta}=\left\{\frac{u}{u+i}e^{i\theta}\,\middle|\,u\in(-\infty,\infty)\right\}$ *is equal to r, and that pass throughout* $e^{i\theta}$, $-th\ re^{i\theta}$ *and* $th\ re^{i\theta}$ (see Figure 2).

3. FRAGMENTS OF THE GEOMETRY OF MÖBIOUS MAPPINGS

The Möbious group on the unit disk D is the group of all conformal automorphisms of the unit disc D, i.e., $G = G(D) = \left\{ e^{i\theta} \frac{z-a}{1-\overline{a}z} \mid a \in D,\ z \in \mathbb{C},\ \theta \in [0, 2\pi) \right\}$.

The set

$$H_D^{\theta} = \left\{ g_a^{\theta} = g_a^{\theta}(z) = \frac{z + ae^{i\theta}}{1 + ae^{-i\theta}z} \mid a \in (-1, 1) \right\},\ \theta \in [0, \pi) \text{ is fixed,}$$

stand for the hyperbolic subgroup of G with fixed points $e^{i\theta}$ *and* $-e^{i\theta}$,

$$P_D^{\theta} = \left\{ g_u^{\theta} = g_u^{\theta}(z) = \frac{(u+i)z - ue^{i\theta}}{-(u-i) + ue^{-i\theta}z} \mid u \in (-\infty, +\infty) \right\},\ \theta \in [0, 2\pi) \text{ is fixed,}$$

is the parabolic subgroup of G with fixed point $e^{i\theta}$,

and finally

$$E_D^{\theta} = \left\{ g_{\theta}^{z_0} = g_{\theta}^{z_0}(z) = \frac{\left(1 - |z_0| e^{i\theta}\right) z - z_0\left(1 - e^{i\theta}\right)}{\overline{z_0}\left(1 - e^{i\theta}\right) z + e^{-i\theta} - |z_0|^2} \mid \theta \in [0, 2\pi) \right\},\ z_0 \in D \text{ is fixed,}$$

is the elliptic subgroup of G with fixed point z_0.

Since the hyperbolic distance is invariant with respect to $g \in G$ and from the definition of the groups H_D^{θ} and P_D^{θ} and sets P_D^{θ}, $\Delta_H(\theta, r)$, O_0^{θ} and $\Delta_O(\theta, r)$ we have the following statements:

Lemma 9. (i) $\Delta_H(\theta, r) = \bigcup_{g \in H_D^{\theta}} g\left(D_h(0, r)\right) = \bigcup_{g \in H_D^{\theta}} g\left(D_{ph}(0, thr)\right),\ r \in (0, +\infty)$

(ii) $\Delta_P(\theta, r) = \bigcup_{g \in P_D^{\theta}} g\left(D_h(0, r)\right) = \bigcup_{g \in P_D^{\theta}} g\left(D_{ph}(0, thr)\right),\ r \in (0, +\infty)$.

The set $A,\ A \subset D,$ is the stabilisator of the group H_D^{θ} if $g(A) = A$, for every $g \in H_D^{\theta}$.

Lemma 10. *For every* $g \in H_D^{\theta}$ *we have* $g(P_{\theta}) = P_{\theta}$, *i.e., the diameter* P_{θ} *is stabilisator of the group* H_D^{θ}.

Lemma 11. *For every* $g \in H_D^{\theta}$ *and* $r \in (0, +\infty)$ *we have* $g\left(\Delta_H(\theta, r)\right) = \Delta_H(\theta, r)$, *i.e., the set* $\Delta_H(\theta, r)$ *is also the stabilisator of the group* H_D^{θ}.

Lemma 12. *For every* $g \in P_D^{\theta}$ *we have* $g\left(O_0^{\theta}\right) = O_0^{\theta}$, *i.e., the ori-cycle* O_0^{θ} *is the stabilisator of the group* P_D^{θ}.

Lemma 13. *For every* $g \in P_D^{\theta}$ *and* $r \in (0,+\infty)$ *we have* $g\left(\Delta_O(\theta,r)\right) = \Delta_O(\theta,r)$, *i.e.,* $\Delta_O(\theta,r)$ *is the stabilisator of the group* P_D^{θ}.

For $g \in G(D)$ denote $g^n(z) = \underbrace{g\ \left(g\left(\ldots\left(g\right.\right.\right.}_{n \text{ puta}}(z)\ldots)\ )$, $g^0(z) = i$, i is the identity and $g^{-n}(z) = \left(g^{-1}\right)^n(z) = \underbrace{g^{-1}\left(g^{-1}\left(\ldots\left(g^{-1}\right.\right.\right.}_{n \text{ puta}}(z)\ldots)\ )$, $n \in \mathbb{N}$.

Lemma 14 (see [2] on p. 73)**.**
(i) *Let* $g \in H_D^{\theta}$. *For fixed points* $e^{i\theta}$ *and* $-e^{i\theta}$ *there holds* $g^n(z) \xrightarrow[n\to\infty]{} e^{i\theta}$ *and* $g^{-n}(z) \xrightarrow[n\to\infty]{} -e^{i\theta}$, *where we mean uniform convergence on compacts sets of the disk D.*
(ii) *Let* $g \in P_D^{\theta}$. *Then for fixed point* $e^{i\theta}$ *we have* $g^n(z) \xrightarrow[n\to\infty]{} e^{i\theta}$, *where we also mean the uniform convergence of compact subsets of the disk D.*

Therefore, the point $e^{i\theta}$ is an attraction point for $g \in H_D^{\theta}$, and $-e^{i\theta}$ is repulsive point for g, i.e. it is an attraction point for g^{-1}. If $g \in P_D^{\theta}$ then the attraction point for $g \in P_D^{\theta}$.

For $g \in H_D^{\theta}$, and $\theta \in [0,\pi)$ fixed, $g \neq i$, denote $H_g^{\theta} = \left\{ g^n \middle| n \in \mathbb{Z} \right\}$. The set H_g^{θ} sa with composition operation is the cyclic subgroup of the group H_D^{θ}. If $g \in P_D^{\theta}$, then the set $P_g^{\theta} = \left\{ g^n \middle| n \in \mathbb{Z} \right\}$ with composition of functions is the cyclic subgroup of P_D^{θ}.

Let $\Delta_g(\theta,r) = \underset{n\in\mathbb{Z}}{\cup} g^n\left(D_h(0,r)\right)$, $r \in (0,+\infty)$.

Further, from the property of invariance of the hyperbolic distance with respect to $g \in G$ we have:

Lemma15. $\Delta_g(\theta,r) = \underset{n\in\mathbb{Z}}{\cup}\left(D_h\left(g^n(0),r\right)\right)$, $r \in (0,+\infty)$.

Lemma16. *Let* $g \in H_D^{\theta}$, $g \neq i$. *For every* $r \in (0,+\infty)$ *there exists* $r_1 \in (0,+\infty)$ *such that* $\Delta_H(\theta,r) \subset \Delta_g(\theta,r_1)$, *and* $\Delta_g(\theta,r) \subset \Delta_H(\theta,r)$ *for every* $r \in (0,+\infty)$.

Proof of lemma 16. Let $g \in H_D^{\theta}$ be arbitrary and let it be fixed and $g \neq i$. Let $z \in \Delta_H(\theta,r)$. There exists $a \in (0,+\infty)$ such that $z \in D_h\left(ae^{i\theta},r\right)$. Since $ae^{i\theta} \in P_{\theta}$ and $g^n(0) \in P_{\theta}$ for every $n \in \mathbb{Z}$, there exists $N \in \mathbb{Z}$ such that $ae^{i\theta}$ is between $g^N(0)$ and $g^{N+1}(0)$ or is equalt to one of that points. Let $0 < M = d_h\left(0, g(0)\right) = d_h\left(g^n(0), g^{n+1}(0)\right)$, $n \in \mathbb{Z}$. Then we have

$$d_h\left(g^N(0),z\right)\le d_h\left(g^N(0),ae^{i\theta}\right)+d_h\left(ae^{i\theta},z\right)\le d_h\left(g^N(0),g^{N+1}(0)\right)+d_h\left(ae^{i\theta},z\right)<M+r$$

Therefore, for every $z\in\Delta_H(\theta,r)$ there exists $N\in\mathbb{Z}$ such that $d_h\left(g^N(0),z\right)<M+r$, where M and r are independent on z and N.

Since $\Delta_g(\theta,M+r)=\underset{n\in\mathbb{Z}}{\cup}\left(D_h\left(g^n(0),M+r\right)\right)$ (by Lemma 14) and $D_h\left(g^N(0),M+r\right)\subset\Delta_g(\theta,M+r)$, we obtain $z\in\Delta_g(\theta,M+r)$. If we take $r_1=M+r$, it follows $\Delta_H(\theta,r)\subset\Delta_g(\theta,r_1)$.

Now $\Delta_g(\theta,r)\subset\Delta_H(\theta,r)$, $r\in(0,+\infty)$ follows from Lemma 9 and Lemma 15. $\square$

Lemma 17. *Let* $g\in P_D^\theta$, $g\ne i$. *For every* $r\in(0,+\infty)$ *there exists* $r_1\in(0,+\infty)$ *such that* $\Delta_P(\theta,r)\subset\Delta_g(\theta,r_1)$, *and* $\Delta_g(\theta,r)\subset\Delta_P(\theta,r)$ *for every* $r\in(0,+\infty)$.

Lemma 17 may be proved in a similar way as Lemma 16, instead of diameter P_θ one has to take the oricycle O_0^θ .

We will further consider the domains: $\tilde{\Delta}_H(\theta,r)=\bigcup\limits_{a\in[0,1]}D_h\left(ae^{i\theta},r\right)$ and $\tilde{\tilde{\Delta}}_H(\theta,r)=\bigcup\limits_{a\in[-1,0]}D_h\left(ae^{i\theta},r\right)$, $r\in(0,+\infty)$, we call them the hypercyclic domains in D and $\tilde{\Delta}_O(\theta,r)=\underset{u\in(0,+\infty)}{\cup}D_h\left(\frac{u}{u+i}e^{i\theta},r\right)$ and $\tilde{\tilde{\Delta}}_O(\theta,r)=\underset{u\in(-\infty0,)}{\cup}D_h\left(\frac{u}{u+i}e^{i\theta},r\right)$, $r\in(0,+\infty)$, which will be called the oricyclic domains in D.

Lemma 18. *Let* $g_a\in H_D^\theta$, $g_a\ne i$, *for which* $e^{i\theta}$ *is an attraction fixed point. Then for every* $r\in(0,+\infty)$ *there exists* $r_1\in(0,+\infty)$ *such that* $\bigcup\limits_{n=0}^{\infty}g_a^n\left(D_h(0,r)\right)\subset\tilde{\Delta}_H(\theta,r)\subset\bigcup\limits_{n=0}^{\infty}g_a^n\left(D_h(0,r_1)\right)$.

Lemma 19. *Let* $g_u\in P_D^\theta$, $g_u\ne i$, *for which* $e^{i\theta}$ *is fixed attraction point. Then for every* $r\in(0,+\infty)$ *there exists* $r_1\in(0,+\infty)$ *such that* $\bigcup\limits_{n=0}^{\infty}g_u^n\left(D_h(0,r)\right)\subset\tilde{\Delta}_O(\theta,r)\subset\bigcup\limits_{n=0}^{\infty}g_u^n\left(D_h(0,r_1)\right)$, *u>0, and* $\bigcup\limits_{n=0}^{\infty}g_u^n\left(D_h(0,r)\right)\subset\tilde{\tilde{\Delta}}_O(\theta,r)\subset\bigcup\limits_{n=0}^{\infty}g_u^n\left(D_h(0,r_1)\right)$, *u<0.*

Lemma 18 and lemma 19 may be proved in a similar way as Lemma 16.

4. CLASSICAL RESULTS FOR ASYMPTOTIC AND ANGULAR LIMIT VALUES OF ANALYTIC FUNCTIONS AT A POINT

For $A\subset D,\ \overline{A}\cap\Gamma=\{e^{i\theta}\}$, we denote by $\overline{A}$ the closure of the set A, and $C(f,A,e^{i\theta})=\left\{\omega \,\middle|\, \omega\in\Omega,\ (z_n)\subset A,\ \lim\limits_{n\to\infty} z_n=e^{i\theta},\ \lim\limits_{n\to\infty} f(z_n)=\omega\right\}$ is the boundaty set of a function $f:D\to\Omega$ corresponding to the point $e^{i\theta}$ allong the set A. It is known that $C(f,A,e^{i\theta})=\overline{C(f,A,e^{i\theta})}$.

The symbol $\varphi_n \rightrightarrows_K \varphi$ denotes the uniform convergence on the set $K\subset D$, of the sequence (φ_n) of functions $\varphi_n:D\to\overline{\mathbb{C}},\ n\in\mathbb{N}$, to $\varphi:D\to\overline{\mathbb{C}}$.

If $A=\Delta(e^{i\theta},\alpha)$ is a Stolz angle in the disk D with the vertex at the point $e^{i\theta}$, then $C(f,\Delta(e^{i\theta},\alpha),e^{i\theta})$ is the boundary set of the function f along the angle $\Delta(e^{i\theta},\alpha)$. If for every $\alpha, 0<\alpha<\frac{\pi}{2}$, $C(f,\Delta(e^{i\theta},\alpha),e^{i\theta})=\{\omega\}$, the $e^{i\theta}$ is the Fatou point of f, and $\omega\in\Omega$ is the unique nontangential boundary value.

We always denote by γ the simple Jordan curve in the disk D with endpoint in $e^{i\theta}$. If $C(f,\gamma,e^{i\theta})=\{\omega\}$, $\omega\in\Omega$, then ω is an asympthotic boundary value of the function f in the point $e^{i\theta}$ along the curve γ.

We give now the classical assymptotic results and nontangential of analytic functions.

Theorem of Lindelöf (see [12, 23])**.** *If $f:D\to\mathbb{C}$ is a bounded analytic function. If $C(f,\gamma,e^{i\theta})=\{\omega\}$, $\omega\in\mathbb{C}$, then $C(f,\Delta(e^{i\theta},\alpha),e^{i\theta})=\{\omega\}$, i.e., $e^{i\theta}$ is the Fatou point of function f.*

There are many proofs of the Lindelöf theorem. A proof based on maximum principle of analytic functions may be found in [23].

One generalization of the Lindelöf theorem is given by Lehto and Virtanen in [11]. The used results from normal function theory and results in harmonic function theory and harmonic measure.

For a family of functions $\mathfrak{I}=\left\{f \,\middle|\, f:O\to\overline{\mathbb{C}}\right\}$ we say that it is *normal family* on a domain *O*, $O\subset\mathbb{C}$, if for every sequence (f_n) in that familty $\mathfrak{I}$ there exists a subsequence (f_{n_k}) which convege uniformly on compact subsets of *O* to a function $f:O\to\overline{\mathbb{C}}$. This is normality in the sense of $\Im$ of Montel. The family of functions $\Im=\left\{f\,\middle|\,f:O\to\overline{\mathbb{C}}\right\}$ is normal in the point $z\in O$ if it is normal familiy in a neiborhoud of *z*.

It is well known that a family of functions $\Im=\left\{f\,\middle|\,f:O\to\overline{\mathbb{C}}\right\}$ is normal family in the domain *O* if and only if it is normal in every point in the domain O (see [16, 20]).

If $O\subset\overline{\mathbb{C}}$, i.e. if $\infty\in O$, then the family of functions $\Im=\left\{f\,\middle|\,f:O\to\overline{\mathbb{C}}\right\}$ is normal in the point ∞ if we have normality of the family $\Im'=\left\{f\left(\frac{1}{z}\right)\,|\,f\in\Im\right\}$ in 0. The family of functions $\Im=\left\{f\,\middle|\,f:O\to\overline{\mathbb{C}}\right\}$ is normal on *O* if it is normal in every point of the domain *O*. The theory of normal functions is well exposed in [16, 20].

If $f:D\to\mathbb{C}$ is a bounded analytic maping, then the family $\left\{f\circ g\,\middle|\,g\in G\right\}$ is normal family of functions on the disk D.

Theorem of Lehto and Virtanen (see [11])**.** *Let $f:D\to\overline{\mathbb{C}}$ be a meromorphic function. If $\left\{f\circ g\,\middle|\,g\in G\right\}$ is normal family of functions on the disk D and $C\left(f,\gamma,e^{i\theta}\right)=\{\omega\}$, $\omega\in\mathbb{C}$, then we have $C\left(f,\Delta\left(e^{i\theta},\alpha\right),e^{i\theta}\right)=\{\omega\}$, i.e., $e^{i\theta}$ is the Fatou point of the function f.*

For the proof of the theroem Lehto and Virtanen used the results from harmonic function theory and harmonic measures (the theorem on two constants) and the propery of the normal meromorphic functions (see [3, 11]).

A meromorphic function $f:D\to\overline{\mathbb{C}}$ for which the family $\left\{f\circ g\,\middle|\,g\in G\right\}$ is normal family of functions on D is the class of very well understood normal meromorphic functions N which contains the Bloch class of holomorphic functions denoted by B.

In the following theorems proved by Bagemihl and Seidel [1], it is proved the existence of angular boundary values under weaker asymptotical conditions then these in the preceding theorems. But these theorems are based on the theorems of Lehto and Virtanen.

Theorem of Bagemihl and Seidel 1 (see [1])**.** *Let $f:D\to\overline{\mathbb{C}}$ be a meromorphic function. If $\left\{f\circ g\,\middle|\,g\in G\right\}$ is a normal family of functions on the disk D and if for every $z\in D$ we have $f(z)\neq w,\ w\in\overline{\mathbb{C}}$, and if there exists a sequence (z_n), $z_n\in D$, $n\in\mathbb{N}$, such that:*

$\lim_{n\to\infty} z_n = e^{i\theta}$, $d_h(z_n, z_{n+1}) < M$, $n \in \mathbb{N}$, i $\lim_{n\to\infty} f(z_n) = \omega$, $\omega \in \overline{\mathbb{C}}$, *then* $C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{\omega\}$ *for every* α, $0 < \alpha < \frac{\pi}{2}$, *i.e.,* $e^{i\theta}$ *is the Fatou point of the function f.*

Theorem of Bagemihl and Seidel 2 (see [1])**.** *Let* $f : D \to \overline{\mathbb{C}}$ *be a meromorphic function. If* $\left\{ f \circ g \,\middle|\, g \in G \right\}$ *is a normal family of functions on the disk D and if there exists a sequence* (z_n), $z_n \in D$, $n \in \mathbb{N}$, *such that* $\lim_{n\to\infty} z_n = e^{i\theta}$, $\lim_{n\to\infty} d_h(z_n, z_{n+1}) = 0$, i $\lim_{n\to\infty} f(z_n) = \omega$, $\omega \in \overline{\mathbb{C}}$, *then* $C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{\omega\}$ *for every* $\alpha, 0 < \alpha < \frac{\pi}{2}$, *i.e.,* $e^{i\theta}$ *is the Fatou point of function f.*

Bagemihl and **Seidel** [1] constructed an analytic functions in order to show that the condition concerning the hypervbolic distance $d_h(z_n, z_{n+1})$ in Theorem 5 and Theorem 6 is not possible to remove.

In the following theorem proved by Gavrilov and Burkova in [8], it is proved the existence of angular boundary values for the broader class of meromorphic functions then the class in the theorem of Lehto and Virtanen. In [Gavrilov and Burkova 11] it is given an example of meromorphic function for which $\left\{ f \circ g_a^\theta \,\middle|\, g_a^\theta \in H_D^\theta \right\}$ is normal on *D* but the family $\left\{ f \circ g \,\middle|\, g \in G \right\}$ is not normal on *D*.

A construction is based on the theorem which says that for a meromorphic function $f : D \to \overline{C}$ the family $\left\{ f \circ g \,\middle|\, g \in G \right\}$ is normal on the disku *D* if and only if the disk *D* does not contain the so called P-sequences for the function f, dok je $\left\{ f \circ g_a^\theta \,\middle|\, g_a^\theta \in H_D^\theta \right\}$ is normal family on the disk *D* if and only if in the domain $\Delta_g(\theta, r) \subset \Delta_H(\theta, r)$, $r \in (0, +\infty)$, does not exist the P-sequences for the function f.

A sequence (z_n), $z_n \in D$, $\lim_{n\to\infty} |z_n| = 1$, is a *P-sequence* for a function $f : D \to \overline{C}$ if for every subsequence $\left(z_{n_k}\right)_{k\in N}$ and for every ε, $0 < \varepsilon < 1$, the function f on $\bigcup_{k\in N} D_h\left(z_{n_k}, \varepsilon\right)$, takes inifinity many times all velues in $\overline{C}$, except possibly at most two (see definition, Gavrilov[6]).

In the sequel we will need the following theorems concerning the *P*-sequences:

Theorem on P-sequences 1 (see [6, Lemma 1]). *Let* (z_n) *be a P-sequence for a meromorphic function* $f:D\to\overline{C}$. *If for a sequence* $(z_n')\subset D$ *we have* $\lim\limits_{n\to\infty} d_h\left(z_n,z_n'\right)=0$, *then the sequence* (z_n') *is P-sequence for f.*

Theorem on P-sequences 2 (see [26]). *Let* $f:D\to\overline{C}$ *be a meromorphic fuction on D and let* $(z_n)\subset D$ *a sequence such that* $\lim\limits_{n\to\infty}|z_n|=1$ *and* $\lim\limits_{n\to\infty} f(z_n)=c$ *for some a* $c\in\overline{\mathbb{C}}$. *Further, let* $\left(z_n'\right)\subset D$ *be a sequence such that* $\lim\limits_{n\to\infty}\left|z_n'\right|=1, \lim\limits_{n\to\infty} d_h\left(z_n,z_n'\right)=0$, *and* $(f(z_n))$ *does not sonverge to a c as* $n\to\infty$. *Then* (z_n) *and* $\left(z_n'\right)$ *are both P-sequences of the fuction f.*

Theorem of Gavrilov and Burkova (see [8]). *Let* $f:D\to\overline{\mathbb{C}}$ *be a meromorphic function. If* $\left\{f\circ g_a^\theta \,\middle|\, g_a^\theta\in H_D^\theta\right\}$ *is a normal family on the disk D and* $C\left(f,\gamma,e^{i\theta}\right)=\{\omega\}$, $\omega\in\mathbb{C}$, *then* $C\left(f,\Delta\left(e^{i\theta},\alpha\right),e^{i\theta}\right)=\{\omega\}$ *for every* $\alpha, 0<\alpha<\frac{\pi}{2}$, *i.e.,* $e^{i\theta}$ *is the Fatou point of the function f.*

A proof of theorem of Gavrilov and Burkova goes in the same way as the, by using the result from harmonic function theory as well as using properties of harmonic measure, as in the proof of theorem Lehto – Virtanen.

In [1] are given theorems that are analigies to the theorems of Bagemil and Seidel for meromorphic functions on *D* i.e., functions for which $\left\{f\circ g_a^\theta \,\middle|\, g_a^\theta\in H_D^\theta\right\}$ is normal on the disk *D*.

5. MAIN RESULT

The main lemma 1. *For any function* $f:D\to\overline{\mathbb{C}}$, *any compact set K,* $K\subset D$, *and any mapping* $g_a\in H_D^\theta$, $g_a\neq i$, *the following conditions are equivalent:*

i) $f\circ(g_a)^n \rightrightarrows_K c$;

ii) $C\left(f,\bigcup\limits_{n=0}^{\infty} g_a^n(K),e^{i\theta}\right)=\{c\}$.

Proof of main lemma 1. Let $c\in\mathbb{C}$.

i)⇒ ii). From i) we have

$$(\forall\varepsilon>0)(\exists N_1=N_1(\varepsilon))(\forall n\geq N_1)(\forall z\in K)\left(\left|f\circ g_a^n(z)-c\right|<\varepsilon\right) \qquad (4)$$

i.e., $f\left(\bigcup_{n=N_1}^{\infty} g_a^n(K)\right) \subset \{w \in \mathbb{C} \mid |w-c| < \varepsilon\}.$

From lemma 8 we have

$$(\forall \delta > 0)(\exists N_2 = N_2(\delta))(\forall n \geq N_2)(\forall z \in K)\left(\left|g_a^n(z) - e^{i\theta}\right| < \delta\right), \qquad (5)$$

i.e.,
$$\left(\forall z \in \bigcup_{n=N_1}^{\infty} g_a^n(K)\right)\left|z - e^{i\theta}\right| < \delta. \qquad (5')$$

Let (z_n) be any sequence in $\bigcup_{n=1}^{\infty} g_a^n(K)$ for which $\lim_{n\to\infty} z_n = e^{i\theta}$. From (5), i.e., from (5′) we obtain

$$(\exists N_3 = N_3((z_n), N_2))(\forall n \geq N_3)\left(z_n \in \bigcup_{k=N_2}^{\infty} g_a^k(K)\right). \qquad (6)$$

If $N_2 \leq N_1$, then we have $\bigcup_{n=N_2}^{\infty} g_a^n(K) \subset \bigcup_{n=N_1}^{\infty} g_a^n(K)$, from this and from (6) it follows that $z_n \in \bigcup_{n=N_1}^{\infty} g_a^n(K)$ for every $n \geq N_3$. Having in mind now (4) it follows that $(\forall n \geq N_3)(|f(z_n) - c| < \varepsilon)$, which means that $\lim_{n\to\infty} f(z_n) = c.$

If $N_1 \leq N_2$, then from the sequence (z_n), except $z_1, \ldots, z_{N_3}$, remove those that are in the set $\bigcup_{k=1}^{N_1} g_a^k(K)$, there are only finite many of them. Therefore, there exists N_4 such that $z_n \in \bigcup_{n=N_1}^{\infty} g_a^n(K)$ for every $n \geq N_4$. Now, according to (4) we obtain $(\forall n \geq N_3)(|f(z_n) - c| < \varepsilon)$, i.e., in this case we also have $\lim_{n\to\infty} f(z_n) = c.$

Therefore, for every sequence (z_n) in $\bigcup_{n=0}^{\infty} g_a^n(K)$ we have $\lim_{n\to\infty} f(z_n) = c$, do we may conclude that $C\left(f, \bigcup_{n=0}^{\infty} g_a^n(K), e^{i\theta}\right) = \{c\}.$

ii)⇒ i). From ii) we have $\forall (z_n) \subset \bigcup_{n=0}^{\infty} g_a^n(K) \wedge \lim_{n\to\infty} z_n = e^{i\theta} \Rightarrow \lim_{n\to\infty} f(z_n) = c$ i.e.,

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon))\left(\forall z \in \bigcup_{n=0}^{\infty} g_a^n(K)\right)\left(\left|z - e^{i\theta}\right| < \delta \Rightarrow |f(z) - c| < \varepsilon\right) \qquad (7)$$

Since $g_a^n \rightrightarrows_K e^{i\theta}$, for $\delta > 0$ the exists $N = N(\delta)$ such that for any $n \geq N$ and every $z \in K$ holds $\left|g_a^n(z) - e^{i\theta}\right| < \delta$, i.e.,

$$\bigcup_{n=N}^{\infty} g_a^n(K) \subset \left\{ z \in D \middle| \left| z - e^{i\theta} \right| < \delta \right\}. \tag{8}$$

From (7) and (8) it follows that $f\left(\bigcup_{n=N_1}^{\infty} g_a^n(K)\right) \subset \left\{ w \in \mathbb{C} \middle| \left| w - c \right| < \varepsilon \right\}$, and therefore $(\forall n \geq N)(\forall z \in K)\left(\left| f\left(g_a^n(z)\right) - c \right| < \varepsilon\right)$, i.e., $f\circ(g_a)^n \rightrightarrows_K c$.

If $c = \infty \in \overline{\mathbb{C}}$, the proof goes in the same way as in the case $c \in \mathbb{C}$ instead of the Eucilean metric we have to take the spherical distance. $\square$

Main lemma 2. *For any function* $f : D \to \overline{\mathbb{C}}$*, and a compact set K,* $K \subset D$*, and any mapping* $g_u \in P_D^\theta$, $g_u \neq i$*, the following conditions are equivalent:*

i) $f\circ(g_u)^n \rightrightarrows_K c$, $c \in \mathbb{C}$.

ii) $C\left(f, \bigcup_{n=0}^{\infty} g_u^n(K), e^{i\theta}\right) = \{c\}$.

Main Lemma 2 may be proved in the same way as Main Lemma 1.

6. APPLICATIONS

For $g_a \in H_D^\theta$, $g_a \neq i$, for which $e^{i\theta}$ is an attraction fixed point $H_{g_a}^\theta = \left\{ g_a^n \middle| n \in \mathbb{N} \cup \{0\} \right\}$, $a \in (-1,1)$ is fixed, is the hyperbolic semigroup of G with fixed attraction point $e^{i\theta}$, and $P_{g_u}^\theta = \left\{ g_u^n \middle| n \in \mathbb{N} \cup \{0\} \right\}$, $u \in (-\infty, +\infty)$ is fixed, is the parabolic semigroup of G with attraction fixed point $e^{i\theta}$.

6.1. Angular boundary values of meromorphic functions

Theorem 1. *Let* $f : D \to \overline{\mathbb{C}}$ *be a meromorphic function. If*

$$\left\{ f \circ g \middle| g \in H_{g_a}^\theta \right\} = \left\{ f \circ g_a^n \middle| n \in \mathbb{N} \cup \{0\} \right\},\ a \in (-1,1)$$

(a is fixed), is normal family of functions on the disk D, γ *is simple Jordan curve with one endpoint in* $e^{i\theta}$ *and* $\gamma \subset \tilde{\Delta}_H(\theta, r)$ *and* $C\left(f, \gamma, e^{i\theta}\right) = \{c\}$, $c \in \overline{\mathbb{C}}$*, then*

$$C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{c\} \text{ for every } \alpha, 0 < \alpha < \frac{\pi}{2},$$

i.e., $e^{i\theta}$ *is the Fatou point of the function f.*

Proof of Theorem 1. Since $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$, $a \in (-1,1)$, a is fixed, is a normal family of functions on the disk D, there exists a sequence $\left(f \circ g_a^{n_k} \right)$ which uniformly on compact sete convegre to a meromorphic function φ on $\overline{D_{r_1}}$ . .i.e., $f\circ(g_a)^{n_k} \rightrightarrows_{\overline{D_{r_1}}} \varphi$.

Since $\gamma \subset \tilde{\Delta}_H(\theta, r)$, the sets $\gamma \cap g_a^{n_k}\left(D_{r_1} \setminus D_r \right)$, $0 < r < r_1 < 1$, $n \in \mathbb{N}$, are made of two simple curves. By γ_k we denote one of them Then we have $\gamma_k \cap \gamma_{k+1} = \varnothing$, and $\left[\Gamma_k = g_a^{-n_k}(\gamma_k) \right] \cap \left[\Gamma_{k+1} = g_a^{-n_{k+1}}(\gamma_{k+1}) \right] = \varnothing$, $n \in \mathbb{N}$, since the Moebius transforms g_a^n are bijections

For every $m \in \mathrm{N}$ let us select a sequence $\left(z_k^m \right)$, $z_k^m \in \Gamma_k$, such that $\lim\limits_{k\to\infty} z_k^m = z_0^m \in \overline{D}_{ph}(0, r_1)$ and $z_0^i \neq z_0^j$ for $i \neq j$. We will show that $\varphi\left(z_0^m \right) = c$, $c \in \overline{C}$, for every $m \in \mathrm{N}$.

For every $m \in \mathrm{N}$ there holds

$$d_S\left(\varphi\left(z_0^m \right), c \right) \leq d_S\left(\varphi\left(z_0^m \right), \varphi\left(z_k^m \right) \right) + d_S\left(\varphi\left(z_k^m \right), f_{n_k}\left(z_k^m \right) \right) + d_S\left(f_{n_k}\left(z_k^m \right), c \right). \quad (9)$$

Let ε be any positive real number. Since of coninuily of φ we have $d_S\left(\varphi\left(z_0^m \right), \varphi\left(z_k^m \right) \right) < \frac{\varepsilon}{3}$, if k is enough big.

Since the sequence $\left(f_{n_k} \right)$ converge uniformly on compact sets of the disk D to φ, we have $d_S\left(\varphi(z), f_{n_k}(z) \right) < \frac{\varepsilon}{3}$ for every $z \in \overline{D_{r_1}}$ and enough big k.

Since $z_k^m \in \overline{D_r}$ we have $d_S\left(\varphi\left(z_k^m \right), f_{n_k}\left(z_k^m \right) \right) < \frac{\varepsilon}{3}$. Since $z_k^m \in \Gamma_k$ it follows that $\varphi_{n_k}\left(z_k^m \right) = w_k^m \in \gamma_k \subset \gamma$ and $\lim\limits_{k\to\infty} w_k^m = e^{i\theta}$.

Since c is the asymptotic value of f snd since $\lim\limits_{k\to\infty} f_{n_k}\left(w_k^m \right) = \lim\limits_{k\to\infty} f \circ \varphi_{n_k}\left(z_k^m \right) = \lim\limits_{k\to\infty} f_{n_k}\left(z_k^m \right) = c$, for enough big k we have $d_S\left(f_{n_k}\left(z_k^m \right), c \right) < \frac{\varepsilon}{3}$ for every $m \in \mathrm{N}$.

From (9) and obtained inequality it follows that $d_S\left(\varphi\left(z_0^m \right), c \right) < \varepsilon$ for every m. Since ε is any number, we have $\varphi\left(z_0^m \right) = c$ for every m.

Since the sequence $\left(z_0^m \right)$ is in $\overline{D_{r_1}}$ and i $\overline{D_{r_1}}$ has an accumulation point, from the uniqness theorem we have $\varphi \equiv c$.

Therefore, we have proved that any sequence in the family $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\}$ which is uniformly convergent on compact sets in D, is convergent to the constant c.

Now we will show that any sequence in the family $\left\{ f\circ g_a^n \,\middle|\, n\in\mathbb{N}\cup\{0\}\right\}$ converge uniformly on compact sets of D the the constant c. Assume conctrary, that thereexist a sequence postoji (f_n), $f_n\in\left\{ f\circ g_a^n \,\middle|\, n\in\mathbb{N}\cup\{0\}\right\}$, which uniformly on compacts does not converge to the constant c. Then there exists a number $\varepsilon>0$ such that for every $k\in\mathrm{N}$ we have $n_k\in\mathrm{N}$ and $z_{n_k}\in\overline{D_r}$ such that $d_S\left(f_{n_k}\left(z_{n_k}\right),c\right)\geq\varepsilon$. Since tha family $\left\{ f\circ g_a^n \,\middle|\, n\in\mathbb{N}\cup\{0\}\right\}$ is normal, f_{n_k} has a subsequence $f_{n_{kl}}$ which uniformly on compact sets of *D* converge, according to the preceding consideration it follows that it converge to the constant c, which is contrary with the assumption $d_S\left(f_{n_k}\left(z_k^m\right),c\right)\geq\varepsilon$. This contradiction shows that every sequence in $\left\{ f\circ g_a^n \,\middle|\, n\in\mathbb{N}\cup\{0\}\right\}$ uniformly on compact sets of *D* converge to the constant c. Having in mind the Lemma 1 it follows that $C\left(f,\bigcup_{n=0}^{\infty} g_a^n(K),e^{i\theta}\right)=\{c\}$, for every compact set K, $K\subset D$, and form Lemma 5 and Lemma 18 we have that $C\left(f,\Delta(\theta,\alpha),e^{i\theta}\right)=\{c\}$ for every α, $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, i.e., the function f in the point $e^{i\theta}$ has angular boundary value c. □

In [27] it is proved that $\left\{ f\circ g_a^n \,\middle|\, n\in\mathbb{N}\cup\{0\}\right\}$, $a\in(-1,1)$ is normal family on D if and only if in the domain $\tilde{\Delta}_H(\theta,r)$, $r\in(0,+\infty)$, does not exist P-sequences for *f*.

Theorem 2. *Let* $f:D\to\overline{\mathbb{C}}$ *be a meromorphic function. If* $\left\{ f\circ g_a^n \,\middle|\, n\in\mathbb{N}\cup\{0\}\right\}$, $a\in(-1,1)$, *where a is fixed, normal family of functions on the disk D and if for a sequence* $(z_n)\subset\tilde{\Delta}_H(\theta,r)$ *holds* $\lim_{n\to\infty} z_n=e^{i\theta}$ *and* $\lim_{n\to\infty} f(z_n)=c$, $c\in\overline{\mathbb{C}}$, *then for any sequence* (r_n) *for which* $\lim_{n\to\infty} r_n=0$ *and* $\bigcup_{n=1}^{\infty} D_h(z_n,r_n)\subset\tilde{\Delta}_H(\theta,r_1)$ *for* $r_1>0$, $C\left(f,\bigcup_{n=1}^{\infty} D_h(z_n,r_n),e^{i\theta}\right)=\{c\}$.

Theorem 2 follows directly from theorem on P-sequences which is formulated in the Section 4 and the criteria for normality formulated above of the familiy of functions $\left\{ f\circ g_a^n \,\middle|\, n\in\mathbb{N}\cup\{0\}\right\}$, $a\in(-1,1)$ on D.

Theorem 3. *Let* $f:D\to\overline{\mathbb{C}}$ *be meromorphic function. If* $\left\{ f\circ g_a^n \,\middle|\, n\in\mathbb{N}\cup\{0\}\right\}$, $a\in(-1,1)$ *where a is fixed, is normal family of functions on the disk D and if for a sequence* $(z_n)\subset\tilde{\Delta}_H(\theta,r)$ *holds:* $\lim_{n\to\infty} z_n=e^{i\theta}$, $\lim_{n\to\infty} d_h(z_n,z_{n+1})=0$ *and* $\lim_{n\to\infty} f(z_n)=c$, $c\in\overline{\mathbb{C}}$, *then* $C\left(f,\Delta\left(e^{i\theta},\alpha\right),e^{i\theta}\right)=\{c\}$ *for every* $\alpha, 0<\alpha<\frac{\pi}{2}$, *i.e.,* $e^{i\theta}$ *is the Fatou point of the function f.*

Proof of Theorem 3. Let $x_n = 2d_h\left(z_n, z_{n+1}\right)$. Then we have $\lim\limits_{n\to\infty} x_n = 0$. If $D_h\left(z_n, x_n\right) = \left\{z \middle| \ d_h\left(z, z_n\right) < x_n\right\}$, then form Theorem 2 follows that $C\left(f, \bigcup\limits_{n=1}^{\infty} D_h\left(z_n, x_n\right), e^{i\theta}\right) = \{c\}\ c \in \overline{\mathbb{C}}$. Since the curve (poligonal line) $\gamma = \overline{z_1 z_2 \ldots z_n \ldots e^{i\theta}} \subset \bigcup\limits_{n=1}^{\infty} D_h\left(z_n, x_n\right)$, we have $C\left(f, \gamma, e^{i\theta}\right) = \{c\}$. Since it is possible to chose r>0, such that $\bigcup\limits_{n=1}^{\infty} D_h\left(z_n, x_n\right) \subset \tilde{\Delta}_H(\theta, r)$, from Theorem 1 we conclude $C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{c\}$ for every $\alpha, 0 < \alpha < \frac{\pi}{2}$, i.e., $e^{i\theta}$ is the Fatou point of *f*. ☐

Theorem 4. *Let* $f : D \to \overline{\mathbb{C}}$ *be meromorphic function such that* $f(z) \neq c,\ c \in \overline{\mathbb{C}},\ z \in D$. *If* $\left\{f \circ g_a^n \middle| \ n \in \mathbb{N} \cup \{0\}\right\}$, $a \in (-1,1)$ *, a is fixed, normal family f functions on th disk D and if holds:* $\lim\limits_{n\to\infty} z_n = e^{i\theta}$ *and* $\lim\limits_{n\to\infty} f\left(g_a^n(0)\right) = c,\ c \in \overline{\mathbb{C}}$, *then we have* $C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{c\}$ *for every* $\alpha, 0 < \alpha < \frac{\pi}{2}$, *i.e.,* $e^{i\theta}$ *is the Fatou point of f.*

Proof of Theorem 4. Form normality of meromorphic functions $\left\{f \circ g_a^n \middle| \ n \in \mathbb{N} \cup \{0\}\right\}$ and the condition $\lim\limits_{n\to\infty} f\left(g_a^n(0)\right) = c,\ c \in \overline{\mathbb{C}}$, from Hurwitz theorem (see [20]) $fog_u^n \rightrightarrows_K c$ for every compact set $K \subset D$. If we take $K = \overline{D_h(0, r)}$, 0<r<1, then from the Main Lemma 1, Lemma 5 and Lemma 18 we have $C\left(f, \Delta\left(e^{i\theta}, \alpha\right), e^{i\theta}\right) = \{c\}$ for every $\alpha, 0 < \alpha < \frac{\pi}{2}$, i.e., $e^{i\theta}$ is the Fatou point of f. ☐

6.2. Tangentialy oricyclic boundary values of meromorphic functions

If for every $r \in (0, +\infty)$ holds $C\left(f, \tilde{\Delta}_o(\theta, r), e^{i\theta}\right) = \{\omega\}$, $\omega \in \overline{\mathbb{C}}$, then we will call ω the upper oricyclic boundary value of f in the point $e^{i\theta}$. On the other hand, if for every $r \in (0, +\infty)$ holds $C\left(f, \tilde{\tilde{\Delta}}_o(\theta, r), e^{i\theta}\right) = \{\omega\}$, $\omega \in \overline{\mathbb{C}}$, $\omega \in \overline{\mathbb{C}}$, then we will call ω the lower o oricyclic boundary value for f in the point $e^{i\theta}$. If $C\left(f, \tilde{\Delta}_o(\theta, r) \cup \tilde{\tilde{\Delta}}_o(\theta, r), e^{i\theta}\right) = \{\omega\}$ then we call se ω the oricyclic boundary value of f in the point $e^{i\theta}$.

Theorem 5. *Let* $f : D \to \overline{\mathbb{C}}$ *be meromorphic function. If*

$$\left\{ f\circ g \,\middle|\, g\in P_{g_u}^{\theta} \right\} = \left\{ f\circ g_u^n \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(0,\infty),$$

u is fixed, normal family of functions on the disk D and if for a sequence $(z_n)\subset\tilde{\Delta}_O(\theta,r)$ holds $\lim\limits_{n\to\infty} z_n = e^{i\theta}$ and $\lim\limits_{n\to\infty} f(z_n)=c,\ c\in\overline{\mathbb{C}}$, then for every sequence (r_n) for which $\lim\limits_{n\to\infty} r_n = 0$ and $\bigcup\limits_{n=1}^{\infty} D_h(z_n,r_n)\subset\tilde{\Delta}_O(\theta,r_1)$ for a $r_1>0,\ C\left(f,\bigcup\limits_{n=1}^{\infty} D_h(z_n,r_n),e^{i\theta}\right)=\{c\}$.

Theorem 5′. *Let $f:D\to\overline{\mathbb{C}}$ be a meromorphic function. If*

$$\left\{ f\circ g \,\middle|\, g\in P_{g_u}^{\theta} \right\} = \left\{ f\circ g_u^n \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(-\infty,0)\ ,$$

where u is fixed, normal family of functions on the disk D and if for a sequence $(z_n)\subset\tilde{\tilde{\Delta}}_o(\theta,r)$ holds: $\lim\limits_{n\to\infty} z_n = e^{i\theta}$ and $\lim\limits_{n\to\infty} f(z_n)=c,\ c\in\overline{\mathbb{C}}$, then for every (r_n) for which $\lim\limits_{n\to\infty} r_n = 0$ and $\bigcup\limits_{n=1}^{\infty} D_h(z_n,r_n)\subset\tilde{\tilde{\Delta}}_o(\theta,r)$ for $r_1>0,\ C\left(f,\bigcup\limits_{n=1}^{\infty} D_h(z_n,r_n),e^{i\theta}\right)=\{c\}$.

U [27] it is proved that $\left\{ f\circ g_u^n \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(-\infty,0)$ is normal family on the disk D if and only if in the domain $\tilde{\tilde{\Delta}}_o(\theta,r)\,\tilde{\Delta}_O(\theta,r),\ r\in(0,+\infty)$, does not exist P-sequences for the function f. On the other hand, the family $\left\{ f\circ g_u^n \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(0,+\infty)$ is normal on the disk D if and only if in the domain $\tilde{\Delta}_O(\theta,r),\ r\in(0,+\infty)$, does not exist P-sequences for the function f.

Theorem 5 and Thorem 5′ follows directly from theorem on P-sequences 1 which is formulated in Section 4 and the above formulated criterion for normality of the family of functions $\left\{ f\circ g_u^n \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(-\infty,0)$ $\left\{ f\circ g \,\middle|\, g\in P_{g_u}^{\theta} \right\} = \left\{ f\circ g_u^n \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(-\infty,0)$.

Theorem 6. *Let $f:D\to\overline{\mathbb{C}}$ be a meromorphic function. If*

$$\left\{ f\circ g \,\middle|\, g\in P_{g_u}^{\theta} \right\} = \left\{ f\circ g_u^n \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(0,\infty),\ u \text{ is fixed,}$$

is normal family on the disk D, γ a simple Jordan curve with one endpoint in $e^{i\theta}$ and $\gamma\subset\tilde{\Delta}_O(\theta,r)$ and $C\left(f,\gamma,e^{i\theta}\right)=\{\omega\},\ \omega\in\mathbb{C}$, then $C\left(f,\tilde{\Delta}_O(\theta,r),e^{i\theta}\right)=\{\omega\}$ for every $r\in(0,+\infty)$, i.e., ω is the upper oricyclic boundary value for the function f in the point $e^{i\theta}$.

Theorem 6′. *Let* $f:D\to\overline{\mathbb{C}}$ *be a meromorphic function. If*

$$\left\{ f\circ g \,\middle|\, g\in P^{\theta}_{g_u} \right\}=\left\{ f\circ g^{n}_{u} \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(-\infty,0)\ ,\ u \text{ is fixed,}$$

is normal family of functions on the disku D, γ *is simple Jordan curve with one endpoint in* $e^{i\theta}$ *and* $\gamma\subset\tilde{\tilde{\Delta}}_{o}(\theta,r)$ *and* $C\left(f,\gamma,e^{i\theta}\right)=\{\omega\}$, $\omega\in\mathbb{C}$, *then* $C\left(f,\tilde{\tilde{\Delta}}_{o}(\theta,r),e^{i\theta}\right)=\{\omega\}$ *for every* $r\in(0,+\infty)$, *i.e.,* ω *is the lower oricyclic boundary value of the function f in the point* $e^{i\theta}$.

Theorem 6 and Theorem 6′ may be proved using the Main Lemma 2 and Lemma 19 in the same way as Theorema 1 is derived from the Main Lemma 1 and Lemma 18.

Theorem 7 and Theorem 7′ may be proved using Theorem 6 and Theorem 6′ in the same way as Theorem 3 using Theorem 1.

Theorem 7. *Let* $f:D\to\overline{\mathbb{C}}$ *be a meromorphic function. If*

$$\left\{ f\circ g \,\middle|\, g\in P^{\theta}_{g_u} \right\}=\left\{ f\circ g^{n}_{u} \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(0,+\infty)\ ,\ u \text{ is fixed,}$$

is a normal family of functions on the disk D and if for a sequence $(z_n)\subset\tilde{\Delta}O(\theta,r)$ *holds:* $\lim\limits_{n\to\infty}z_n=e^{i\theta}$, $\lim\limits_{n\to\infty}d_h(z_n,z_{n+1})=0$ *and* $\lim\limits_{n\to\infty}f(z_n)=c$, $c\in\overline{\mathbb{C}}$, *then* $C\left(f,\tilde{\Delta}O(\theta,r),e^{i\theta}\right)=\{\omega\}$ *for every* $r\in(0,+\infty)$, *i.e.,* ω *is the upper oricyclic boundary value of function f in the point* $e^{i\theta}$.

Theorem 7′. *Let* $f:D\to\overline{\mathbb{C}}$ *be a meromorphic function. If*

$$\left\{ f\circ g \,\middle|\, g\in P^{\theta}_{g_u} \right\}=\left\{ f\circ g^{n}_{u} \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\},\ u\in(-\infty,0)\ ,\ u \text{ is fixed,}$$

is normal family of functions on the disk D and if for a sequence $(z_n)\subset\tilde{\tilde{\Delta}}_{o}(\theta,r)$ *holds:* $\lim\limits_{n\to\infty}z_n=e^{i\theta}$, $\lim\limits_{n\to\infty}d_h(z_n,z_{n+1})=0$ *and* $\lim\limits_{n\to\infty}f(z_n)=c$, $c\in\overline{\mathbb{C}}$, *then* $C\left(f,\tilde{\tilde{\Delta}}_{o}(\theta,r),e^{i\theta}\right)=\{\omega\}$ *for every* $r\in(0,+\infty)$, *i.e.,* ω *is the lower oricyclic boundary value of the function f in the point* $e^{i\theta}$.

Theorem 8. *Let* $f:D\to\overline{\mathbb{C}}$ *be a meromorphic function such that* $f(z)\neq c$, $c\in\overline{\mathbb{C}}$, $z\in D$. $\left\{ f\circ g \,\middle|\, g\in P^{\theta}_{g_u} \right\}=\left\{ f\circ g^{n}_{u} \,\middle|\, n\in\mathbb{N}\cup\{0\} \right\}$, $u\in(0,+\infty)$ *, u is fixed, normal family of functions on D and if* $\lim\limits_{n\to\infty}z_n=e^{i\theta}$ *and* $\lim\limits_{n\to\infty}f\left(g^{n}_{u}(0)\right)=c$, $c\in\overline{\mathbb{C}}$, *then we have* $C\left(f,\tilde{\Delta}O(\theta,r),e^{i\theta}\right)=\{\omega\}$ *for every* $r\in(0,+\infty)$, *i.e.,* ω *is the upper oricyclic boundary value of the function f in the point* $e^{i\theta}$.

Theorem 8′. *Let* $f: D \to \overline{\mathbb{C}}$ *be a meromorphic function such that* $f(z) \neq c,\ c \in \overline{\mathbb{C}},\ z \in D$. $\left\{ f \circ g \,\middle|\, g \in P_{g_u}^{\theta} \right\} = \left\{ f \circ g_u^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\},\ u \in (-\infty, 0)$ *, u is fixed, normal family of functions on the disk D and if* $\lim_{n\to\infty} z_n = e^{i\theta}$ *and* $\lim_{n\to\infty} f\left(g_u^n(0)\right) = c,\ \ c \in \overline{\mathbb{C}}$*, then* $C\left(f, \tilde{\tilde{\Delta}}_o(\theta, r), e^{i\theta} \right) = \{\omega\}$ *for every* $r \in (-\infty, 0)$*, i.e..* ω *is the lower oricyclic boundary value of the function f in* $e^{i\theta}$.

Theorem 8 and Theorem 8′ may be proved using the Main Lemma 2 and Lemma 19 in the same way as Theorem 4 is derived from the Main Lemma 1, Lemma 5 and Lemma 18.

7. CONSTRUCTION OF ONE EXAMPLE

We will construct an example of meromorphic function $f: D \to \overline{\mathbb{C}}$ for which $\left\{ f \circ g_a^n \,\middle|\, n \in \mathbb{N} \cup \{0\} \right\},\ \ a \in (-1, 1)$, a is fixed, is normal family of functions on the disk *D*, and $\left\{ f \circ g_a^{\theta} \,\middle|\, g_a^{\theta} \in H_D^{\theta} \right\}$ is not normal family of functions on *D*. This construction is similar as one in the work [8].

Let $z_k = -\rho_k e^{i\theta},\ \rho_k > 0,\ k \in \mathbb{N},$ be such that $\lim_{k\to\infty} \rho_k = 1$ and $\lim_{k\to\infty} d(z_k, z_{k+1}) = 0.$ The elements of the sequence (z_k) are in the set $P_\theta \cap \Delta_{g^{-1}}(-\theta, r)$. Let a sequence (ε_k) be a such one that we have:

$0 < \varepsilon_{k+1} < \varepsilon_k$; $\lim_{k\to\infty} \varepsilon_k = 0$; $D(z_k, \varepsilon_k) \cap D(z_{k+1}, \varepsilon_{k+1}) = \varnothing, k \in \mathbb{N}$; $\lim_{k\to\infty}\left(\sup_{z \in D(z_k, \varepsilon_k)} d(z_k, z_{k+1}) \right) = 0$; $\sum_{k=1}^{\infty} \varepsilon_k < +\infty$.

Let $a_k = \varepsilon_k^3,\ k \in \mathbb{N}$, and $f(z) = \sum_{k=1}^{\infty} a_k (z - z_k)^{-1}$. The function f is meromorphic on the disk D, with the poles in $z_k,\ k \in \mathbb{N}$. Since $f(z_k) = \infty,\ \ \left| f(z_k + \varepsilon) \right| < M,\ k \in \mathbb{N},$ and $\lim_{n\to\infty} d_h(z_k, z_k + \varepsilon) = 0$ from Theorem 2 on *P*-sequences it follows that $(z_k) \subset \Delta_{g^{-1}}(-\theta, r) \subset \Delta(\theta, r)$ is *P*-sequence of *f*. Therefore, we may conclude that $\left\{ f \circ g_a^{\theta} \,\middle|\, g_a^{\theta} \in H_D^{\theta} \right\}$ is not normal family of functions on the disk *D*.

Since for every $z', z'' \in D \setminus \bigcup_{k=1}^{\infty} D(z_k, \varepsilon_k)$ we have $\left| f(z') - f(z'') \right| \leq \left| z_k{}' - z_k{}'' \right| \sum_{k=1}^{\infty} \varepsilon_k = C < +\infty$ and since $\Delta_g(\theta, r),\ r > 0,$ contains finite number of points z_k, and since $\Delta_g(\theta, r),\ r > 0,$ is invariant set with respect to $g_a^n,\ n \in \mathbb{N},$ it follows that $\limsup\limits_{\Delta_g(\theta,r)\ni z\to e^{i\theta}} \left|f(z)\right| = c_f(r) < \infty,\ 0<r<1$. Therefore, for every $r, 0<r<1,$ the function f is bounded on $O_r \cap \Delta_g(\theta,r),$ where $O_r = \left\{z \,\middle\|\, \left|z-e^{i\theta}\right| < 1-r\right\}$ so we have that $\left\{f\circ g_a^n \,\middle|\, n\in\mathbb{N}\cup\{0\}\right\},\ a\in(-1,1)$ is normal family on the disku D (see [20], p. 35, Montel's theorem).

CONCLUSION

In this paper it is given a new approach in deriving theorems from the theory of asymptotical behavior of analytic functions. Namely, our theorems are proved using some results from the dynamic and the geometry of Möebious mappings and classical uniqueness theorem for analytic mappings, but in the preceding time these theorems were proved by using the approach and the results from the theory of harmonic mappings and harmonic measure theory.

The Main Lemma 1 and the Main Lemma 2 prove that the necessary condition for a function $f:D\to\overline{\mathbb{C}}$ to has the angular or oricyclic boundary value in $e^{i\theta}$ is that the following two families of functions

$$\left\{f\circ g_a^n \,\middle|\, n\in\mathbb{N}\cup\{0\}\right\},\ a\in(-1,1),\ \left\{f\circ g \,\middle|\, g\in P_{g_u}^{\theta}\right\} = \left\{f\circ g_u^n \,\middle|\, n\in\mathbb{N}\cup\{0\}\right\},\ u\in(-\infty,0),$$

are normal on the disk D.

The constructed example in Section 7 shows that the angular boundary values exist for broader class of meromorphic functions then the class considered in the theorems of Lehto-Virtanen and Gavrilov-Burkova. We have proved theorems of type of Bagemihl-Seidel for broader class of functions.

From Theorem 6 and Theorem 6′ it follows that the upper and the lower oricyclic boundary values of meromorphic function $f:D\to\overline{\mathbb{C}}$ in $e^{i\theta}$ are equal $\omega,\ \omega\in\overline{\mathbb{C}},$ then f has tangential – oricyclic boundary value ω in $e^{i\theta}$. In general case it is possible to occur that one of these boundary values exists but the other not. This may be proved by an example which may be constructed in a similar way as the example in the Section 7.

For further consideration it remains to consider is it possible to use the approach of this paper in order to derive results concerning the asymptotic behavior of harmonic functions on the unit disk D in the complex plane $\mathbb{C}$.

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INVERSE PROBLEMS FOR STURM – LIOUVILLE OPERATOR WITH POTENTIAL FUNCTIONS FROM $L_2[0,\pi]$

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Summary. This paper deals with non-self-adjoint second-order differential operators with two constant delays. We consider four boundary value problems $D_{i,k}, i=0,1, k=1,2$

$$-y''(x)+q_1(x)y(x-\tau_1)+(-1)^i q_2(x)y(x-\tau_2)=\lambda y(x), x\in[0,\pi]$$

$$y'(0)-hy(0)=0, \qquad y'(\pi)+H_k y(\pi)=0,$$

where $\frac{\pi}{3}\le\tau_2<\frac{\pi}{2}\le 2\tau_2\le\tau_1<\pi,\ h,H_1,H_2\in R\backslash\{0\}$ and λ is a spectral parameter. We assume that q_1,q_2 are real-valued potential functions from $L_2[0,\pi]$ such that $q_1(x)=0, x\in[0,\tau_1)$ and $q_2(x)=0,\ x\in[0,\tau_2)$. The inverse spectral problem of recovering operators from their spectra has been studied. We prove that delays τ_1,τ_2 and parameters h,H_1,H_2 are uniquely determined from the spectra. Then we prove that potentials are uniquely determined by Volterra linear integral equations.

1 INTRODUCTION

The theory of differential equations with delays is a very important branch of the theory of ordinary differential equations and has been studied in detail in [1] and the references therein. For a number of results relating to the inverse spectral problems for classical Sturm-Liuville operators we refer the reader to [2], while some aspects of the direct and inverse problems for operators with a delay can be found in [3] - [13]. While there are a number results about both direct and inverse problems for operators with one delay, there are just a few results related to the operators with two or more delays (see [14]-[18]). The motivation behind this paper is to initiate further research in the inverse spectral theory for differential operators with delays. In what follows, we always take $i=0,1$ and $k=1,2$. In this paper we consider the boundary value problems $D_{i,k}$

$$-y''(x)+q_1(x)y(x-\tau_1)+(-1)^i q_2(x)y(x-\tau_2)=\lambda y(x), x\in[0,\pi] \tag{1}$$

$$y'(0)-hy(0)=0, \tag{2}$$

$$y'(\pi)+H_k y(\pi)=0 \tag{3}$$

where $\frac{\pi}{3}\le\tau_2<\frac{\pi}{2}\le 2\tau_2\le\tau_1<\pi,\ h,H_1,H_2\in R\backslash\{0\}$ and λ is a spectral parameter. We assume that q_1,q_2 are real-valued potential functions from $L_2[0,\pi]$ such that

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$q_1(x) = 0,\ x \in [0, \tau_1)$ and $q_2(x) = 0,\ x \in [0, \tau_2)$. We study the inverse spectral problem of recovering operators from the spectra of $D_{i,k}$. Let $\left(\lambda_{n,i,k}\right)_{n=0}^{\infty}$ be the eigenvalues of $D_{i,k}$. The inverse problem is formulated as follows.

Inverse problem: Given $\left(\lambda_{n,i,k}\right)_{n=0}^{\infty}$, determine delays τ_1, τ_2, parameters h, H_1, H_2 and potential functions q_1, q_2.

To solve this inverse problem, we use the method of Fourier coefficients. This method based on determination of direct relations between Fourier coefficients of the potentials or some functions containing the potentials, and Fourier coefficients of some known functions.

In Section 2, we study the spectral properties of the boundary value problems $D_{i,k}$. In Section 3, we prove that delays and parameters are uniquely determined from the spectra. Then we prove that potentials are uniquely determined by the system of two Volterra linear integral equations.

2 SPECTRAL PROPERTIES

One can easily show that differential equation (1) under the initial condition (2) and conditions $q_1(x) = 0, x \in [0, \tau_1)$ and $q_2(x) = 0,\ x \in [0, \tau_2)$ is eqivalent to the integral equation

$$\begin{aligned} y_i(x,z) = cosxz + \frac{h}{z} sinzx + \frac{1}{z}\int_{\tau_1}^{x} q_1(t) sinz(x-t) y(t-\tau_1, z)dt + \\ + (-1)^i \frac{1}{z}\int_{\tau_2}^{x} q_2(t) sinz(x-t) y(t-\tau_2, z)dt \end{aligned} \tag{4}$$

Here and in the sequel $\lambda = z^2$. By the method of steps it can be easily verified that the solution of integral equation (4) on $(\tau_1, \pi]$ is

$$\begin{aligned} y_i(x,z) = coszx + \frac{h}{z} sinzx + \frac{1}{z}\left(b_{sc}^{(1)}(x,z) + (-1)^i b_{sc}^{(2)}(x,z)\right) + \\ + \frac{h}{z^2}\left(b_{s^2}^{(1)}(x,z) + (-1)^i b_{s^2}^{(2)}(x,z)\right) + \frac{1}{z^2} b_{s^2c}^{(2)}(x,z) + \frac{h}{z^3} b_{s^3}^{(2)}(x,z) \end{aligned} \tag{5}$$

where

$$b_{sc}^{(k)}(x,z) = \int_{\tau_k}^{x} q_k(t) sinz(x-t) cosz(t-\tau_k)dt,$$

$$b_{s^2}^{(i)}(x,z) = \int_{\tau_k}^{x} q_k(t) sinz(x-t) sinz(t-\tau_k)dt,$$

$$b_{s^3}^{(2)}(x,z) = \int_{2\tau_2}^{x} q_2(t) sinz(x-t) b_{s^2}^{(2)}(t-\tau_2, z)dt,$$

$$b_{s^2c}^{(2)}(x,z) = \int_{2\tau_2}^{x} q_2(t) sinz(x-t) b_{sc}^{(2)}(t-\tau_2, z)dt.$$

Denote

$$\Delta_{i,k}(\lambda) = F_{i,k}(z) = y_i{}'(\pi, z) + H_k y(\pi, z).$$ [1]

From (5) we obtain

$$F_{i,k}(z) = \left(-z + \frac{hH_k}{z}\right) sin\pi z + (h + H_k) cos\pi z + b_{c^2}^{(1)}(z) + (-1)^i b_{c^2}^{(2)}(z) +$$

$$+\frac{h}{z}\left(b_{cs}^{(1)}(z) + (-1)^i b_{cs}^{(2)}(z)\right) + \frac{H_k}{z}\left(b_{sc}^{(1)}(z) + (-1)^i b_{sc}^{(2)}(z)\right) +$$

$$+\frac{H_k h}{z^2}\left(b_{s^2}^{(1)}(z) + (-1)^i b_{s^2}^{(2)}(z)\right) + \frac{1}{z} b_{csc}^{(2)}(z) + \frac{h}{z^2} b_{cs^2}^{(2)}(z) + \frac{H_k}{z^2} b_{s^2c}^{(2)}(z) + \frac{Hh}{z^3} b_{s^3}^{(2)}(z)$$

where

$$b_{cs}^{(k)}(z) = \int_{\tau_k}^{\pi} q_k\,(t) cosz(x - t) sinz(t - \tau_k) dt,$$
$$b_{c^2}^{(k)}(z) = \int_{\tau_k}^{\pi} q_k\,(t) cosz(x - t) cosz(t - \tau_k) dt$$

$$b_{s^2c}^{(2)}(z) = \int_{2\tau_2}^{\pi} q_2\,(t) sinz(x - t) b_{sc}^{(2)}(t - \tau_2, z\,) dt,$$
$$b_{csc}^{(2)}(z) = \int_{2\tau_2}^{\pi} q_2\,(t) cosz(x - t) b_{sc}^{(2)}(t - \tau_2, z\,) dt,$$
$$b_{cs^2}^{(2)}(z) = \int_{2\tau_2}^{\pi} q_2\,(t) cosz(x - t) b_{s^2}^{(2)}(t - \tau_2, z\,) dt.$$

To simplify further consideration we define so called *the transitional functions* $\tilde{q}_i$ as follows

$$\tilde{q}_i(t) = \begin{cases} q_1\left(t + \frac{\tau_1}{2}\right) + (-1)^i q_2\left(t + \frac{\tau_2}{2}\right), t \in \left[\frac{\tau_1}{2}, \pi - \frac{\tau_1}{2}\right] \\ (-1)^i q_2\left(t + \frac{\tau_2}{2}\right), t \in \left[\frac{\tau_2}{2}, \tau_2\right) \cup \left(\pi - \frac{\tau_1}{2}, \pi - \frac{\tau_2}{2}\right]. \\ 0, t \in \left[0, \frac{\tau_2}{2}\right) \cup \left(\pi - \frac{\tau_2}{2}, \pi\right] \end{cases} \tag{6}$$

Let us also define functions $K^{(2)}$ and $U^{(2)}$ by

$$K^{(2)}(t) = q_2(t + \tau_2)\int_{\tau_2}^{t} q_2(s) ds - q_2(t)\int_{t+\tau_2}^{\pi} q_2(s) ds - \int_{t+\tau_2}^{\pi} q_2(s - t) q_2(s) ds,$$
$$t \in [\tau_2, \pi - \tau_2], \qquad K^{(2)}(t) = 0, t \in [0, \tau_2) \cup (\pi - \tau_2, \pi],$$

and

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[1] Below, instead of the argument (π, z) we write argument (z)

$$U^{(2)}(t)=q_2(t+\tau_2)\int_{\tau_2}^{t}q_2(s)ds-q_2(t)\int_{t+\tau_2}^{\pi}q_2(s)ds+\int_{t+\tau_2}^{\pi}q_2(s-t)q_2(s)ds,$$
$$t\in[\tau_2,\pi-\tau_2],\qquad U^{(2)}(t)=0, t\in[0,\tau_2)\cup(\pi-\tau_2,\pi].$$

and introduce notations

$$J_1^{(k)}=\int_{\tau_k}^{\pi}q_i\,(t)dt,\qquad J_2^{(2)}=\int_{2\tau_2}^{\pi}q_2\,(t)\left(\int_{\tau_2}^{t-\tau_2}q_2\,(s)ds\right)dt$$

and functions

$$\tilde{a}_{i,c}(z)=\int_0^{\pi}\tilde{q}_i(t)cosz(\pi-2\,t)dt,\quad \tilde{a}_{i,s}(z)=\int_0^{\pi}\tilde{q}_i(t)sinz(\pi-2\,t)dt,$$
$$k_s(z)=\int_0^{\pi}K^{(2)}(t)\,sinz(\pi-2t)dt,\quad k_c(z)=\int_0^{\pi}K^{(2)}(t)\,cosz(\pi-2t)dt,$$
$$u_s(z)=\int_0^{\pi}U^{(2)}(t)\,sinz(\pi-2t)dt,\quad u_c(z)=\int_0^{\pi}U^{(2)}(t)\,cosz(\pi-2t)dt.$$

One can easily show that folowing relations hold

$$\int_{\tau_2}^{\pi-\tau_2}K^{(2)}(t)\,dt=-J_2^{(2)},\qquad \int_{\tau_2}^{\pi-\tau_2}U^{(2)}(t)\,dt=J_2^{(2)}. \tag{7}$$

Using aforementioned tags and relations (7), we can rewrite characteristic functions $F_{i,k}(z)$ as follows

$$F_{i,k}(z)=\left(-z+\frac{hH_k}{z}\right)sin\pi z+(h+H_k)cos\pi z+\frac{1}{2}\left(\tilde{a}_{i,c}(z)+J_{i,c}(z)\right)+$$
$$+\frac{h}{2z}\left(-\tilde{a}_{i,s}(z)+J_{i,s}(z)\right)+\frac{H_k}{2z}\left(\tilde{a}_{i,s}(z)+J_{i,s}(z)\right)+\frac{hH_k}{2z^2}\left(\tilde{a}_{i,c}(z)-J_{i,c}(z)\right)$$
$$+\frac{1}{4z}\left(J_{2,s}(z)-u_s(z)\right)-\frac{h}{4z^2}\left(J_{2,c}(z)+k_c(z)\right)-\frac{H_k}{4z^2}\left(J_{2,c}(z)-u_c(z)\right)$$
$$-\frac{hH_k}{4z^3}\left(J_{2,s}(z)+k_s(z)\right) \tag{8}$$

where

$$J_{i,c}(z)=J_1^{(1)}cosz(\pi-\tau_1)+(-1)^iJ_1^{(2)}cosz(\pi-\tau_2),$$
$$J_{i,s}(z)=J_1^{(1)}sinz(\pi-\tau_1)+(-1)^iJ_1^{(2)}sinz(\pi-\tau_2)$$
$$J_{2,c}(z)=J_2^{(2)}cosz(\pi-2\tau_2),\qquad J_{2,s}(z)=J_2^{(2)}sinz(\pi-2\tau_2).$$

The functions $F_k(z)$ are entire in λ of order $1/2$. Using (8), by the well known method (see [2]), we obtain the asymptotic formulas for $\left(\lambda_{n,i,k}\right)_{n=0}^{\infty}$ of $D_{i,k}$:

$$\lambda_{n,i,k} = n^2 + \frac{2}{\pi}(h + H_k) + \frac{J_1^{(1)}}{\pi} cosn\tau_1 + (-1)^i \frac{J_1^{(2)}}{\pi} cosn\tau_2 + o(1), n \to \infty. \tag{9}$$

Now, by Hadamard's factorization theorem, from the spectra of $D_{i,k}$, we can construct the characteristic functions $F_{i,k}$. The next lemma holds.

Lemma 2.1. The specification of spectrum $\left(\lambda_{n,i,k}\right)_{\mathrm{n=0}}^{\infty}$ of the boundary value problems $\mathrm{D_{i,k}}$ uniquely determines the characteristic functions $F_{i,k}(z)$ by the formulas

$$F_{i,k}(z) = \pi\left(\lambda_{0,i,k} - z^2\right) \prod_{n=1}^{\infty} \frac{\lambda_{n,i,k} - z^2}{n^2}. \tag{10}$$

3 MAIN RESULTS

Lemma 3.1. The delays τ_k, integrals $J_1^{(k)}$ and sums $h + H_k$ are uniquely determined by eigenvalues $\left(\lambda_{n,i,k}\right)_{n=0}^{\infty}$.

Proof. Let us consider the sequences

$$\rho_{n,k} = \tfrac{1}{2}\left(\lambda_{n,0,k} + \lambda_{n,1,k}\right)$$

and

$$\sigma_n = \frac{1}{2}\left(\lambda_{n,0,1} - \lambda_{n,1,1}\right).$$

From (9) we obtain the next asymptotic formulas

$$\rho_{n,k} = n^2 + \frac{2}{\pi}(h + H_k) + \frac{J_1^{(1)}}{\pi} cosn\tau_1 + o(1)$$

and

$$\sigma_n = \frac{J_1^{(2)}}{\pi} cosn\tau_2 + o(1).$$

Obviously, the delays τ_1, τ_2 and integrals $J_1^{(1)}, J_1^{(2)}$ can be determined from sequences $\left(\rho_{n,k}\right)_{n=0}^{\infty}$ and $(\sigma_n)_{n=0}^{\infty}$ in the same way as for the operators with one delay (see [13]). Lemma 3.1. is proved. □

Lemma 3.2. Parameters h and H_k are uniquely determined by eigenvalues $\left(\lambda_{n,0,k}\right)_{n=0}^{\infty}$.

Proof. By virtue of Lemma 3.1., functions $J_{0,c}(z)$ and $J_{0,s}(z)$ are known. Since the characteristic functions are uniquely determined by the spectra, putting $\lambda = \left(\frac{4m+1}{2}\right)^2$ into functions $F_{0,k}$ from (10), we can define functions

$$F^*{}_{0,k}(m) = F_{0,k}\left(\frac{4m+1}{2}\right) + \frac{4m+1}{2} - \frac{1}{2}J_{0,c}\left(\frac{4m+1}{2}\right) - \frac{H_k+h}{4m+1}J_{0,s}\left(\frac{4m+1}{2}\right).$$

Then, using the form of the characteristic functions $F_{0,k}$ from (8), we get

$$h = \frac{1}{2}\lim_{m\to\infty}\frac{4m+1}{H_2-H_1}\left(F^*{}_{0,2}(m) - F^*{}_{0,1}(m)\right)$$

At the end, we determine H_k from $h + H_k$, thus proving Lemma 3.2. □

In order to recover the potential functions from the spectra by the method of Fourier coefficients, we should transform the characteristic functions (8). For this purpose, we use the method of partial integration in (8), once in integrals $\tilde{a}_{i,s}(z), \tilde{a}_{i,c}(z), u_s(z)$ and $u_c(z)$, and twice in the integrals $k_c(z)$ and $k_s(z)$. This is where the next function appears

$$K^{(2)*}(t) = \begin{cases} \int_{\tau_2}^{t} K^{(2)}(u)\,du, t \in [\tau_2, \pi-\tau_2] \\ 0, t \in [0,\tau_2) \cup (\pi-\tau_2, \pi] \end{cases}.$$

One can show that following relation holds

$$\int_{\tau_2}^{\pi-\tau_2}\left(\int_{\tau_2}^{t} K^{(2)}(u)\,du\right)dt = -(\pi-2\tau_2)J_2^{(2)}.$$

Then we obtain the characteristic functions in the form

$$\begin{aligned} F_{i,k}(z) = {} & \left(-z + \frac{H_k h}{z}\right)sin\pi z + (h+H_k)cos\pi z + \frac{1}{2}\left(\tilde{a}_{i,c}(z) + \frac{H_k}{z}\tilde{a}_{i,s}(z)\right) - \\ & -h\left(\tilde{q}_{i,c}{}^{(1)}(z) + \frac{H_k}{z}\tilde{q}_{i,s}{}^{(1)}(z)\right) - \frac{1}{2}\left(u_c^*(z) + \frac{H_k}{z}u_s^*(z)\right) + h\left(k_c^{**}(z) + \frac{H_k}{z}k_s^{**}(z)\right) \\ & + \frac{J_{i,c}(z)}{2} + \frac{2h+H_k}{2z}J_{i,s}(z) + \frac{1}{2z}\left(1 - \frac{H_k h}{z^2}\right)J_{2,s}(z) + \frac{h}{2z}(\pi-2\tau_2)J_2^{(2)}sinz(\pi-2\tau_2) \\ & + \frac{H_k h}{2z^2}(\pi-2\tau_2)J_2^{(2)}cosz(\pi-2\tau_2) \end{aligned} \quad (11)$$

where

$$\tilde{q}_{i,c}{}^{(1)}(z) = \int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}} \left(\int_{\frac{\tau_2}{2}}^{t} \tilde{q}_i(s)\, ds \right) cosz(\pi - 2t)dt,$$

$$\tilde{q}_{i,s}{}^{(1)}(z) = \int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}} \left(\int_{\frac{\tau_2}{2}}^{t} \tilde{q}_i(s)\, ds \right) sinz(\pi - 2t)dt\,,$$

$$u_c^*(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} U^{(2)}(s)\, ds \right) cosz(\pi - 2t)dt,$$

$$u_s^*(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} U^{(2)}(s)\, ds \right) sinz(\pi - 2t)dt$$

and

$$k_c^{**}(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} K^{(2)*}(s)\, ds \right) cosz(\pi - 2t)dt,$$

$$k_s^{**}(z) = \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} K^{(2)*}(s)\, ds \right) sinz(\pi - 2t)dt.$$

In order to recover the potential functions from the spectra, at the beginning we define functions

$$A_i(z) = \frac{2}{H_2 - H_1}\Big(H_2 F_{i,1}(z) - H_1 F_{i,2}(z)\Big) + 2zsin\pi z - 2hcos\pi z - J_{i,c}(z) - $$

$$- \frac{2hJ_1^{(2)}}{z} sinz(\pi - \tau_2) \tag{12}$$

and

$$B_i(z) = \frac{2z}{H_2 - H_1}\Big(F_{i,2}(z) - F_{i,1}(z)\Big) - 2hsin\pi z - 2zcos\pi z - J_{i,s}(z). \tag{13}$$

From (11) we obtain

$$A_i(z) = \tilde{a}_{i,c}(z) - 2h\tilde{q}_{i,c}{}^{(1)}(z) - u_c^*(z) + 2hk_c^{**}(z) + \alpha(z) \tag{14}$$

$$B_i(z) = \tilde{a}_{i,s}(z) - 2h\tilde{q}_{i,s}{}^{(1)}(z) - u_s^*(z) + 2hk_s^{**}(z) + \beta(z) \tag{15}$$

where

$$\alpha(z) = \frac{J_2^{(2)}}{z}\,(h(\pi - 2\tau_2) + 1)sinz(\pi - 2\tau_2)$$

and

$$\beta(z) = \frac{hJ_2^{(2)}}{z^2}\big(z(\pi - 2\tau_2)cosz(\pi - 2\tau_2) - sinz(\pi - 2\tau_2)\big).$$

One can easily show that

$$\lim_{z\to 0}\beta(z) = 0,$$

and

$$\lim_{z\to 0}\alpha(z) = J_2^{(2)}(h(\pi-2\tau_2)+1)(\pi-2\tau_2).$$

Put $z = m, m \in N$ into (14) and (15) and denote

$$A_{2m,i} = \frac{2}{\pi}(-1)^m A_i(m), \quad B_{2m,i} = \frac{2}{\pi}(-1)^{m+1}B_i(m).$$

Then we obtain

$$A_{2m,i} = \frac{2}{\pi}\tilde{a}_{2m,i} - \frac{4}{\pi}h\tilde{q}^{(1)}_{2m,i,c} - \frac{2}{\pi}u^*_{2m,c} + \frac{4}{\pi}hk^{**}_{2m,c} - \frac{2J_2^{(2)}}{\pi m}(h(\pi-2\tau_2)+1)sin2m\tau_2, \tag{16}$$

$$B_{2m} = \frac{2}{\pi}\tilde{b}_{2m,i} - \frac{4}{\pi}h\tilde{q}^{(1)}_{2m,i,s} - \frac{2}{\pi}u^*_{2m,s} + \frac{4}{\pi}hk^{**}_{2m,s} - \frac{2hJ_2^{(2)}}{\pi m^2}(m(\pi-2\tau_2)cos2m\tau_2 + sin2m\tau_2) \tag{17}$$

where

$$\tilde{a}_{2m,i} = \int_0^\pi \tilde{q}_i(t)cos2mtdt, \qquad \tilde{b}_{2m,i} = \int_0^\pi \tilde{q}_i(t)sin2mtdt,$$

$$u^*_{2m,s} = \int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}}\left(\int_{\frac{\tau_2}{2}}^{t} U^{(2)}(s)\,ds\right)sin2mtdt,$$

$$u^*_{2m,c} = \int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}}\left(\int_{\frac{\tau_2}{2}}^{t} U^{(2)}(s)\,ds\right)cos2mtdt$$

$$k^{**}_{2m,c} = \int_{\tau_2}^{\pi-\tau_2}\left(\int_{\tau_2}^{t} K^{(2)*}(s)\,ds\right)cos2mtdt,.$$

$$k^{**}_{2m,s} = \int_{\tau_2}^{\pi-\tau_2}\left(\int_{\tau_2}^{t} K^{(2)*}(s)\,ds\right)sin2mtdt.$$

Denote $A_{0,i} = \frac{2}{\pi}\lim_{m\to 0}A_i(m)$.

Then we obtain

$$A_{0,i} = \frac{2}{\pi}\tilde{a}_{0,i} - \frac{4}{\pi}h\tilde{q}^{(1)}_{0,i,c} - \frac{2}{\pi}u^*_{0,c} + \frac{4}{\pi}hk^{**}_{0,c} + \frac{2J_2^{(2)}}{\pi}(h(\pi-2\tau_2)+1)(\pi-2\tau_2). \tag{18}$$

Since sequences $\{A_{2m,i}\}$ and $\{B_{2m,i}\}$ belong to the space l_2, by virtue of Riesz-Fischer theorem, there exist functions f_i from $L_2[0,\pi]$ such that

$$f_i(t) = \frac{A_{0,i}}{2} + \sum_{m=1}^{\infty} A_{2m,i} cos2mt + B_{2m,i} sin2mt\, , t \in [0,\pi]$$

Now multiplying (18) with $\frac{1}{2}$, (16) with $cos2mt$ and (17) with $sin2mt$, and then summing-up from $m = 1$ to $m = \infty$, we get the system of integral equations

$$\tilde{q}_i(t) - 2h \int_{\frac{\tau_2}{2}}^{t} \tilde{q}_i(s)\, dt_2 - \int_{\tau_2}^{t} U^{(2)}(s) ds + 2h \int_{\tau_2}^{t} K^{(2)*}(s)\, ds + \Phi(t) = f_i(t) \tag{19}$$

where

$$\Phi(t) = -\frac{2J_2^{(2)}}{\pi}(h(\pi - 2\tau_2) + 1) \sum_{m=1}^{\infty} \frac{sin2m\tau_2}{m} cos2mt -$$

$$-\frac{2hJ_2^{(2)}}{\pi}(\pi - 2\tau_2) \sum_{m=1}^{\infty} \frac{cos2m\tau_2}{m} sin2mt - \frac{2hJ_2^{(2)}}{\pi} \sum_{m=1}^{\infty} \frac{sin2m\tau_2}{m^2} sin2mt.$$

After summing and subtracting integral equations (19), and then introducing substitution of variables, we get the system of integral equations

$$q_1(x) - 2h \int_{\tau_1}^{x} q_1(u)\, du - \int_{\tau_2 + \frac{\tau_1}{2}}^{x} U^{(2)}\left(u - \frac{\tau_1}{2}\right) du + 2h \int_{\tau_2 + \frac{\tau_1}{2}}^{x} K^{(2)*}\left(u - \frac{\tau_1}{2}\right) du +$$

$$+\Phi\left(x - \frac{\tau_1}{2}\right) = \frac{1}{2}\left(f_0\left(x - \frac{\tau_1}{2}\right) + f_1\left(x - \frac{\tau_1}{2}\right)\right) \tag{20}$$

and

$$q_2(x) - 2h \int_{\tau_2}^{x} q_2(u)\, du = \frac{1}{2}\left(f_0\left(x - \frac{\tau_2}{2}\right) - f_1\left(x - \frac{\tau_2}{2}\right)\right). \tag{21}$$

Finally, we come to our main result.

Theorem 3.1. Let $q_k \in L_2[\tau_i, \pi]$, $q_k(x) = 0$ for $x \in [0, \tau_k)$.
If $\frac{\pi}{3} \le \tau_2 < \frac{\pi}{2} \le 2\tau_2 \le \tau_1 < \pi$, then integral equations (20) and (21) have unique solutions $q_1 \in L_2[\tau_1, \pi]$ and $q_2 \in L_2[\tau_2, \pi]$, respectively.

Proof. Obviously, the integral equation (21) has a unique solution q_2 on (τ_2, π). Then we obtain that integrals $\int_{\tau_2 + \frac{\tau_1}{2}}^{x} U^{(2)}\left(u - \frac{\tau_1}{2}\right) du$ and $\int_{\tau_2 + \frac{\tau_1}{2}}^{x} K^{(2)*}\left(u - \frac{\tau_1}{2}\right) du$ are known too, as well as the integral $J_2^{(2)}$. For sums appearing in the function Φ, we have

$$\sum_{m=1}^{\infty}\frac{sin2m\tau_2}{m}cos2mt=\begin{cases}-\tau_2, & t\in(\tau_2,\pi-\tau_2),\\ \frac{\pi}{2}-\tau_2, & t\in(0,\tau_2)\cup(\pi-\tau_2,\pi),\\ \frac{\pi}{4}-\tau_2, & t=\tau_2, t=\pi-\tau_2\end{cases}$$

$$\sum_{m=1}^{\infty}\frac{cos2m\tau_2}{m}sin2mt=\begin{cases}-t, & t\in(0,\tau_2)\\ \frac{\pi}{2}-t, & t\in(\tau_2,\pi-\tau_2)\\ \pi-t, & (\pi-\tau_2,\pi)\\ \frac{\pi}{4}-\tau_2, & t=\tau_2,\\ -\frac{\pi}{4}+\tau_2, & t=\pi-\tau_2\end{cases}$$

and

$$\sum_{m=1}^{\infty}\frac{sin2m\tau_2}{m^2}sin2mt=\begin{cases}(\pi-2\tau_2)t, & t\in(0,\tau_2)\\ \tau_2(\pi-2t), & t\in(\tau_2,\pi-\tau_2)\\ (\pi-2\tau_2)(t-\pi), & (\pi-\tau_2,\pi)\\ (\pi-2\tau_2)\tau_2, & t=\tau_2\\ -(\pi-2\tau_2)\tau_2, & t=\pi-\tau_2\end{cases}$$

Then for $x\in(\tau_1,\pi)$ we obtain linear integral equation

$$q_1(x)-2h\int_{\tau_1}^{x}q_1(u)\,du=\int_{\tau_2+\frac{\tau_1}{2}}^{x}U^{(2)}\left(u-\frac{\tau_1}{2}\right)du-2h\int_{\tau_2+\frac{\tau_1}{2}}^{x}K^{(2)*}\left(u-\frac{\tau_1}{2}\right)du$$
$$-\Phi\left(x-\frac{\tau_1}{2}\right)+\frac{1}{2}\left(f_0\left(x-\frac{\tau_1}{2}\right)+f_1\left(x-\frac{\tau_1}{2}\right)\right)$$

which has a unique solution q_1 on (τ_1,π). Theorem is proved. □

4 CONCLUSION

Inverse spectral problems for classical Sturm-Liouville operators have been studied completely, while the inverse problems for differential operators with delays have not been studied enough. The main results for classical Sturm-Liouville operators is presented in [2] while some of the results for differential operators with delay can be found in ([3],[4],[5],[10],[11],[12],[13]). The class of operators with two delays has been least studied, but some of the results for this class of operators are presented in ([16], [17]). The motivation behind this paper is to initiate further research in the inverse spectral theory for differential operators with delays. We studied the inverse spectral problem of recovering operators from the spectra of $D_{i,k}$. To solved this inverse problem, we used the method of Fourier coefficients. This method is based on determination of direct relations between Fourier coefficients of the potentials or some functions containing the potentials, and Fourier coefficients of some known functions. We studied the spectral properties of the boundary value problems and proved that delays and parameters are uniquely determined from the spectra. Then we proved that potentials are uniquely determined by the system of two Volterra linear integral equations.

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AMENABILITY OF $A \oplus_T X$ AS AN EXTENSION OF BANACH ALGEBRA

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Summary. Let $A, X, \mathfrak{U}$ be Banach algebras and A be a Banach $\mathfrak{U}$-bimodule also X be a Banach $A-\mathfrak{U}$-module. In this paper we study the relation between module amenability, weak module amenability and module approximate amenability of Banach algebra $A \oplus_T X$ and that of Banach algebras A, X. Where T: $A \times A \to X$ is a bounded bi-linear mapping with specific conditions.

1 INTRODUCTION

The notation of amenability of Banach algebras was introduced by B.Johnson in [9]. A Banach algebra A is amenable if every bounded derivation from A into any dual Banach A-bimodule is inner, equivalently if $H(A,\ X^*) = \{0\}$ for any Banach A-bimodule X, where $H(A,\ X^*)$ is the first Hochschild co- homology group of A with coefficient in X^*. Also, a Banach algebra A is weakly amenable if $H(A,\ A^*) = \{0\}$. Bade, Curtis and Dales introduced the notion of weak amenability on Banach algebras in [5]. They considered this concept only for commutative Banach algebras. After a while, Johnson defined the weak amenability for arbitrary Banach algebras [8].

For a morphism $T\colon B \to A$ from a Banach algebra B to a commutative Banach algebra A. The notion of module amenability of Banach algebras was introduced by Amini in [1]. Amini and Ebrahimi Bagha in [3] studied the concept of weak module amenability. In [10] the notation of module approximate amenability and contractibility as modules over of another Ba- nach algebra was introduced for the notion of Banach algebras.

M. Sangani-Monfared in [11] defined a product on $A \times B$ and obtained the Banach algebra $A \times_\theta B$ using a character $\theta \in \sigma(B)$, for Banach algebras in a fairly general setting.

Later, S.J. Bhatt and P.A. Dabhi in [6] defined a product on $A \times B$ and obtained a Banach algebra $A \times_T B$ for a morphism $T : B \to A$ from a Banach algebra B to a commutative Banach algebra A.

The first and the second authors generalized all these constructions, and de- fined the module Lau product $A \times_\alpha B$ for Banach algebras A and B such that A is a Banach B-bimodule. They studied the ideal amenability of $A \times_\alpha B$ in [4].

T.Yazdan panah in [12] studied the concept of expanded modular of Banach algebra denoted by $A \oplus_T X$. He showed that $A \oplus_T X$ is amenable if and only if A is amenable and $X = \{0\}$. In this paper, we define a new Banach algebra different from of all above Banach algebras, named $A \oplus_T X$ in section 2. Then, some required basic properties of the following part are studied. In section 3, as the main section of paper, we study the relationship between module amenability of $A \oplus_T X$ and module amenability of A and X. We show that If $T(A,\ 0) = X$ and $A^2 = A$, then the module amenability of A implies module amenability of $A \oplus_T X$. Furthermore, it's

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conversly obtained that the module amenability of $A \oplus_T X$ implies module amenability of A and moreover if $T(A,\ 0) = X$, then X also is module amenable. In sectiones 4 and 5 respectively we study the relationship between weak module amenability (based as definition in [1] and [2]) and module approximte amenability of $A \oplus_T X$ and weak module amenability and module approximte amenability of A, X.

2 DEFINITIONS AND BASIC PROPERTIES

Throughout this paper it's assumed that $\mathfrak{U}$ be a Banach algebra, A be a Banach $\mathfrak{U}$-bimodule and X be a Banach A-U-bimodule. Module actions are assumed as follow too:

$$A \times \mathfrak{U} \to A; (a,\quad \alpha) \longmapsto a \circ \alpha, \mathfrak{U} \times A \to A; (\alpha,\quad a) \longmapsto \alpha \cdot a.$$
$$X \times \mathfrak{U} \to X; (x,\quad \alpha) \longmapsto x \vartriangle \alpha, \mathfrak{U} \times X \to X; (\alpha,\quad x) \longmapsto \alpha \nabla x.$$
$$X \times A \to X; (x,\quad a) \longmapsto x \circ a, A \times X \to X; (a,\quad x) \longmapsto a \cdot x.$$

Consider the bounded bilinear map $T : A \times A \to X$, which has the following properties:

$a \cdot T(a_1 a_2, 0) = T(aa_1, 0) \circ a_2, T(a_1 a_2, 0) = T(a_1, 0)T(a_2, 0)$,

$T(\alpha \cdot a,\quad \alpha \nabla x) = \alpha \cdot T(a,\quad x), T(a \circ \alpha,\quad x \vartriangle \alpha) = T(a,\quad x) \cdot \alpha,$

$\| T(a,\ 0) \| = \| a \|$, for all $a, a_1, a_2 \in A, x \in X, \alpha \in \mathfrak{U}$.

Module extension $A \oplus X$, with the product

$$(a,\ x)(a_1,\ x_1) = (aa_1,\ a \cdot x_1 + x \circ a_1 + T(aa_1, 0))$$

and the norm $\| (a,\ x) \| = \| a \| + \| x \|$ is a Banach algebra denoted by $A \oplus_T X$.

Definition 2.1 *The bounded map $D\colon A \to X^*$ with $D(a + b) = D(a) + D(b), D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in A$, and $D(\alpha \cdot a) = \alpha \cdot D(a), D(a \cdot \alpha) = D(a) \cdot \alpha (\alpha \in \mathfrak{U},\ a \in A)$, is called module derivation.*

Note that X^* is also Banach module over A and $\mathfrak{U}$ with compatible actions under the canonical actions of A and $\mathfrak{U}$, $\alpha \cdot (a \cdot f) = (\alpha \cdot a) \cdot f, (a \in A,\ \alpha \in \mathfrak{U},\ f \in X^*)$, and the same for right action. Here the canonical actions of A and $\mathfrak{U}$ on X^* are defined by $(\alpha \cdot f)(x) = f(x \vartriangle \alpha), (a \cdot f)(x) = f(x \circ a), (\alpha \in \mathfrak{U},\ a \in A,\ f \in X^*,\ x \in X)$ and it's the same for right actions. As in [1] we call A- module X which have a compatible $\mathfrak{U}$-action as above, a $A - \mathfrak{U}$ modules, above assertion is to say that if X is an $A - \mathfrak{U}$- module, then so is X^*. Also we use the notation $Z_{\mathfrak{U}}(A,\ X^*)$ for the set of all module derivations $D\colon A \to X^*$, and $N_{\mathfrak{U}}(A,\ X^*)$ for those which are inner and $H_{\mathfrak{U}}(A,\ X^*)$ for the quotient group.

Proposition 2.2 *$A \oplus_T X$ is a Banach $\mathfrak{U}$- bimodule.*

Proof. Consider the module actions as follow:

$\mathfrak{U} \times (A \oplus_T X) \to A \oplus_T X; \alpha \cdot (a,\ x) = (\alpha \cdot a,\ \alpha \nabla x)$, and $(A \oplus_T X) \times \mathfrak{U} \to A \oplus_T X; (a,\ x) \bullet \alpha = (\alpha \circ a,\ \alpha \vartriangle x)$. It is easy to check the satification of the properties. ■

Proposition 2.3 *If Y is an A-U-module, then $Y \oplus \{O\}$ is a Banach $A \oplus_T X - \mathfrak{U}$-bimodule.*

Proof. Assume that the module actions on Y, are as follows:

$\mathfrak{U} \times Y \to Y; (\alpha,\ y) \longmapsto \alpha \vartriangle y, Y \times \mathfrak{U} \to Y; (y,\ \alpha) \longmapsto y \bullet \alpha.$ And $A \times Y \to Y; (a,\ y) \longmapsto a \cdot y, Y \times A \to Y; (y,\ a) \longmapsto y.a$. Define the module actions as: $(Y \oplus \{0\}) \times \mathfrak{U} \to Y \oplus$

$\{0\}; (y,\ 0) \bullet \alpha = (y \bullet \alpha,\ 0), \mathfrak{U} \times (Y \oplus \{0\}) \to Y \oplus \{0\};\quad \alpha \bullet (y,\ 0) = (\alpha \vartriangle y,\ 0)$. And $(A \oplus_T X) \times (Y \oplus \{0\}) \to Y \oplus \{0\}; (a,\ x).(y,\ 0) = (a.y,\ 0), (Y \oplus \{0\}) \times (A \oplus_T X) \to Y \oplus \{0\}; (y,\ 0) \circ (a,\ x) = (y.\ a,\ 0)$. We only need to show that the actions are compatible.

$$\begin{aligned} 1)\alpha.\ ((a,\ x).(y,\ 0)) &= \alpha.\ (a.y,\ 0) \\ &= (\alpha \vartriangle (a.y),\ 0) = ((\alpha \cdot a) \cdot y,\ 0) \\ &= ((\alpha \cdot a,\ \alpha \nabla x) \cdot (y,\ 0) = (\alpha \cdot (a,\ x)).(y,\ 0)\ . \end{aligned}$$

$$\begin{aligned} 2)((a,\ x) \cdot (y,\ 0)) \bullet \alpha &= (a \cdot y,\ 0) \bullet \alpha \\ &= ((a \cdot y) \bullet \alpha,\ 0)) = (a \cdot (y \bullet \alpha),\ 0) \\ &= (a,\ x) \cdot (y \bullet \alpha,\ 0) = (a,\ x) \cdot ((y,\ 0) \bullet \alpha)\ . \end{aligned}$$

$$\begin{aligned} 3)(\alpha.\ (y,\ 0)) \cdot (a,\ x) &= (\alpha \vartriangle y,\ 0) \cdot (a,\ x) \\ &= ((\alpha \vartriangle y).\ a,\ 0) = (\alpha \vartriangle (y.\ a),\ 0)) \\ &= \alpha.\ (y.\ a,\ 0) = \alpha.\ ((y,\ 0) \circ (a,\ x)) \end{aligned}$$

■

Proposition 2.4 *Let* $M \oplus N$ *be a Banach* $A \oplus_T X - \mathfrak{U}$*-bimodule, then* M *is a Banach* $A - \mathfrak{U}$*-bimodule.*

Proof. Consider the map $Q_M : M \oplus N \to M; (m,\ n) \longmapsto m$ and define the module actions as: $M \times \mathfrak{U} \to M; (m,\ n) \longmapsto m \cdot \alpha = Q_M((m,\ 0) \bullet \alpha), \mathfrak{U} \times M \to M; (\alpha,\ m) \longmapsto \alpha \circ m = Q_M(\alpha \cdot (m,\ 0\ M \times A \to M; (m,\ a) \longmapsto m.a = Q_M((m,\ 0) \circ (a,\ 0))$ and $A \times M \to A; (a,\ m) \longmapsto a \bullet m = Q_M((a,\ 0) \cdot (m, 0))$ ■

Proposition 2.5 *Let* M *be a Banach* $A - \mathfrak{U}$*-module and* N *be a Banach* $X - \mathfrak{U}$*-bimodule,* $M \oplus N$ *is a Banach* $A \oplus_T X - \mathfrak{U}$*-bimodule.*

Proof. Given module actions on $M \oplus N$ as follows:

$(M \oplus N) \times \mathfrak{U} \to M \oplus N; (m,\ n) \cdot \alpha = (m.\alpha,\ n\nabla\alpha), \mathfrak{U} \times (M \oplus N) \to M \oplus N;\quad \alpha \bullet (m,\ n) = (\alpha \bullet m,\ \alpha \vartriangle m), (M \oplus N) \times (A \oplus_T X) \to (M \oplus N); (m,\ n) \cdot (a,\ x) = (m\ \alpha,\ n.\ T(a,\ 0\ (A \oplus_T X) \times (M \oplus N) \to M \oplus N; (a,\ x) \bullet (m,\ n) = (a \bullet m,\ T(a,\ 0) \odot n)$.

■

Proposition 2.6 *For each* $(f,\ g) \in M^* \oplus N^*, (a,\ x) \in A \oplus_T X, (m,\ n) \in M \oplus N$ *we have* $(f,\ g) \cdot (a,\ x) = (f \cdot a,\ g \cdot T(a,\ 0))$ *and* $(a,\ x).(f,\ g) = (a.f,\ T(a,\ 0).g)$.

Proof.

$$\begin{aligned} \langle (f,\ g).(a,\ x),\ (m,\ n) \rangle &= \langle (f,\ g),\ (a,\ x) \bullet (m,\ n) \rangle \\ &= \langle (f,\ g),\ (a \circ m,\ T(a,\ 0) \odot n) \rangle \\ &= \langle f,\ a \circ m \rangle + \langle g,\ T(a,\ 0) \odot n \rangle \\ &= \langle f.a,\ m \rangle + \langle g.T(a,\ 0),\ n \rangle \\ &= \langle (f.a,\ g.T(a,\ 0\ (m,\ n))) \end{aligned}$$

■

Proposition 2.7 *If* N *is a Banach* $X - \mathfrak{U}$*- bimodule, then is a Banach* $A - \mathfrak{U}$*-bimodule.*

Proof. The module actions are defined as follow:

$A \times N \to N; a.n = T(a,\ 0) \odot n$ and $N \times A \to N; n \bullet a = n.T(a,\ \odot)$. ■

3 MODULE AMENABILITY

Lemma 3.1 *$D \in Z_{\mathfrak{U}}(A \oplus_T X,\ M^* \oplus N^*)$ if and only if there are $D_1 \in Z_{\mathfrak{U}}(A,\ M^*), D_3 \in Z_{\mathfrak{U}}(X,\ N^*), R \in Z_{\mathfrak{U}}(A,\ N^*)$ and linear map $D_2 : X \to M^*$ such that*

1) $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D_3(x))$,
2) $D_2(a \bullet x) = a \cdot D_2(x)$,
3) $D_2(x \circ a) = D_2(x) \cdot a,$
4) $R(bd) = R(b) \cdot T(d,\ 0) + T(b,\ 0) \cdot R(d) = R(b).d + b.R(d)$,
5) $D_2(T(ab,\ 0)) = 0,$
6) $D_3(a.x) = T(a,\ 0).D_3(x)$,
7) $D_3(x \circ a) = D_3(x).T(a,\ 0)$,
8) $D_3(T(ab,\ 0)) = 0.$

Proof. Suppose that $D \in Z_{\mathfrak{U}}(A \oplus_T X,\ M^* \oplus N^*)$ then there are $d_1 : A \oplus \tau X \to M^*$, $d_2 : A \oplus_T X \to N^*$ such that $\mathrm{D} = (d_1,\ d_2)$, Set

$$D_1 : A \to M^*; D_1(a) = d_1(a,\ 0),$$

$$D_2 : X \to N^*; D_2(x) = d_1(0,\ x),$$

$$D_3 : X \to N^*; D_3(x) = d_2(0,\ x), R : A \to N^*; R(a) = d_2(a,\ 0).$$

Now

$$\begin{aligned} D(a,\ x) &= (d_1,\ d_2)((a,\ 0) + (0,\ x)) = (d_1,\ d_2)(a,\ 0) + (d_1,\ d_2)(0,\ x) \\ &= (d_1(a,\ 0),\ d_2(a,\ 0)) + ((d_1(0,\ x),\ d_2(0,\ x)) \\ &= (d_1(a,\ 0) + d_1(0,\ x)) + (d_2(a,\ 0) + d_2(0,\ x)) \\ &= (D_1(a) + D_2(x),\ R(a) + D_3(x)), \end{aligned} \tag{1}$$

Now

$$\begin{aligned} D((a,\ x)(m,\ x')) &= D(am,\ a \cdot x' + x \circ m + T(am,\ 0)) \\ &= (D_1(am) + D_2(a \cdot x') + D_2(x \circ m) + D_2(T(am,\ 0)), R(am) + D_3(a.x') \\ &\quad + D_3(x \circ m) + D_3(T(am, 0))), \end{aligned} \tag{2}$$

since D is module derivation so

$$\begin{aligned} D((a,\ x)(m,\ x')) &= D(a,\ x) \cdot (m,\ x') + (a,\ x) \cdot D(m,\ x') \\ &= (D_1(a) + D_2(x),\ R(a) + D_3(x)).(m,\ x') \\ &\quad + (a,\ x).(D_1(m) + D_2(x),\ R(m) + D_3(x')) \\ &= ((D_1(a) \cdot m + D_2(x)) \cdot m + a \cdot D_2(x') + D_2(x).m,\ R(a) \cdot T(m,\ 0) \\ &\quad + T(a,\ 0).R(m) + D_3(x).T(m,\ 0) + T(a,\ 0).D_3(x')). \end{aligned} \tag{3}$$

In 3, 2 Take $x = x' = 0$ to get $D_1 \in Z_{\mathfrak{U}}(A,\ M^*)$, (5), (4) and (8). Take $\mathrm{a} = 0$ to get (3) and (6). Take $\mathrm{m} = 0$ to get (2), (7). And in a similar way we can get other parameters. Conversely is in a same way. ■

Corollary 3.2 *Let $X = \{0\}$ and D, D_1 and R be as in perivious lemma, then $D = \delta_{(f,g)}$ if and only if $D_1 = \delta_f$ and $g = \overline{\delta}_g$. Where $\overline{\delta}_g(a) = gT(a,\ 0) - T(a,\ 0) \cdot g$.*

Proof. Since $X = \{0\}$ and $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D(x))$ so $D(a,\ 0) = (D_1(a),\ R(a))$. If $D = \delta_{(f,g)}$ then

$$\begin{aligned} D(a,\ 0) &= \delta_{(f,g)}(a,\ 0)\\ &= (f,\ g) \cdot (a,\ 0) - (a,\ 0) \cdot (f,\ g)\\ &= (f \cdot a,\ g \cdot T(a,\ 0)) - (a \cdot f,\ T(a,\ 0) \cdot g)\\ &= (f \cdot a - a \cdot f,\ g \cdot T(a,\ 0) - T(a,\ 0) \cdot g) = (\delta_f(a),\ \overline{\delta}_g(a))\ . \end{aligned}$$

So $D_1 = \delta_f$ and $R = \overline{\delta}_g$. Conversely

$$\begin{aligned} D(a,\ 0) &= (D_1(a),\ R(a))\\ &= (\delta_f(a),\ \overline{\delta}_g(a))\\ &= (f \cdot a - a \cdot f,\ g \cdot T(a,\ 0) - T(a,\ 0) \cdot g)\\ &= (f,\ g) \cdot (a,\ 0) - (a,\ 0) \cdot (f,\ g)\\ &= \delta_{(f,g)}(a,\ 0)\ . \end{aligned}$$

■

Theorem 3.3 *The module amenability of $A \oplus_T X$ implies module amenability of A. Moreover if $T(A,\ 0) = X$, then X is also module amenable.*

Proof. Assume that M, N are Banach $A - \mathfrak{U}$-bimodule and Banach $X - \mathfrak{U}$-bimodule respectively. Let $D_1: A \to M^*$ and $D_2: X \to N^*$ be module derivations. By Proposition 2.5, $M \oplus N$ is a Banach $A \oplus_T X - \mathfrak{U}$- bimodule. Define $D' : A \oplus_T X \to M^* \oplus N^*; D'(a,\ x) = (D_1(a), D_2(T(a,0))$. Now

$$\begin{aligned} D'((a,\ x)(m,\ x')) &= D'\ (am,\ a.x' + x \circ m + T(am,\ 0))\\ &= (D_1(am), D_2(T(am, 0)))\\ &= (D_1(a).m + a.D_1(m),\ D_2(T(a,\ 0)).T(m,\ 0) + T(a,\ 0).D_2(T(m,\ 0)))\\ &= (D_1(a),\ D_2(T(a,\ 0))).(m,\ x') + (a,\ x).(D_1(m),\ D_2(T(m,\ 0)))\\ &= D'(a,\ x).(m,\ x') + (a,\ x).D'(m,\ x)). \end{aligned}$$

Also

$$\begin{aligned} D'(\alpha \cdot (a,\ x)) &= D'(\alpha \cdot a,\ \alpha \nabla x)\\ &= ((D_1(\alpha \cdot a),\ D_2(T(\alpha \cdot a,\ 0)))\\ &= ((\alpha \cdot D_1(a),\ \alpha \cdot D_2(T(a,\ 0)))\\ &= (\alpha \cdot D'(a,\ x)\ . \end{aligned}$$

And

$$\begin{aligned} D'((a,\ x) + (m,\ x') &= D'((a + m,\ x + x'))\\ &= (D_1(a + m),\ D_2(T(a + m),\ 0))\\ &= (D_1(a) + D_1(m),\ D_2(T(a,\ 0)) + D_2(T(m,\ 0))\\ &= (D_1(a),\ D_2(T(a,\ 0)) + (D_1(m),\ D_2(T(m,\ 0))\\ &= D'(a,\ x) + D'(m,\ x'). \end{aligned}$$

So D' is a module derivation. Sice $A \oplus_T X$ is module amenable, there exists $(f,\ g) \in M^* \oplus N^*$ such that $D' = \delta_{(f,g)}$. Thus

$$\begin{aligned} D'(a,\ x) &= \delta_{(f,g)}(a,\ x)\\ &= (f,\ g) \cdot (a,\ x) - (a,\ x) \cdot (f,\ g)\\ &= (f \cdot a + g \cdot T(a,\ 0)) - (a \cdot f,\ T(a,\ 0) \cdot g)\\ &= (f \cdot a - a \cdot f,\ g \cdot T(a,\ 0) - T(a,\ 0) \cdot g)\ . \end{aligned}$$

Consequently $D_1(a) = f \cdot a - a \cdot f$ i.e. $D_1 = \delta_f$ and $D_2(T(a,\ 0)) = \delta_g(T(a, 0))$. Since $T(A,\ 0) = X$, $D_2(x) = \delta_g(x)$ for all $x \in X$. ■

Theorem 3.4 *The module amenability of* A *implies module amenability of* $A \bigoplus_T \{0\}$.

Proof. Let $M \bigoplus N$ be a Banch $A \bigoplus_T X - \mathfrak{U}$-bimodule and $D: A \bigoplus_T X \to M^* \bigoplus N^*$ be a module derivation. By lemma 3.1, there are D_1, D_2, D_3 and R such that $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D_3(x))$. Since here $X = \{0\}$ so $D(a,\ 0) = (D_1(a),\ R(a))$. Module amenability of A implies there exist $f \in M^*$ such that $D_1 = \delta_f$ and since N^* is an A-U- bimodule and R is module derivation, there exist $g \in N^*$ such that $R = \delta_g$. Thus $D = \delta_{(f,g)}$. ■

Theorem 3.5 *If* $T(A,\ 0) = X$ *and* $A^2 = A$, *then the module amenability of* A *implies module amenability of* $A \bigoplus_T X$.

Proof. Let $M \bigoplus N$ be a Banch $A \bigoplus_T X - \mathfrak{U}$-bimodule and $D: A \bigoplus_T X \to M^* \bigoplus N^*$ be a module derivation. By lemma 3.1, there are D_1, D_2, D_3 and R such that $D(a,\ x) = (D_1(a) + D_2(x),\ R(a) + D_3(x))$. Since here $T(A,\ 0) = X$ and $A^2 = A$ so $D(a,\ x) = (D_1(a),\ R(a))$. Module amenability of A implies there exist $f \in M^*$ such that $D_1 = \delta_f$ and since N^* is an $A - \mathfrak{U}$-bimodule and R is module derivation, there exist $g \in N^*$ such that $R = \delta_g$. Thus $D = \delta_{(f,g)}$. ■

Example 3.6 *Let* $\mathbb{N}$ *be the set of positive integers. Consider* $S = (\mathbb{N},\ \vee)$ *with the maximum operation* $m \vee n = max\{m,\ n\}$, *then* S *is a amenable countable, abelian inverse semigroup with the identity 1. Clearly* $E_S = S$. *This semigroup is denoted by* $\mathbb{N}_\vee$. $l^1(\mathbb{N}_\vee)$ *is unital with unit* δ_1. *Since* $\mathbb{N}_\vee$ *is amenable and* $l^1(\mathbb{N}_\vee)$ *is unital so* $l^1(\mathbb{N}_\vee)$ *is module amenable (as an* $l^1(\mathbb{N}_\vee) - l^1(\mathbb{N}_\vee)$*)-bimodule. Define* $T: l^1(S) \times l^1(S) \to l^1(S)$; $T(\delta_x,\ \delta_y) = \begin{cases} \delta_x, & \delta_y = 0 \\ \delta_{x \vee y}, & \delta_y \neq 0. \end{cases}$ *Then* $l^1(\mathbb{N}_\vee) \bigoplus_T l^1(\mathbb{N}_\vee)$ *is module amenable.*

4 WEAK MODULE AMENABILITY

The Banach algebra A is called weak module amenable (as an $\mathfrak{U}$-bimodule), if $H_{\mathfrak{U}}(A,\ X) = \{0\}$, where X is a commutative $\mathfrak{U}$-submodule of A^*([2]).

Theorem 4.1 *The weak module amenability of* $A \bigoplus_T X$ *implies weak module amenability of* A. *In addition if* $T(A,\ 0) = X$ *then* X *is also weak module amenable.*

Proof. Assume that M, N are commutative $\mathfrak{U}$-submodule of A^* and X^*, respectively. we can show that $M \bigoplus N$ is a commutative $\mathfrak{U}$-submodule of $(A \bigoplus_T X)^*$. Let $D_1 \in Z_{\mathfrak{U}}(A,\ M)$ and $D_2 \in Z_{\mathfrak{U}}(X,\ N)$. Define $D : A \bigoplus_T X \to M \bigoplus N; D(a,\ x) = (D_1(a), D_2(T(a, 0)))$, it is easy to see that $D \in Z_{\mathfrak{U}}(A \bigoplus \tau X, M \bigoplus N)$. Since $A \bigoplus_T X$ is weak module amenable there is $(f,\ g) \in M \bigoplus N$ such that $D = \delta_{(f,g)}$ and

$$\begin{aligned} (D_1(a), D_2(T(a,\quad 0))) &= D(a,\quad x) \\ &= \delta_{(f,g)}(a,\ x) \end{aligned}$$

$$
\begin{aligned}
&= (f,\ g)\cdot(a,\ x) - (a,\ x)\cdot(f,\ g)\\
&= (f\cdot a - a\cdot f,\ g\cdot T(a,\ 0) - T(a,\ 0)\cdot g)\\
&= (\delta_f(a),\ \delta_g(T(a,\ 0)))
\end{aligned}
$$

Hence A, X are weak module amenable. ■

Theorem 4.2 *The weak module amenability of* A *implies the weak module amenability of* $A \oplus_T \{0\}$.

Proof. Suppose that $M \oplus N$ is a commutative Banach $\mathfrak{U}$-submodule of $(A \oplus \tau\{0\})^*$, and $D \in Z_{\mathfrak{U}}(A \oplus \tau\{0\},\ M \oplus N)$. Then M and N are commutative $\mathfrak{U}$-submodule of A^*. Since $D \in Z_{\mathfrak{U}}(A \oplus \tau\{0\},\ M \oplus N)$, by lemma 3.1 there are $D_1 \in Z_{\mathfrak{U}}(A,\ M)$, and $R \in Z_{\mathfrak{U}}(A,\ N)$, such that $D(a,\ 0) = (D_1(a),\ R(a))$. Since A is weak module amenable so there are $m \in M$ and $n \in N$ such that $D_1 = \delta_m, R = \delta_n$, where $\delta_n(a) = a\cdot n - n \bullet a = T(a,\ 0) \circledcirc n - n. T(a,\ 0)$.

Now

$$
\begin{aligned}
D(a,\quad x) &= (D_1(a),\quad R(a))\\
&= (\delta_m(a),\ \delta_n(a))\\
&= (a\cdot m - a \bullet m,\ T(a,\ 0) \circledcirc n - n.\ T(a,\ 0))\\
&= (a,\ 0)\cdot(m,\ n) - (m,\ n)\circ(a,\ 0)\\
&= \delta_{(m,n)}(a,\ 0)\ .
\end{aligned}
$$

■

Theorem 4.3 *If* $T(A,\ 0) = X$ *and* $A^2 = A$, *then the weak module amenability of* A *implies the weak module amenability of* $A \oplus_T X$.

Proof. The proof is as above theorem. ■

Example 4.4 *Let* $S = \mathbb{N}_\vee$ *be as in Example 3.6, since* $l^1(S)$ *is* $l^1(S) - l^1(S) -$ *module and* $l^1(S)$ *is weak module amenable. Let* $T\colon l^1(S) \times l^1(S) \to l^1(S)$ *have the properties as above theorems, then* $l^1(S) \oplus_T l^1(S)$ *is weak module amenable.*

5 MODULE APPROXIMATE AMENABILITY

Let A be as above, then A is module approximately amenable (as an $\mathfrak{U}$- bimodule), if for any commutative Banach $A - \mathfrak{U}$-bimodule X, each module derivation $D\colon A \to X^*$ is approximately inner.

A derivation $D\colon A \to X$ is said to be approximately inner if there exists a net $(x_i)_i \subseteq X$ such that $D(a) = \lim_i(a\cdot x_i - x_i\cdot a), a \in A$.([10]) .

Lemma 5.1 *Let* D_1, R, D_3 *and* D_2 *are such as in the Lemma 3.1, and* $D(a,\ b) = (D_1(a) + D_2(b),\ R(a) + D_3(b))$. *If* $T(A,\ 0) = X$ *and* $A^2 = A$ *then: D is approximately inner if and only if* D_1 *and* R *are approximately inner.*

Proof. Assume that M is a commutative $A - \mathfrak{U}$-bimodule and also N is commutative $X - \mathfrak{U}$-bimodule, then $M \oplus N$ is a commutative $A \oplus_T X - \mathfrak{U}$-bimodule. Let D be approximately inner so there is $(f_i,\ g_i)_i \subseteq M^* \oplus N^*$ such that

$$\begin{aligned} D(a,\quad x) &= T(a',\quad 0)\\ &= \lim_i((a,\ x)\cdot(f_i,\ g_i)-(f_i,\ g_i)\cdot(a,\ x))\\ &= \lim_i((a\cdot f_i,\ T(a,\ 0)\cdot g_i)-(f_i\cdot a,\ g_i.T(a,\ 0)))\\ &= \lim_i(a\cdot f_i-f_i\cdot a,\ T(a,\ 0)\cdot g_i-g_i\cdot T(a,\ 0)), \end{aligned}$$

i.e. $D(a)=\lim_i(a\cdot f_i-f_i\cdot a)$ and $R(a)=\lim_i(T(a,\ 0)\cdot g_i-g_i\cdot T(a,0))$.

Conversely, let $D_1(a)=\lim_{i\in I}(a\cdot f_i-f_i\cdot a)$ and $R(a)=\lim_{j\in J}(T(a,\ 0)\cdot g_j-g_j\cdot T(a,0))$

Since the index sets $(I,\ \leq),(J,\ \leq)$ are ordered sets, so the set $\Lambda=I\times J=\{(i,\ j)\ :\ i\in I,\ j\in J\}$ is ordered as follows

$$(i,\ j)\leq(i',\ j')\Leftrightarrow(i\leq i',\ j\leq j).$$

For $\lambda=(i,\ j)\in\Lambda$ set $t_\lambda=(f_i,\ g_j)$. Let $\epsilon>0$ be given. Since $D_1(a)=\lim_i(a\cdot f_i-f_i\cdot a)$ and $R(a)=\lim_j(T(a,\ 0)\cdot g_j-g_j\cdot T(a,0))$ there are $i_0\in I,j_0\in J$ such that

1) For all $i\geq i_0,\| D_1(a)-(a\cdot f_i-f_i\cdot a)\|\leq\frac{\epsilon}{3}$.

2) For all $j\geq j_0,\| R(a)-(T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\leq\frac{\epsilon}{3}$.

Now set $\lambda_0=(i_0,\ j_0)$, then for all $\lambda\geq\lambda_0$, since $D(a,\ T(a',\ 0))=(D_1(a),\ R(a))$, we have

$$\begin{aligned} \| D(a,\quad x)-((a,\quad x)\cdot t_\lambda-t_\lambda\cdot(a,\quad x))\| &=\| D(a,\quad x)-((a,\quad x)\cdot(f_i,\ g_j)-(f_i,\quad g_j)\cdot(a,\quad x))\|\\ &=\| D(a,\ x)-(a\cdot f_i-f_i\cdot a,\ T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\\ &=\|(D_1(a),\ R(a))-(a\cdot f_i-f_i\cdot a,\ T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\\ &=\|((D_1(a)-(a\cdot f_i-f_i\cdot a)),R(a)-(T(a,\ 0)\cdot g_j-g_j\cdot T(a,0)))\|\\ &\leq\| D_1(a)-(a\cdot f_i-f_i\cdot a)\|+\| R(a)-(T(a,\ 0)\cdot g_j-g_j\cdot T(a,\ 0))\|\\ &<\epsilon. \end{aligned}$$

Hence $D(a,\ x)=\lim_\lambda((a,\ x)\cdot t_\lambda-t_\lambda(a,\ x$ where $x=T(a',\ 0)$ i.e. D is approximately inner. ■

Theorem 5.2 *If $A\oplus\tau X$ is module approximately amenable then A is module approximately amenable. Furthermore, if $T(A,\ 0)=X$ also X is module approximately amenable.*

Proof. In an argument as in the proof of Theorem 3.3 and the application, the usage of above lemma. ■

Theorem 5.3 *If $T(A.O)=X$ and $A^2=A$ then the module approximate amenability of A implies the module approximate amenability of $A\oplus_T X$.*

Proof. Let $M\oplus N$ be a commutative $A\oplus_T X-\mathfrak{U}$-bimodule and $D\in Z_{\mathfrak{U}}(A\oplus\tau X,M^*\oplus N^*)$. There are $D_1\in Z_{\mathfrak{U}}(A,\ M^*),D_3\in Z_{\mathfrak{U}}(X,\ N^*),R\in Z_{\mathfrak{U}}(A,\ N^*)$ and $D_2:X\to N^*$ such that $D(a,\ x)=(D_1(a)+D_2(x),\ R(a)+D(x))$ and since $T(A,\ 0)=X$ and $A^2=A$ we have $D(a,\ x)=(D_1(a),\ R(a))$. Since A,X are module approximate amenable, so D_1 and R are approximatly inner. Thus by the above lemma, D is approximately inner. ■

Example 5.4 *Let S be an amenable inverse semigroup such that the set of idempotents E_S be equal to S and $l^1(S)$ has approximately unit. Since S is amenable, $l^1(S)$ is module approximately amenable, [10]. Also $l^1(S)$ is $l^1(S) - l^1(S)$-bimodule, thus $l^1(S) \oplus_T l^1(S)$ is module approximately amenable. Where $T: l^1(S) \times l^1(S) \to l^1(S)$ is defined by*

$$T(\delta_x,\ \delta_y) = \begin{cases} \delta_x, & \delta_y = 0 \\ \delta_{xy}, & \delta_y \neq 0. \end{cases}$$

6 CONCLUSIONS

The module amenability of $A \oplus_T X$ implies module amenability of A and The module amenability of A implies module amenability of $A \oplus_T \{0\}$. Also If $T(A,\ 0) = X$ and $A^2 = A$, then the module amenability of A implies module amenability of $A \oplus_T X$. meanwhile, The weak module amenability of $A \oplus_T X$ implies weak module amenability of A. On the contrary, if $T(A,\ 0) = X$ and $A^2 = A$, then the weak module amenability of A implies the weak module amenability of $A \oplus_T X$.

Considering approximately, if $A \oplus \tau X$ is module approximately amenable then A is module approximately amenable. On the contrary, if $T(A,\ 0) = X$ and $A^2 = A$ then the module approximate amenability of A implies the module approximate amenability of $\oplus_T X$. For example, we have S be an amenable inverse semigroup such that the set of idempotents E_S be equal to S and $l^1(S)$ has approximately unit. Since S is amenable, $l^1(S)$ is module approximately amenable.

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