MATHEMATICA MONTISNIGRI
Vol XXVII (2013) 9-37

ON MULTIPLIERS OF SOME NEW ANALYTIC $M_\alpha^{p,q}$, $M^{p,\infty,\alpha}$ AND $M^{\infty,p,\alpha}$ TYPE SPACES AND RELATED SPACES ON THE UNIT POLYDISC

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Abstract. We study certain new spaces of coefficient multipliers of new analytic Lizorkin-Triebel type spaces $M_\alpha^{p,q}$, $M^{p,\infty,\alpha}$ and $M^{\infty,p,\alpha}$ and related analytic spaces in the unit polydisc with some restriction on parameters. Our results extend some previously known assertions on coefficient multipliers of classical analytic Bergman A_α^p and analytic weighted Hardy H_α^p type spaces in the unit disk. Many results are new even in onedimensional case of unit disk. We define and study also spaces of multipliers of some new analytic Besov type spaces in polydisk.

1. Introduction

The goal of this paper is to continue the investigation of spaces of coefficient multipliers of analytic $M_\alpha^{p,q}$, and related $F_\alpha^{p,q}$ Lizorkin - Triebel type spaces in the unit polydisc including $q=\infty$ and $p=\infty$ limit cases. This probably started before in [16], [17]. These $F_\alpha^{p,q}$ and $M_\alpha^{p,q}$ type spaces including limit $p=\infty$ and $q=\infty$ cases serve as a very natural extension of the classical Hardy and Bergman spaces in the unit polydisc simultaneously. Spaces of Hardy and Bergman type in higher dimension were studied intensively by many authors during past several decades, see for example [21], [5], [7], [12], [13], [14] and references therein. The investigation in this new direction of $F_\alpha^{p,q}$, and related $M_\alpha^{p,q}$ type classes in the unit polydisc was

2000 *Mathematics Subject Classification.* Primary 30H10.
Key words and phrases. Multipliers, polydisc, analytic functions, analytic Bergman type, mixed norm, Lizorkin-Triebel type spaces in polydisk.

2 Supported by MNTR Serbia, Project 174017.

started in recent papers of the second author, see [16], [17], [18]. We note that complete analogues of these classes and their pointwise multipliers in the unit ball were studied also in recent papers of Ortega and Fabrega, see [12], [13], [14] and references therein. See also [1], [7], [20] for some other properties of these new analytic $F_\alpha^{p,q}$, $M_\alpha^{p,q}$ type classes and related spaces in the unit disk or subframe and expanded disk. Below we list notations and definitions which are needed and in the next section we formulate and prove main results of this note. The paper will be devoted mainly to the study of multipliers of $M_\alpha^{p,q}$ classes including limit cases $q=\infty$ and $p=\infty$ and related $T_s^{p,q}$ spaces.

Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ be the unit disc in $\mathbb{C}$, $\mathbb{T}=\partial\mathbb{D}=\{z\in\mathbb{C}:|z|=1\}$, $\mathbb{D}^n$ is the unit polydisc in $\mathbb{C}^n$, $\mathbb{T}^n\subset\partial\mathbb{D}^n$ is the distinguished boundary of $\mathbb{D}^n$, $\mathbb{Z}_+=\{n\in\mathbb{Z}:n\geq 0\}$, $\mathbb{Z}_+^n$ is the set of all multi indexes and $I=[0,1)$.

We use the following notation: for $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ and $k=(k_1,\dots,k_n)\in\mathbb{Z}_+^n$ we set $z^k=z_1^{k_1}\cdots z_n^{k_n}$ and for $z=(z_1,\dots,z_n)\in\mathbb{D}^n$ and $\gamma\in\mathbb{R}$ we set $(1-|z|)^\gamma=(1-|z_1|)^\gamma\cdots(1-|z_n|)^\gamma$ and $(1-z)^\gamma=(1-z_1)^\gamma\cdots(1-z_n)^\gamma$. Next, for $z\in\mathbb{R}^n$ and $w\in\mathbb{C}^n$ we set $wz=(w_1z_1,\dots,w_nz_n)$. Also, for $k\in\mathbb{Z}_+^n$ and $a\in\mathbb{R}$ we set $k+a=(k_1+a,\dots,k_n+a)$. For $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ we set $\overline{z}=(\overline{z}_1,\dots,\overline{z}_n)$. For $k\in(0,+\infty)^n$ we set $\Gamma(k)=\Gamma(k_1)\cdots\Gamma(k_n)$.

The Lebesgue measure on $\mathbb{C}^n\cong\mathbb{R}^{2n}$ is denoted by $dV(z)$, normalized Lebesgue measure on $\mathbb{T}^n$ is denoted by $d\xi=d\xi_1\dots d\xi_n$ and $dR=dR_1\dots dR_n$ is the Lebesgue measure on $[0,+\infty)^n$.

The space of all functions holomorphic in $\mathbb{D}^n$ is denoted by $H(\mathbb{D}^n)$. Every $f\in H(\mathbb{D}^n)$ admits an expansion $f(z)=\sum_{k\in\mathbb{Z}_+^n}a_kz^k$. For $\beta>-1$ the operator of fractional differentiation is defined by

$$D^\beta f(z)=\sum_{k\in\mathbb{Z}_+^n}\frac{\Gamma(k+\beta+1)}{\Gamma(\beta+1)\Gamma(k+1)}a_kz^k,\ z\in\mathbb{D}^n. \tag{1}$$

For $f \in H(\mathbb{D}^n)$, $0 < p < \infty$ and $r \in I^n$ we set

$$M_p(f,r) = \left(\int_{\mathbb{T}^n} |f(r\xi)|^p d\xi\right)^{1/p}, \tag{2}$$

with the usual modification to include the case $p = \infty$. For $0 < p \le \infty$ we have analytic Hardy classes in the polydisc:

$$H^p(\mathbb{D}^n) = \{f \in H(\mathbb{D}^n) : \|f\|_{H^p} = \sup_{r \in I^n} M_p(f,r) < \infty\}. \tag{3}$$

For $n = 1$ these spaces are well studied. The topic of multipliers of Hardy spaces in polydisc is relatively new, see for example [21], [22]. These spaces are Banach spaces for $p \ge 1$ and complete metric spaces for all other positive values of p. Also, for $0 < p \le \infty$, $0 < q < \infty$ and $\alpha > 0$ we have mixed (quasi) norm spaces, defined below.

$$A_\alpha^{p,q}(\mathbb{D}^n) = \left\{f \in H(\mathbb{D}^n) : \|f\|^q_{A_\alpha^{p,q}} = \int_{I^n} M_p^q(f,R)(1-R)^{\alpha q - 1} dR < \infty\right\}. \tag{4}$$

If we replace the integration by I^n above by integration by unit interval I then we get other similar to these $F_\alpha^{p,q}$ analytic spaces but on subframe and we denote them by $B_\alpha^{p,q}$ These spaces are Banach spaces for cases when both p and q are bigger than one, and they are complete metric spaces for all other values of parameters. We refer the reader for these classes in the unit ball and the unit disk to [21], [12], [13], [14] and references therein. Multipliers between $A_\alpha^{p,q}$ spaces on the unit disc were studied in detail in [8].

As is customary, we denote positive constants by C, sometimes we indicate dependence of a constant on a parameter by using a subscript, for example C_q.

We define now the main objects of this paper.

For $0 < p, q < \infty$ and $\alpha > 0$ we consider Lizorkin - Triebel spaces $F^{p,q}_{\alpha}(\mathbb{D}^n) = F^{p,q}_{\alpha}$ consisting of all $f \in H(\mathbb{D}^n)$ such that

$$\|f\|^p_{F^{p,q}_{\alpha}} = \int_{\mathbb{T}^n} \left(\int_{I^n} |f(R\xi)|^q (1-R)^{\alpha q - 1} dR \right)^{p/q} d\xi < \infty. \tag{5}$$

It is not difficult to check that those spaces are complete metric spaces, if $\min(p, q) \geq 1$ they are Banach spaces. If we replace the integration by I^n above by integration by unit interval I then we get other similar to these $F^{p,q}_{\alpha}$ analytic spaces, but on subframe and we denote them by $T^{p,q}_{\alpha}$. If we replace the integration by T^n above by integration by unit circle T then we get other similar to these $F^{p,q}_{\alpha}$ analytic spaces, but on expanded disk and we denote them by $M^{p,q}_{\alpha}$.

All these $T^{p,q}_{\alpha}$ and $M^{p,q}_{\alpha}$ classes are new (including limit cases when one of indexes is equal to ∞) and so far there are only several papers in literature devoted to the study of their properties. It is not difficult to check that those spaces are complete metric spaces, if $\min(p, q) \geq 1$ they are Banach spaces. We note that this scale of spaces includes, for $p = q$, weighted Bergman spaces $A^p_{\alpha} = F^{p,p}_{\frac{\alpha+1}{q}}$, see [5] for a detailed account of these spaces. On the other hand, for $q = 2$ these spaces coincide with Hardy - Sobolev spaces namely $H^p_{\alpha} = F^{p,2}_{\frac{\alpha+1}{2}}$, for this well known fact see [21], [1] and references therein. Finally, for $\alpha \geq 0$ and $\beta \geq 0$ we set

$$A^{\infty,\infty}_{\alpha,\beta}(\mathbb{D}^n) = \{f \in H(\mathbb{D}^n) : \|f\|_{A^{\infty,\infty}_{\alpha,\beta}} = \sup_{r \in I^n} (M_{\infty}(D^{\alpha} f, r))(1-r)^{\beta} < \infty\}. \tag{6}$$

This space is a Banach space. For all positive values of p and s we introduce the following three new spaces. We note first replacing q by ∞ in a usual way we will arrive at some other spaces (limit case of $F^{p,q}_{\alpha}$ classes). The limit space case $F^{p,\infty,s}(\mathbb{D}^n)$ is defined as a space of all analytic functions f in the polydisc such that the function $\phi(\xi) = \sup_{r \in I^n} |f(r\xi)|(1-r)^s$, $\xi \in \mathbb{T}^n$ is in $L^p(\mathbb{T}^n, d\xi)$.

If we replace the "integration" by I^n above by integration by unit interval I then we get other similar to these analytic spaces $F^{p,\infty,s}(\mathbb{D}^n)$ but on subframe and we denote them by $T^{p,\infty}_{\alpha}$. If we replace the integration by torus T^n above by integration by unit circle T then we get other similar to these analytic spaces $F^{p,\infty,s}(\mathbb{D}^n)$but on expanded disk and we denote them by $M^{p,\infty,\alpha}$. Finally, the limit space case $A^{p,\infty,s}(\mathbb{D}^n)$, $s, p \in (0,\infty)$, is the space of all analytic functions f in the polydisc such that

$$\sup_{r\in I^n} M_p(f,r)(1-r)^s < \infty.$$

These last spaces the so called weighted Hardy spaces are well studied by many authors for many decades see for this for example [5] and many references there.

Obviously the limit $F^{p,\infty,s}$ space is embedded in the last space we defined. This simple observation concerns also $T^{p,\infty,\alpha}$, $M^{p,\infty,\alpha}$ and will be used in this note. It can be checked by standard manner that these four classes of spaces are Banach spaces for all $p \geq 1$ and they are complete metric spaces for all other positive values of p.

The following definition of coefficient multipliers is well known in the unit disk. We provide a natural extension to the polydisc setup.

Definition 1. Let X and Y be quasi normed subspaces of $H(\mathbb{D}^n)$. A sequence $c = \{c_k\}_{k\in\mathbb{Z}_+^n}$ is said to be a coefficient multiplier from X to Y if for any function $f(z) = \sum_{k\in\mathbb{Z}_+^n} a_k z^k$ in X the function $h = M_c f$ defined by $h(z) = \sum_{k\in\mathbb{Z}_+^n} c_k a_k z^k$ is in Y. The set of all multipliers from X to Y is denoted by $M_T(X,Y)$.

The problem of characterizing the space of multipliers (pointwise multipliers and coefficient multipliers) between various spaces of analytic functions is a classical problem in complex function theory, there is vast literature on this subject, see [4], [5], [8], [9] and references therein. Note our results do not provide characterizations but all are in higher dimension for certain new classes of functions which were never considered before and they present obvious interest from our point of view. In this paper at the same time we are looking for some extensions of some already known theorems, namely we are interested in spaces of multipliers acting into analytic

Lizorkin-Triebel $F^{p,q}_\alpha$, $T^{p,q}_\alpha$, $M^{p,q}_\alpha$ spaces and all related spaces we defined above in the unit polydisc and from these spaces into certain well studied classes like mixed norm spaces, Bergman spaces and Hardy spaces. We note that the analogue of this problem of description of spaces of multipliers in $\mathbb{R}^n$ for various functional spaces in $\mathbb{R}^n$ was considered previously by various authors in recent decades, for these we refer the reader to [23]. We note the study of spaces of multipliers of nonanalytic classes of functions is also a large area of research.

2. On coefficient multipliers of analytic Lizorkin - Triebel type $M^{p,q}_\alpha$, $F^{p,q}_\alpha$ and $T^{p,q}_\alpha$ spaces in the unit polydisc

We start this section with several lemmas which will play an important role in the proofs of all our results. We note that these assertions can serve as direct natural extensions of previously known one dimensional results to the case of several complex variables. The following lemma in dimension one is well-known, see [16], [5], [6] and references therein. This is a several variables version of the so called Littlewood - Paley formula, see [6], which allows to pass from integration on the unit circle to integration over unit disk. We omit a simple proof which follows from a straightforward technical computation based on orthonormality of the trigonometric system, similarly to the classical planar case.

Lemma 1 ([17]). *For $f, g \in H(\mathbb{D}^n)$, $r \in I^n$ and $\alpha > 0$ we have*

$$\int_{\mathbb{T}^n} f(r\bar{t})g(rt)dt = (2\alpha)^n \prod_{j=1}^n r_j^{-2\alpha} \times \tag{7}$$

$$\times \int_0^{r_1} \cdots \int_0^{r_n} \int_{\mathbb{T}^n} D^{\alpha+1} g(R\xi) f(r\bar{\xi}) \prod_{j=1}^n (r_j^2 - R_j^2)^\alpha R_1 \cdots R_n dR d\xi.$$

The first part of the following lemma was stated in [17]. This lemma provides direct connection between mixed norm spaces and standard A^p_α Bergman classes in higher dimensions. A detailed proof of the first part can be found in the cited paper of the second author and the proof of the second part is just a modification of that proof. We again omit details refereing the reader to [17].

Lemma 2. *Let* $0 < \max(p,q) \le s \le 1$ *and* $\alpha > 0$. *Then we have*

$$\left(\int_{\mathbb{D}^n} |f(w)|^s (1-|w|)^{s(\alpha+\frac{1}{p})-2} dV(w)\right)^{1/s} \le C\|f\|_{F_\alpha^{p,q}}, \qquad f \in F_\alpha^{p,q}(\mathbb{D}^n), \tag{8}$$

$$\left(\int_{\mathbb{D}^n} |f(w)|^s (1-|w|)^{s(\alpha+\frac{1}{p})-2} dV(w)\right)^{1/s} \le C\|f\|_{A_\alpha^{p,q}}, \qquad f \in A_\alpha^{p,q}(\mathbb{D}^n). \tag{9}$$

The following lemma is crucial for all proofs of necessity of multiplier conditions in our main results. It provides explicit estimates of the denominator of the Bergman kernel in various (quasi) norms in polydisc which appear in this note. Again these estimates are known and can be found in [16], [17], [18]. Let us note that the Bergman kernel in polydisc is a product of n one dimensional Bergman kernels. This fact often allows one to reduce calculations in several variables to the already classical one dimensional case, see for example [5] and references therein.

Lemma 3 ([16])**.** *Let* $0 < p, q < \infty$ *and set*

$$f_R(z) = \frac{1}{(1-Rz)^{\beta+1}}, \ \beta > -1, \quad R \in I^n, \quad z \in \mathbb{D}^n. \tag{10}$$

Then we have the following partially known (quasi) norm estimates:

$$\|f_R\|_{A_\alpha^{p,q}(\mathbb{D}^n)} \le \frac{C}{(1-R)^{\beta-\alpha-1/p+1}}, \ \beta > \alpha - 1 + \frac{1}{p}, \tag{11}$$

$$\|f_R\|_{F_\alpha^{p,q}(\mathbb{D}^n)} \le \frac{C}{(1-R)^{\beta-\alpha-1/p+1}}, \ \beta > \alpha - 1 + \frac{1}{p}. \tag{12}$$

Lemma 4 ([16])**.** *Let* $0 < p, q < \infty$ *and set*

$$f_R(z) = \frac{1}{(1-Rz)^{\beta+1}}, \ \beta > -1, \quad R \in I^n, \quad z \in \mathbb{D}^n. \tag{13}$$

Then we have the following partially known (quasi) norm estimates:

$$\|f_R\|_{A^{p,\infty,\alpha}(\mathbb{D}^n)} \le \frac{C}{(1-R)^{\beta-\alpha-1/p+1}}, \ \beta > \alpha - 1 + \frac{1}{p}; \tag{14}$$

$$\|f_R\|_{F^{p,\infty,\alpha}(\mathbb{D}^n)} \leq \frac{C}{(1-R)^{\beta-\alpha-1/p+1}}, \ \beta > \alpha - 1 + \frac{1}{p}. \tag{15}$$

The following two lemmas are new even in case of unit disk.

Lemma 5. *Let* $0 < p, q < \infty$ *and set*

$$f_R(z) = \frac{1}{(1-Rz)^{\beta+1}}, \ \beta > -1, \ R \in I^n, \ z \in \mathbb{D}^n. \tag{16}$$

Then we have the following (quasi) norm estimates:

$$\|f_R\|_{T^{p,\infty,\alpha}(\mathbb{D}^n)} \leq \frac{C}{(1-R)^{\beta-\frac{\alpha}{n}-1/p+1}}, \ \beta > \frac{\alpha}{n} + 1/p - 1, \tag{17}$$

$$\|f_R\|_{T^{\infty,q,\alpha}(\mathbb{D}^n)} \leq \frac{C}{(1-R)^{\beta-\frac{\alpha}{n}+1}}, \ \beta > \frac{\alpha}{n} - 1, \tag{18}$$

$$\|f_R\|_{T^{p,q}_{\alpha}(\mathbb{D}^n)} \leq \frac{C}{(1-R)^{\beta-\frac{\alpha}{n}-1/p+1}}, \ \beta > \frac{\alpha}{n} + 1/p - 1. \tag{19}$$

Note that if we consider M type spaces then we always assume below $R \in I$.

Lemma 6. *Let* $0 < p, q < \infty$ *and set*

$$f_R(z) = \frac{1}{(1-Rz)^{\beta+1}}, \ \beta > -1, \ R \in I, \ z \in \mathbb{D}^n. \tag{20}$$

Then we have the following (quasi) norm estimates:

$$\|f_R\|_{M^{p,q}_{\alpha}(\mathbb{D}^n)} \leq \frac{C}{(1-R)^{(\beta-\alpha+1)n-1/p}}, \ \beta > \alpha - 1 + \frac{1}{np}, \tag{21}$$

$$\|f_R\|_{M^{\infty,q,s}(\mathbb{D}^n)} \leq \frac{C}{(1-R)^{(\beta-s+1)n}}, \ \beta > s - 1, \tag{22}$$

$$\|f_R\|_{M^{p,\infty,\alpha}(\mathbb{D}^n)} \leq \frac{C}{(1-R)^{(\beta-\alpha+1)n-1/p}}, \ \beta > \alpha - 1 + \frac{1}{np}. \tag{23}$$

The following lemma follows from Lemma 2, but we include it here to stress its importance for later proofs.

Lemma 7 ([5]). *If $0 < v \leq 1$, $q > \frac{1}{v} - 2$ and $t > 0$ then we have, for every $f \in H(\mathbb{D}^n)$:*

$$(24)\quad \int_{\mathbb{D}^n} |f(w)|^t (1-|w|)^q dV(w) \leq C \left(\int_{\mathbb{D}^n} |f(w)|^{vt} (1-|w|)^{2v-2+qv} dV(w) \right)^{1/v}.$$

The proof of this lemma shows that if we replace here $(1-|w|)^q$ by $(1-|w|r)^q$, where $r \in I^n$, then the integrand on the right hand side changes to

$$(1-|w|r)^{qv}(1-|w|)^{2v-2}|f(w)|^{vt},$$

and conditions on parameters in this estimate will be

$$\frac{1}{2} < v \leq 1.$$

The proof of this last statement is a modification of the well-known proof of Lemma 7, therefore we omit easy details. This will be used below by us in the proof of one of the main results of this paper taken from [17].

The first assertions we formulate provide conditions which are necessary, but not in general sufficient, in cases when all indexes are different in pairs of mixed norm spaces we consider in this note. By this we mean spaces of coefficient multipliers from $F^{p,q}_\alpha$ spaces to $A^{r,s}_\beta$ spaces in the polydisc, and conversely. Even these assertions are more general than those that are present in the literature, see [5], a large survey article [21] by V. Shvedenko and more recent work [2], [3], [4] by O. Blasco, J. L. Arregui and M. Pavlović.

Since the proof of our main theorem contains in some sense proofs of all preliminary easy observations which we put below before formulation of main theorem we provide only complete sketches of proofs of these observations.

Let $g \in H(\mathbb{D}^n)$, $g(z) = \sum_{k \in \mathbb{Z}^n_+} c_k z^k$. If $c = \{c_k\}_{k \in \mathbb{Z}^n_+}$ is a coefficient multiplier from $A^{p,q}_\alpha$ to $F^{t,s}_\beta$ where $t \leq s$, then the following condition holds:

$$(25)\quad \sup_{r \in I^n} M_t(D^m g, r)(1-r)^{\beta+m-\alpha-\frac{1}{p}+1} < \infty,$$

where m is any number satisfying $m > \alpha - \beta + \frac{1}{p} - 1$.

Moreover the last condition is a necessary condition for the following cases when we consider multipliers from $F^{p,q}_{\alpha}$ to $A^{t,s}_{\beta}$ or from $F^{p,q}_{\alpha}$ to $F^{t,s}_{\beta}$ for $t \leq s$.

However, we add additional conditions when we look at multipliers into Lizorkin-Triebel type spaces. These conditions appear due to the first embedding in the following chain

$$F^{t,s}_{\beta} \hookrightarrow A^{t,s}_{\beta} \hookrightarrow A^{t,\infty}_{\beta}$$

of embeddings. The first embeddings holds for $t \leq s$ and follows directly from Minkowski inequality. The second embedding is a well known classical fact and it holds for all strictly positive s, t and β, see, for example, [5], [12], [13].

In all of the above cases of necessity of condition (25) proof follows directly from Lemma 3 and the above embeddings in combination with the closed graph theorem which is always used in such situations, see the proof of theorem below as a typical example. We remark in addition that in the case of multipliers into Lizorkin-Triebel spaces we should use both embeddings, in the other case only the second one. We omit details referring readers to the proof of the first part of Theorem 1. We will see this necessary condition on a multiplier which we just mentioned holds also when $s \leq t$, moreover we will show that it is sharp in this case.

If $s = \infty$ and $\alpha = \gamma + r$, then the condition (25) is again necessary when considering multipliers from $F^{p,q}_{\gamma}$ into $A^{\infty,\infty}_{r,\beta}$. We again omit a proof which is a modification of arguments of the proof which we see below and which we will repeat several times. We continue this paper with series of necessary conditions, these are new assertions on limit case spaces $F^{p,\infty,\alpha}$, $T^{p,\infty,s}$, $M^{p,\infty,\gamma}$, $M^{\infty,p,\gamma}$ which as we indicated above are defined analogously to the $F^{p,q}_{\alpha}$ spaces, the only difference being replacement of inner (quasi) norm L^q by L^{∞} norm and integration interval or the interval by which the supremum is taken. We also constantly use in arguments an obvious fact that the weighted Hardy spaces $A^{p,\infty}_{\alpha}$ contain $F^{p,\infty,\alpha}$ classes in the polydisc and the easily visible modification of this fact when we replace the integration interval or the interval by which the supremum is taken.

Namely, we look for estimates on the rate of growth of a function g which represents a coefficient multiplier into the above described space. Again proofs are parallel to the one presented in the proof of Theorem 1.

We again assume that $g = \sum_{k \in \mathbb{Z}_+^n} c_k z^k$ is analytic in polydisc, by an estimate of the rate of growth of g we mean the following:

$$\sup_{r \in I^n} M_p(D^m g, r)(1-r)^\tau < \infty, \; p, \tau \in (0, \infty). \tag{26}$$

and

$$\sup_{r \in I} M_p(D^m g, r)(1-r)^\tau < \infty, \; p, \tau \in (0, \infty). \tag{27}$$

Note it is obvious if the first condition holds then the second one also holds with otherτ namely with $n\tau$. The condition (26) (or) and the one after it is typical and often a necessary condition for $\{c_k\}_{k \in \mathbb{Z}_+^n}$ to be a multiplier in many, many cases (not only for this paper) and in that case τ depends on parameters involved and can be explicitly specified for each pair of spaces separately using Lemma 3. We provide below several new results for spaces we study in this note. Note all these results here and below are new even in one dimensional case of unit disk.

Proposition 1. *Assume $0 < v < \infty$ and $0 < p < \infty$. If a sequence $\{c_k\}_{k \in \mathbb{Z}_+^n}$ is a multiplier from H^v to $T^{p,\infty,\gamma}$, then the function $g = \sum_{k \in \mathbb{Z}_+^n} c_k z^k$ satisfies condition (27) with $\tau = (m+1-1/v)n + \gamma$, $\tau_1 = (m+1-1/v)n$, $\tau_1 > 0$.*

Assume $0 < s < \infty$, $0 < p < \infty$, $0 < q < \infty$. If a sequence $\{c_k\}_{k \in \mathbb{Z}_+^n}$ is a multiplier from $F_\alpha^{s,q}$ to $T^{p,\infty,\gamma}$, then the function $g = \sum_{k \in \mathbb{Z}_+^n} c_k z^k$ satisfies condition (27) with $\tau = (m+1-1/s-\alpha)n + \gamma$, $\tau_1 = (m+1-1/s-\alpha)n$, $\tau_1 > 0$.

Assume $0 < s < \infty$, $0 < p < \infty$, $0 < q < \infty$. *If a sequence* $\{c_k\}_{k\in\mathbb{Z}_+^n}$ *is a multiplier from* $A_\alpha^{s,q}$ *to* $T^{p,\infty,\gamma}$, *then the function* $g = \sum_{k\in\mathbb{Z}_+^n} c_k z^k$ *satisfies condition (27) with* $\tau = (m+1-1/s-\alpha)n+\gamma$, $\tau_1 = (m+1-1/s-\alpha)n$, $\tau_1 > 0$.

We provide the proof of only one assertion the rest is similar and we leave that to reader.

Proof. Assume $\{c_k\}_{k\in\mathbb{Z}_+^n} \in M_T\left(F_\alpha^{s,q}(\mathbb{D}^n), T^{p,\infty,\gamma}(\mathbb{D}^n)\right)$. An application of the closed graph theorem gives $\|M_c f\|_{T^{p,\infty,\gamma}} \leq C\|f\|_{F_\alpha^{s,q}}$. Let $w \in \mathbb{D}^n$ and set $f_w(z) = \frac{1}{1-wz^{m+1}}$, $g_w = M_c f_w$. Then we have

$$D^m g_w = D^m M_c f_w = M_c D^m f_w = CM_c \frac{1}{(1-wz)^{m+1}}$$

which, together with the estimate from Lemma 3, gives

$$\|D^m g_w\|_{T^{p,\infty,\gamma}} \leq C \left\| \frac{1}{(1-wz)^{m+1}} \right\|_{F_\alpha^{s,q}} \leq \frac{C}{(1-|w|)^{m-\alpha-1/s+1}}. \tag{28}$$

From here we get what we need. $\square$

The analogue of the assertion we just proved for M - type spaces is the following.

Proposition 2. *Assume* $0 < v < \infty$ *and* $0 < p < \infty$. *If a sequence* $\{c_k\}_{k\in\mathbb{Z}_+^n}$ *is a multiplier from* H^v *to* $M^{p,\infty,\gamma}$, *then the function* $g = \sum_{k\in\mathbb{Z}_+^n} c_k z^k$ *satisfies condition (26) with* $\tau = m+1+\gamma-1/v$, $\tau_1 = m+1-1/v$, $\tau_1 > 0$.

Assume $0 < s < \infty$, $0 < p < \infty$, $0 < q < \infty$. *If a sequence* $\{c_k\}_{k\in\mathbb{Z}_+^n}$ *is a multiplier from* $F_\alpha^{s,q}$ *to* $M^{p,\infty,\gamma}$, *then the function* $g = \sum_{k\in\mathbb{Z}_+^n} c_k z^k$ *satisfies condition (26) with* $\tau = m+1+\gamma-1/s-\alpha$, $\tau_1 = m+1-1/s-\alpha$, $\tau_1 > 0$.

Assume $0 < s < \infty$, $0 < p < \infty$, $0 < q < \infty$. *If a sequence* $\{c_k\}_{k\in\mathbb{Z}_+^n}$ *is a multiplier from* $A_\alpha^{s,q}$ *to* $M^{p,\infty,\gamma}$, *then the function* $g = \sum_{k\in\mathbb{Z}_+^n} c_k z^k$ *satisfies condition (26) with* $\tau = m+1+\gamma-1/s-\alpha$, $\tau_1 = m+1-1/s-\alpha$, $\tau_1 > 0$.

We provide the proof of only one assertion the rest is similar and we leave that to reader.

Proof. Assume $\{c_k\}_{k\in\mathbb{Z}_+^n} \in M_T(F_\alpha^{s,q}(\mathbb{D}^n), M^{p,\infty,\gamma}(\mathbb{D}^n))$. An application of the closed graph theorem gives $\|M_c f\|_{M^{p,\infty,\gamma}} \le C\|f\|_{F_\alpha^{s,q}}$. Let $w \in \mathbb{D}^n$ and set $f_w(z) = \frac{1}{1-wz}$, $g_w = M_c f_w$. Then we have

$$D^m g_w = D^m M_c f_w = M_c D^m f_w = CM_c \frac{1}{(1-wz)^{m+1}},$$

which, together with the estimate from Lemma 3, gives

$$\|D^m g_w\|_{M^{p,\infty\gamma}} \le C \left\| \frac{1}{(1-wz)^{m+1}} \right\|_{F_\alpha^{s,q}} \le \frac{C}{(1-|w|)^{m-\alpha-1/s+1}}. \tag{29}$$

From here we get what we need. □

In next assertions we consider remaining "limit" cases of $M^{\infty,p,\gamma}$, $T^{\infty,p,\gamma}$ spaces. If the $T^{\infty,p,\gamma}$ quazinorm of $D^m g_R$, $R \in (0,1)$ is less or equal to $\frac{C}{(1-R)^\tau}$ for constant C then we say that g function satisfies condition (A), if the $M^{\infty,p,\gamma}$ quazinorm of $D^m g_R$, $R \in (0,1)$ is less or equal to $\frac{C}{(1-R)^\tau}$ for constant C, $R \in I^n$, then we say that g function satisfies condition (B).

Proposition 3. *Assume $0 < v < \infty$ and $0 < p < \infty$. If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from H^v to $T^{\infty,p,\gamma}$, then the function $g = \sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition (A) with $\tau = (m+1-1/v)n$, $\tau > 0$.*

Proposition 4. *Assume $0 < s < \infty$, $0 < p < \infty$, $0 < q < \infty$. If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $F_\alpha^{s,q}$ to $T^{\infty,p,\gamma}$, then the function $g = \sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition (A) with $\tau = (m+1-1/s-\alpha)n$, $\tau > 0$.*

Proposition 5. *Assume $0 < s < \infty$, $0 < p < \infty$, $0 < q < \infty$. If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $A_\alpha^{s,q}$ to $T^{\infty,p,\gamma}$, then the function $g = \sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition (A) with $\tau = (m+1-1/s-\alpha)n$, $\tau > 0$.*

We formulate three more propositions and then provide proofs.

Proposition 6. *Assume* $0 < v < \infty$ *and* $0 < p < \infty$. *If a sequence* $\{c_k\}_{k\in\mathbb{Z}_+^n}$ *is a multiplier from* H^v *to* $M^{\infty,p,\gamma}$, *then the function* $g = \sum_{k\in\mathbb{Z}_+^n} c_k z^k$ *satisfies condition (B) with* $\tau = (m+1-1/v)+\gamma$, $\tau > 0$.

Proposition 7. *Assume* $0 < s < \infty$, $0 < p < \infty$, $0 < q < \infty$. *If a sequence* $\{c_k\}_{k\in\mathbb{Z}_+^n}$ *is a multiplier from* $F_\alpha^{s,q}$ *to* $M^{\infty,p,\gamma}$, *then the function* $g = \sum_{k\in\mathbb{Z}_+^n} c_k z^k$ *satisfies condition (B) with* $\tau = (m+1-1/s-\alpha)+\gamma$, $\tau > 0$.

Proposition 8. *Assume* $0 < s < \infty$, $0 < p < \infty$, $0 < q < \infty$. *If a sequence* $\{c_k\}_{k\in\mathbb{Z}_+^n}$ *is a multiplier from* $A_\alpha^{s,q}$ *to* $M^{\infty,p,\gamma}$, *then the function* $g = \sum_{k\in\mathbb{Z}_+^n} c_k z^k$ *satisfies condition (B) with* $\tau = (m+1-1/s-\alpha)+\gamma$, $\tau > 0$.

Proofs of assertions of this and previous propositions are similar hence we restrict ourselves to the proof of last proposition considering as example only one case here. We consider only one case since proofs of other cases are similar.

Proof. Assume $\{c_k\}_{k\in\mathbb{Z}_+^n} \in M_T(F_\alpha^{s,q}(\mathbb{D}^n), M^{\infty,p,\gamma}(\mathbb{D}^n))$. An application of the closed graph theorem gives $\|M_c f\|_{M^{\infty,p,\gamma}} \le C\|f\|_{F_\alpha^{s,q}}$. Let $w \in \mathbb{D}^n$ and set $f_w(z) = \frac{1}{1-wz}$, $g_w = M_c f_w$. Then we have

$$D^m g_w = D^m M_c f_w = M_c D^m f_w = C M_c \frac{1}{(1-wz)^{m+1}},$$

which, together with the third estimate from Lemma 3, gives

$$\|D^m g_w\|_{M^{\infty,p,\gamma}} \le C \left\| \frac{1}{(1-wz)^{m+1}} \right\|_{F_\alpha^{s,q}} \le \frac{C}{(1-|w|)^{m-\alpha-1/s+1}}. \tag{30}$$

The rest is clear. □

Proofs of all other cases are similarly into these $F^{p,\infty,s}$ and $F^{\infty,p,s}$ we leave this to readers. Note all these results are new even in one dimensional case of unit disk.

The following theorem on multipliers is sharp and it is taken from our previous paper [17] and concerns again analytic $F_\alpha^{p,q}$ type spaces in polydisk (for finite p, q).

We include it for completness of our exposition and for further use, since the sufficiency part leads to some new assertions concerning analytic $T_{\alpha}^{p,q}$ and $M_{\alpha}^{p,q}$ Lizorkin - Triebel type spaces which we introduced above. To add using this theorem below various new results on M and T - type analytic spaces on subframe and expanded disk which we defined above we have to repeat arguments of this theorem using at very last step various embeddings connecting "standard" Bergman- type spaces on polydisk with Bergman - type spaces on subframe and expanded disk(see [19], [20] and see also end of paper).

Theorem 1. *Let* $g \in H(\mathbb{D}^n)$, $g(z) = \sum_{k\in\mathbb{Z}_+^n} c_k z^k$. *Assume* $\frac{t}{2} \leq s \leq t < \infty$, $0 < p, q \leq 1$, $0 < \max(p,q) \leq s \leq 1$, $t(\alpha + \frac{1}{p}) < 2$, $m \in \mathbb{N}$ *and* $m > \frac{2}{t} - 1$.

1^o $\{c_k\}_{k\in\mathbb{Z}_+^n} \in M_T(F_{\alpha}^{p,q}(\mathbb{D}^n), A_{\beta}^{t,s}(\mathbb{D}^n))$ *if and only if*

$$\sup_{r\in I^n} M_t(D^m g, r)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty. \tag{31}$$

2^o $\{c_k\}_{k\in\mathbb{Z}_+^n} \in M_T(A_{\alpha}^{p,q}(\mathbb{D}^n), A_{\beta}^{t,s}(\mathbb{D}^n))$ *if and only if*

$$\sup_{r\in I^n} M_t(D^m g, r)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty. \tag{32}$$

We can remark that the above mentioned embedding $A_{\alpha}^{p,q} \hookrightarrow F_{\alpha}^{p,q}$, $p \geq q$ (which follows from Minkowski's inequality) allows us to deduce sufficiency of condition (32) from part 1^o of the above theorem under additional condition $p \geq q$. However, this additional condition can be dropped with help of the second part of Lemma 2. Hence we will omit the complete proof of second part providing only the complete sketches of it.

Proof. 1^o We are interested only on sufficiency part. The other part almost repeats arguments we had above. Hence we will omit it. Assume (31) holds and choose $f \in F_{\alpha}^{p,q}(\mathbb{D}^n)$. Set $h = M_c f$ and choose $\eta \in \mathbb{T}^n$, $r \in I^n$. Then we have, using

Lemma 1:

$$(33)\quad h(r^2\eta)=\int_{\mathbb{T}^n} f(r\eta\bar{t})g(rt)dt$$
$$=\int_{\mathbb{T}^n} D^m g(R\xi)f(r\eta\bar{\xi})\int_0^{r_1}\cdots\int_0^{r_n}\prod_{j=1}^n (r_j^2-R_j^2)^{m-1}R_1\dots R_n dR d\xi.$$

The last expression can be transformed by a change of variables $R=r\rho$ and we obtain an estimate

$$(34)\quad |h(r^2\eta)|\le C\int_{\mathbb{D}^n}|(D^m g)(r\rho\xi)f(r\rho\eta\bar{\xi})|\prod_{j=1}^n(1-\rho_j^2)^{m-1}d\rho d\xi$$
$$=C\int_{\mathbb{D}^n}|(D^m g)(r\rho\xi)\overline{f}(r\rho\eta\bar{\xi})|\prod_{j=1}^n(1-\rho_j^2)^{m-1}d\rho d\xi.$$

Now we apply Lemma 7 to an analytic function $D^m g(rz)\overline{f}(r\eta\bar{z})$ and obtain

$$(35)\qquad |h(r^2\eta)|^t\le C\int_{\mathbb{D}^n}|(D^m g)(r\rho\xi)|^t|f(r\rho\eta\bar{\xi})|^t\prod_{j=1}^n(1-\rho_j^2)^{t(m+1)-2}d\rho d\xi.$$

Next we integrate over $\eta\in\mathbb{T}^n$. This yields, taking into account (31):

$$(36)\quad \int_{\mathbb{T}^n}|h(r^2\eta)|^t d\eta\le C\int_{\mathbb{T}^n}\int_{I^n}\int_{\mathbb{T}^n}|(D^m g)(r\rho\xi)|^t|f(r\rho\eta\bar{\xi})|^t(1-\rho^2)^{t(m+1)-2}d\rho d\xi d\eta$$
$$=C\int_{\mathbb{T}^n}\int_{I^n}|(D^m g)(r\rho\xi)|^t M_t^t(f,r\rho)(1-\rho^2)^{t(m+1)-2}d\rho d\xi$$
$$\le C\int_{\mathbb{T}^n}\int_{I^n}M_t^t(f,r\rho)(1-r\rho)^{-t(m+1-\alpha+\beta-1/p)}(1-\rho^2)^{t(m+1)-2}d\rho d\xi$$
$$\le C\int_{\mathbb{T}^n}\int_{I^n}M_t^t(f,r\rho)(1-r\rho)^{t(\alpha-\beta+1/p)-2}d\rho d\xi,\quad t(m+1)>2$$
$$\le C\int_{\mathbb{D}^n}|f(rw)|^t(1-r|w|)^{t(\alpha-\beta+1/p)-2}dV(w).$$

Now we apply Lemma 7 with $v=s/t$ and with $(1-r|w|)^q$ instead of $(1-|w|)^q$ where $q=t(\frac{1}{p}-\beta+\alpha)-2$ (see comments after Lemma 7). This gives

$$(37)\quad M_t^s(h,r^2)\le C\int_{\mathbb{D}^n}|f(rw)|^s(1-r|w|)^{s(\alpha-\beta+1/p)-2s/t}(1-|w|)^{2s/t-2}dV(w).$$

Next we integrate both sides over $r \in I^n$. This gives

$$\begin{aligned}
\|h\|^s_{A^{t,s}_\beta} &= \int_{I^n} M^s_t(h,R)(1-R)^{\beta s-1}dR\\
&\le C\int_{I^n}(1-R)^{\beta s-1}\int_{\mathbb{D}^n}|f(Rw)|^s\frac{(1-|w|)^{2s/t-2}}{(1-R|w|)^{-s(\alpha-\beta+1/p)+2s/t}}dV(w)dR\\
&\le C\int_{I^n}(1-R)^{\beta s-1}\int_{I^n}\int_{\mathbb{T}^n}|f(rR\xi)|^s\frac{(1-|w|)^{2s/t-2}}{(1-R|w|)^{-s(\alpha-\beta+1/p)+2s/t}}drdRd\xi\\
&= C\int_{I^n}\int_{I^n}M^s_s(f,rR)(1-R)^{\beta s-1}\frac{(1-|w|)^{2s/t-2}}{(1-R|w|)^{-s(\alpha-\beta+1/p)+2s/t}}drdR\\
&\le C\int_{I^n}(1-r)^{2s/t-2}M^s_s(f,r)\int_{I^n}\frac{(1-R)^{\beta s-1}}{(1-rR)^{s(\beta-\alpha-1/p)+2s/t}}dRdr\\
&\le C\int_{I^n}M^s_s(f,r)(1-r)^{s(\alpha+1/p)-2}dr.
\end{aligned}$$

At the last the assumption $t(\alpha+1/p) > 2$ allowed us to use the following well known estimate:

$$\int_0^1 \frac{(1-R)^\alpha}{(1-Rr)^\lambda}dR \le C(1-r)^{\alpha+1-\lambda}, \qquad 0 \le r < 1,$$

valid for $\alpha > -1$ and $\lambda > \alpha+1$. Finally, Lemma 2, specifically embedding (8), allows us to conclude that $\|h\|_{A^{t,s}_\beta} \le C\|f\|_{F^{p,q}_\alpha}$.

The proof of sufficiency in the $A^{p,q}_\alpha$ case goes along the same lines, the only difference being use of embedding (9) instead of (8) at the last step. □

We remark that the above theorem was announced in [17] without proof.

We add new results on multipliers of analytic $T^{p,q}_\alpha$ and $M^{p,q}_\alpha$ type spaces on subframe and expanded disk which were defined by us above. We note there is no sharp result in this list below. Nevertheless these assertions are new even in case of unit disk and these assertions complete the picture related with this problem from our point of view. To get proofs of these assertions for $T^{p,q}_\alpha$ type classes we have to repeat arguments used above in proofs of previous assertions in combination with the following estimate which can be found in [20].

$$\int \prod_{k=1}^{n} (1 - Rr_k)^{-\alpha} (1 - R)^{\tau} dR \leq C \prod_{k=1}^{n} (1 - r_k)^{-\alpha + \frac{1}{n} + \frac{\tau}{n}},$$

where for all k, $r_k \in (0,1)$, $k = 1, \dots, n$. We note here the same estimate (with modulus) is true if we replace r_k by z_k, $z_k \in C$ for all $k = 1, \dots, n$ where $\tau > -1$ and $\alpha > \frac{1}{n} + \frac{\tau}{n}$. Note as a corollary of this estimate we have the spaces of multipliers of classes on subframe contain the dimension n in all cases we consider below. This differs with what we had above in same situation.

Proposition 9. *Assume $m > -1-\alpha/n+1/q$. If a sequence $\{c_k\}_{k \in \mathbb{Z}_+^n}$ is a multiplier from $T_\alpha^{q,s}$ to $F^{p,\infty,r}$, then the function $g = \sum_{k \in \mathbb{Z}_+^n} c_k z^k$ satisfies condition (26) with $\tau = m + 1 + r + \alpha/n - 1/q,\ p, q, s \in (0, \infty)$.*

Assume $m > -1 - \alpha/n + 1/q$. If a sequence $\{c_k\}_{k \in \mathbb{Z}_+^n}$ is a multiplier from $T_\alpha^{q,s}$ to $A_r^{\infty,\infty}$, then the function $g = \sum_{k \in \mathbb{Z}_+^n} c_k z^k$ satisfies condition (26) with $\tau = m + 1 + r + \alpha/n - 1/q$, $p = \infty,\ q, s \in (0, \infty)$.

Assume $m > -1 - \alpha/n + 1/q$. If a sequence $\{c_k\}_{k \in \mathbb{Z}_+^n}$ is a multiplier from $T_\alpha^{q,s}$ to $A_r^{p,\infty}$, then the function $g = \sum_{k \in \mathbb{Z}_+^n} c_k z^k$ satisfies condition (26) with $\tau = m + 1 + r + \alpha/n - 1/q,\ p, q, s \in (0, \infty)$.

Proposition 10. *Assume $m > -1 - \alpha/n$. If a sequence $\{c_k\}_{k \in \mathbb{Z}_+^n}$ is a multiplier from $T^{\infty,s,\alpha}$ to $F^{p,\infty,r}$, then the function $g = \sum_{k \in \mathbb{Z}_+^n} c_k z^k$ satisfies condition (26) with $\tau = m + 1 + r + \alpha/n,\ p, s \in (0, \infty)$.*

Assume $m > -1-\alpha/n+1/q$. If a sequence $\{c_k\}_{k \in \mathbb{Z}_+^n}$ is a multiplier from $T^{\infty,s,\alpha}$ to $A_r^{\infty,\infty}$, then the function $g = \sum_{k \in \mathbb{Z}_+^n} c_k z^k$ satisfies condition (26) with $\tau = m + 1 + r + \alpha/n$, $p = \infty,\ q, s \in (0, \infty)$.

Assume $m > -1-\alpha/n+1/q$. If a sequence $\{c_k\}_{k \in \mathbb{Z}_+^n}$ is a multiplier from $T^{\infty,s,\alpha}$ to $A_r^{p,\infty}$, then the function $g = \sum_{k \in \mathbb{Z}_+^n} c_k z^k$ satisfies condition (26) with $\tau = m + 1 + r + \alpha/n,\ p, q, s \in (0, \infty)$.

Proposition 11. *Assume $m > -1-\alpha/n+1/q$. If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $T^{q,\infty,\alpha}$ to $F^{p,\infty,r}$, then the function $g=\sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition (26) with $\tau = m+1+r+\alpha/n-1/q,\ p,q,s\in(0,\infty)$.*

Assume $m > -1-\alpha/n+1/q$. If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $T^{q,\infty,\alpha}$ to $A_r^{\infty,\infty}$, then the function $g=\sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition (26) with $\tau = m+1+r+\alpha/n-1/q,\ p=\infty,\ q,s\in(0,\infty)$.

Assume $m > -1-\alpha/n+1/q$. If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $T^{q,\infty,\alpha}$ to $A_r^{p,\infty}$, then the function $g=\sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition (26) with $\tau = m+1+r+\alpha/n-1/q,\ p,q,s\in(0,\infty)$.

Similarly we can find necessary conditions if a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $T_\alpha^{r,s}$ to $A_s^{\infty,p}$, from $T_\alpha^{p,s}$ to H^∞, from $T_\alpha^{p,s}$ to Bl, $T_\alpha^{r,s}$ to $A_\beta^{q,p}$, $T_\alpha^{r,s}$ to $F_\beta^{q,p}$, the last two cases have similarities with the theorem we proved above, so we refer the reader also to that theorem, the Bl is a Bloch space (see, for example, [1], [5]). It is another problem to consider multipliers into T type spaces. To get necessary conditions on multipliers from $F_\alpha^{p,q}$ or $A_\alpha^{p,q}$ to $T_\beta^{r,s}$, $s\le r$ we have to repeat the related implication in simple proof of theorem we provided above a little using estimates for f_R function from lemma above and also the following probably known assertion $sup_{r\in I} M_s(f,r)(1-r)^\beta<\infty$ if $f\in T_\beta^{r,s}$, $s\le r$, with related estimate for quazinorms, see [20]. We leave this procedure to readers.

We find finally some new (even for case of unit disk) estimates for spaces of multipliers of analytic mixed norm $M^{p,q}$ type classes which is one of the main topic of this paper, including limit cases. These results can be formulated as follows. Note we should repeat arguments we used above several times in combination with the estimate in lemma above for f_R function in $M^{\infty,s,\alpha}$, $M^{q,\infty,\alpha}$, $M_\alpha^{q,s}$ classes. We have the following assertions.

Proposition 12. *If a sequence* $\{c_k\}_{k\in\mathbb{Z}^n_+}$ *is a multiplier from* $M^{s,q}_\alpha$ *to* $F^{p,\infty,r}$*, then the function* $g=\sum_{k\in\mathbb{Z}^n_+} c_k z^k$ *satisfies condition (26) with* $R_i=R$*,* $\tau=(m-\alpha+1)n-1/s+r$*,*$\tau>0$*,* $s,q,p\in(0,\infty)$.

If a sequence $\{c_k\}_{k\in\mathbb{Z}^n_+}$ *is a multiplier from* $M^{s,q}_\alpha$ *to* $A^{\infty,\infty}_r$*, then the function* $g=\sum_{k\in\mathbb{Z}^n_+} c_k z^k$ *satisfies condition (26) with* $R_i=R$*,* $\tau=(m-\alpha+1)n-1/s+r$*,* $p=\infty$*,* $\tau>0$*,* $s,q\in(0,\infty)$.

If a sequence $\{c_k\}_{k\in\mathbb{Z}^n_+}$ *is a multiplier from* $M^{s,q}_\alpha$ *to* $A^{p,\infty}_r$*, then the function* $g=\sum_{k\in\mathbb{Z}^n_+} c_k z^k$ *satisfies condition (26) with* $R_i=R$*,* $\tau=(m-\alpha+1)n-1/s+r$*,*$\tau>0$*,* $s,q,p\in(0,\infty)$.

Proposition 13. *If a sequence* $\{c_k\}_{k\in\mathbb{Z}^n_+}$ *is a multiplier from* $M^{\infty,q,\alpha}$ *to* $F^{p,\infty,r}$*, then the function* $g=\sum_{k\in\mathbb{Z}^n_+} c_k z^k$ *satisfies condition (26) with* $R_i=R$*,* $\tau=(m-\alpha+1)n+r$*,* $\tau>0$*,* $p,q,r\in(0,\infty)$.

If a sequence $\{c_k\}_{k\in\mathbb{Z}^n_+}$ *is a multiplier from* $M^{\infty,q,\alpha}$ *to* $A^{\infty,\infty}_r$*, then the function* $g=\sum_{k\in\mathbb{Z}^n_+} c_k z^k$ *satisfies condition (26) with* $R_i=R$*,* $\tau=(m-\alpha+1)n+r$*,* $p=\infty$*,* $\tau>0$*,* $q,r\in(0,\infty)$.

If a sequence $\{c_k\}_{k\in\mathbb{Z}^n_+}$ *is a multiplier from* $M^{\infty,q,\alpha}$ *to* $A^{p,\infty}_r$*, then the function* $g=\sum_{k\in\mathbb{Z}^n_+} c_k z^k$ *satisfies condition (26) with* $R_i=R$*,* $\tau=(m-\alpha+1)n+r$*,* $\tau>0$*,* $p,q,r\in(0,\infty)$.

Proposition 14. *If a sequence* $\{c_k\}_{k\in\mathbb{Z}^n_+}$ *is a multiplier from* $M^{s,\infty,\alpha}$ *to* $F^{p,\infty,r}$*, then the function* $g=\sum_{k\in\mathbb{Z}^n_+} c_k z^k$ *satisfies condition (26) with* $R_i=R$*,* $\tau=(m-\alpha+1)n-1/s+r$*,* $\tau>0$*,* $s,p,r\in(0,\infty)$.

If a sequence $\{c_k\}_{k\in\mathbb{Z}^n_+}$ *is a multiplier from* $M^{s,\infty,\alpha}$ *to* $A^{\infty,\infty}_r$*, then the function* $g=\sum_{k\in\mathbb{Z}^n_+} c_k z^k$ *satisfies condition (26) with* $R_i=R$*,* $\tau=(m-\alpha+1)n-1/s+r$*,* $p=\infty$*,* $\tau>0$*,* $s,r\in(0,\infty)$.

If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ *is a multiplier from* $M^{s,\infty,\alpha}$ *to* $A_r^{p,\infty}$*, then the function* $g=\sum_{k\in\mathbb{Z}_+^n} c_k z^k$ *satisfies condition (26) with* $R_i=R$*,* $\tau=(m-\alpha+1)n-1/s+r$*,* $\tau>0,\ s,p,r\in(0,\infty)$.

We add more remarks for that case where q and s are finite in assertion above. Note similarly we can find necessary conditions if a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $M_\alpha^{r,s}$ to $A_s^{\infty,p}$, from $M_\alpha^{p,s}$ to H^∞, from $M_\alpha^{p,s}$ to Bl, $M_\alpha^{r,s}$ to $A_\beta^{q,p}$ $M_\alpha^{r,s}$ to $F_\beta^{q,p}$, the last two cases have similarities with the theorem we proved above, so we refer the reader also to that theorem. It is anther problem to find conditions for multipliers acting into M type spaces in expanded disk. To get necessary conditions for example on multipliers from $F_\alpha^{p,q}$ or $A_\alpha^{p,q}$ to $M_\beta^{r,s}$, $s\le r$ we have to repeat the related implication in simple proof of theorem we provided above using also the following assertion $sup_{r\in I}M_s(f,r,\dots,r)(1-r)^\tau<\infty$ if $f\in M_\beta^{r,s}$, $s\le r$ with related estimates for quazinorms for some τ and β. This follows immediately from a nice embedding which connects classical Bergman A_α^p spaces in the unit disk with $M_\alpha^{p,q}$ classes, see [19], [20]. We leave this procedure to readers. The cases with infinite indexes can be considered similarly.

3. On coefficient multipliers of analytic $V_{\alpha,1}^{p,s}$, $V_{\alpha,2}^{p,s}$ and $V_{\alpha,3}^{p,s}$ Besov - type spaces in the unit polydisc

We apply our previous result from [19], [20] on analytic spaces related to $\mathcal{R}^s$ differential operator to study certain new spaces of coefficient multipliers of new analytic Besov type spaces $V_{\alpha,1}^{p,q}$, $V_{\alpha,2}^{p,q}$ and $V_{\alpha,3}^{p,q}$ closely related to the same $\mathcal{R}^s$ differential operator in the unit polydisc with some restriction on parameters (see [19], [20]). Our results extend some previously known assertions on coefficient multipliers of classical analytic Bergman A_α^p and analytic weighted Hardy H_α^p type spaces in the unit disk. Moreover some results are new even in onedimensional case of unit disk.

Namely the goal of this section is to continue the interesting investigation of some new analytic Besov - type spaces in polydisk, subframe and expanded disk which was started before in [19], [20] related with so-called $\mathcal{R}^s$ differential operator.

We denote the expanded disk by

$$D^n_* = \{z = (r_1\xi, \dots, r_n\xi) \in \mathbb{D}^n : \xi \in T,\ r_j \in (0,1),\ j = 1, \dots, n\},$$

and the subframe by

$$\widetilde{D^n} = \{z \in \mathbb{D}^n : |z_j| = r,\ r \in (0,1],\ j = 1, \dots, n\}.$$

For $\alpha_j > -1,\ j = 1, \dots, n \quad 0 < p < \infty$, recall that the weighted Bergman space $A^p_{\overrightarrow{\alpha}}(\mathbb{D}^n)$ consists of all holomorphic functions on the polydisk satisfying the condition

$$\|f\|^p_{A^p_{\overrightarrow{\alpha}}} = \int_{\mathbb{D}^n} |f(z)|^p \prod_{i=1}^n (1-|z_i|^2)^{\alpha_i} dV(z) < \infty.$$

For $\alpha_j > -1,\ j = 1, \dots, n,\ p \in (0, \infty]$, the Bergman class on expanded disk is defined by

$$A^p_{\overrightarrow{\alpha}}(D^n_*) = \{f \in H(\mathbb{D}^n) : \|f\|^p_{A^p_{\overrightarrow{\alpha}}(D^n_*)} = \int_0^1 \cdots \int_0^1 \int_{\mathbb{T}} |f(z)|^p \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j} d|z_j| d\xi < \infty\},$$

and similarly the Bergman class on subframe denoted by $A^p_\alpha(\widetilde{D}^n)$, is defined by

$$A^p_\alpha(\widetilde{D}^n) = \{f \in H(\mathbb{D}^n) : \|f\|^p_{A^p_\alpha(\widetilde{D}^n)} = \int_{\mathbb{D}^n} \int_0^1 |f(z)|^p (1-|z|^2)^\alpha dV(z) d|z| < \infty\},$$

where $p \in (0, \infty),\ \alpha > -1$.

In this paper we are interested with spaces of coefficient multipliers of $V^{p,q}_{\alpha,1}$, $V^{p,q}_{\alpha,2}$ and $V^{p,q}_{\alpha,3}$ spaces. We provide a general approach to deal with problem of multipliers on such spaces based heavily on our previous results from [19], [20]. Note again the topic of multipliers in analytic spaces in higher dimension is relatively new and only several papers are devoted to this field of research. We mention again here related to this topic recent papers of author [16], [17] on multipliers of Lizorkin - Treiebl $F^{p,q}_\alpha$ type spaces (including limit $p = \infty$ and $q = \infty$ cases).

These $V_{\alpha,1}^{p,q}$, $V_{\alpha,2}^{p,q}$ and $V_{\alpha,3}^{p,q}$ analytic Besov classes which we define in this note serve as natural extension of the classical Hardy and Bergman spaces in the unit polydisc simultaneously.

See also [7], [19], [20] for some other properties of these new analytic Besov $V_{\alpha,1}^{p,q}$, $V_{\alpha,2}^{p,q}$ and $V_{\alpha,3}^{p,q}$ type spaces related to R^s differential operator on the unit polydisc and related spaces in the unit disk or subframe and expanded disk. Below we list notations and definitions which are needed and in this section we formulate and prove some of main results of this note.

Probably for the first time in literature this $\mathcal{R}^s$ type differential operators appeared in a paper of Lizorkin - Guliev [7] as natural extensions of one dimensional Besov spaces to higher dimension. Later in [19] and [20] it was shown that such Besov type analytic spaces in higher dimension are closely related to analytic spaces on subframe, but not in polydisk. This paper can be seen as continuation of [19], [20] and here we provide applications of some basic results from [19], [20] related with estimates of products of onedimensional Bergman kernels.

We define now one of the main objects of this paper. First if f is an holomorphic with a_k Taylor coefficients then $\mathcal{R}^s f$ is the same holomorphic function, but with $(k_1 + \ldots + k_n + 1)^s a_k$ Taylor coefficients, where s is a real number. It can be easily checked that the last function is an analytic function. For unit disk case ($n = 1$) this operator (the fractional derivative of f function) is very well studied from various points of views by big amount of authors during last several decades. Taking products of onedimensional $(k + 1)^s$ or the same expression via so-called gamma function some authors defined the fractional derivative of an analytic function in polydisk otherwise, see [5]. These two types of fractional derivatives in polydisk are connected. Their connection and properties were given in [20] (see also various references there). We define below various spaces of analytic functions in polydisk, we note each time we can add an additional condition on each such X analytic space (with zero smoothness), we will simply write $\mathcal{R}^s X$ meaning spaces with $\mathcal{R}^s f$ functions, where s is a fixed real number. To define analytic $V^{p,s}_{\alpha,1}$, $V^{p,s}_{\alpha,2}$ and $V^{p,s}_{\alpha,3}$ Besov - type spaces in the unit polydisc we have to note the first space is the Bergman - type space $\mathcal{R}^s A^p_\alpha(\mathbb{D}^n)$ in the polydisk. The other two spaces are the same spaces but on subframe and expanded disk. This means the change of integration interval in quazinorm of classical Bergman - type space $A^p_\alpha(\mathbb{D}^n)$ in polydisk first I^n to I, then T^n to T. These last new Besov - type spaces $V^{p,s}_{\alpha,1}$, $V^{p,s}_{\alpha,2}$ and $V^{p,s}_{\alpha,3}$ and their spaces of coefficient multipliers will be under attention in this paper. We restrict ourselves in this paper only to those values of s which are positive.

Lemma 8. *Let $w = |w|\xi$, $w, z \in \mathbb{D}^n$, $1 - w\overline{z} = \prod_{k=1}^n (1 - w_k \overline{z_k})$, $s \in \mathbb{N} \cup \{0\}$, $\beta > 0$, $p \in (0, \infty)$. Then we have*

$$\int_{\mathbb{T}^n} \left| \mathcal{R}^s \frac{1}{(1 - \xi|w|z)^\beta} \right|^p d\xi \leq C \sum_{\alpha_j \geq 0, \sum \alpha_j = s} \left(\prod_{k=1}^n \frac{1}{(1 - |w_k||z_k|)^{p(\alpha_k + \beta) - 1}} \right),$$

$p > \frac{1}{\min_k \alpha_k + \beta}$.

Corollary 1. *Let $0 < p < \infty$, $s \in \mathbb{N} \cup \{0\}$, $l \in (0, \infty)$, $\gamma > \frac{1}{p} + l$, $w \in \mathbb{D}^n$. Then*

$$\int_{\mathbb{D}^n} \left| \mathcal{R}^s \frac{1}{(1 - w\overline{z})^\gamma} \right|^p (1 - |z|)^{pl-1} dV(z) \leq \sum_{\alpha_j \geq 0, \sum \alpha_j = s} \prod_{k=1}^n \frac{C}{(1 - |w_k|)^{(\alpha_k + \gamma)p - pl - 1}}.$$

Proofs of Lemma 8 and Corollary 1 can be seen in [20].

The following four estimates are very important for this paper and they are heavily based on results from [19], [20] and they open ways for estimates of classical Bergman kernel in Besov type new analytic classes we introduced above. The first two estimates follow as direct corollaries of delicate estimates of $\mathcal{R}^s$ differential operator applied to standard n-products of onedimensional Bergman kernel in classical Hardy H^p and Bergman A^p_α classes in polydisk, which we already provided above, see [19], [20]. Note then in the second pair of estimates below we solve the same problem, but in Bergman classes in expanded disk or subframe.

Lemma 9. *Let* $0 < p, s < \infty$ *and set*

$$f_R(z) = \frac{1}{(1-Rz)^{\beta+1}}, \qquad \beta > -1, \quad R \in I, \quad z \in \mathbb{D}^n. \tag{38}$$

Then we have the following (quasi) norm estimates:

$$\|f_R\|_{V^{p,s}_{\alpha,1}(\mathbb{D}^n)} \leq \frac{C}{(1-R)^{n(\beta+1)-\alpha n-n/p+s}}, \quad \beta > \alpha + 1/p - 1, \tag{39}$$

$$\|f_R\|_{\mathcal{R}^s H^p(\mathbb{D}^n)} \leq \frac{C}{(1-R)^{n(\beta+1)-n/p+s}}, \quad \beta > 1/p - 1. \tag{40}$$

Lemma 10. *Let* $0 < p, s < \infty$ *and set*

$$f_R(z) = \frac{1}{(1-Rz)^{\beta+1}}, \quad \beta > -1, \quad R \in I, \quad z \in \mathbb{D}^n. \tag{41}$$

Then we have the following (quasi) norm estimates:

$$\|f_R\|_{V^{p,s}_{\alpha,2}(\mathbb{D}^n)} \leq \frac{C}{(1-R)^{n(\beta+1)-\alpha-n/p+s}}, \quad \beta > \alpha/n - 1 + 1/p, \tag{42}$$

$$\|f_R\|_{V^{p,s}_{0,3}(\mathbb{D}^n)} \leq \frac{C}{(1-R)^{n(\beta+1)-\alpha n+s-1/p}}, \quad \beta > \alpha - 1 + \frac{1}{np}. \tag{43}$$

As corollaries of these results we can immediately formulate our first results on coefficient multipliers of analytic Besov - type V spaces which we study in this paper.

Proposition 15. *If a sequence* $\{c_k\}_{k\in\mathbb{Z}^n_+}$ *is a multiplier from* $F^{p,q}_\alpha$ *to* $V^{r,s}_{\gamma,1}$, *then the function* $g = \sum_{k\in\mathbb{Z}^n_+} c_k z^k$ *satisfies condition* $M_r(D^\beta \mathcal{R}^s g, \rho)(1-\rho)^{\gamma+\beta-\alpha-1/p+1} < \infty$, *where* $\beta > \alpha + 1/p - 1,\ p, q, r \in (0,\infty)$.

If a sequence $\{c_k\}_{k\in\mathbb{Z}^n_+}$ *is a multiplier from* $A^{p,q}_\alpha$ *to* $V^{r,s}_{\gamma,1}$, *then the function* $g = \sum_{k\in\mathbb{Z}^n_+} c_k z^k$ *satisfies condition* $M_r(D^\beta \mathcal{R}^s g, \rho)(1-\rho)^{\gamma+\beta-\alpha-1/p+1} < \infty$, *where* $\beta > \alpha + 1/p - 1,\ p, q, r \in (0,\infty)$.

If a sequence $\{c_k\}_{k\in\mathbb{Z}^n_+}$ *is a multiplier from* H^p *to* $V^{r,s}_{\gamma,1}$, *then the function* $g = \sum_{k\in\mathbb{Z}^n_+} c_k z^k$ *satisfies condition* $M_r(D^\beta \mathcal{R}^s g, \rho)(1-\rho)^{\gamma+\beta-1/p+1} < \infty$, *where* $\beta > 1/p - 1,\ p, r \in (0,\infty)$.

Proposition 16. *If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $V_{l,1}^{r,s}$ to $A_\alpha^{p,q}$, then the function $g=\sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition $M_p^r(D^\beta g,\rho)(1-\rho)^{r\alpha+((\beta+1)r-rl-1)n+sr}<\infty$, where $\beta>l+1/r-1$, $p,q,r\in(0,\infty)$.*

If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $V_{l,1}^{r,s}$ to $F_\alpha^{p,q}$, then the function $g=\sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition $M_p^r(D^\beta g,\rho)(1-\rho)^{r\alpha+((\beta+1)r-rl-1)n+sr}<\infty$, where $\beta>l+1/r-1$, $p,q,r\in(0,\infty)$.

If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $V_{l,1}^{r,s}$ to H^p, then the function $g=\sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition $M_p^r(D^\beta g,\rho)(1-\rho)^{((\beta+1)r-rl-1)n+sr}<\infty$, where $\beta>l+1/r-1$, $p,r\in(0,\infty)$.

Similar assertions are true by same arguments using Lemma 10 for $V_{\alpha,2}^{p,s}(\mathbb{D}^n)$ and $V_{\alpha,3}^{p,s}(\mathbb{D}^n)$ type spaces. We leave this rather simple procedure to readers.

Proposition 17. *If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $\mathcal{R}^sH^r$ to $A_\alpha^{p,\infty}$, then the function $g=\sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition $M_p^r(D^\beta g,\rho)(1-\rho)^{r\alpha+((\beta+1)r-1)n+sr}<\infty$, where $\beta>1/r-1$, $r,p\in(0,\infty)$.*

If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $\mathcal{R}^sH^r$ to $F^{p,\infty,\alpha}$, then the function $g=\sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition $M_p^r(D^\beta g,\rho)(1-\rho)^{r\alpha+((\beta+1)r-1)n+sr}<\infty$, where $\beta>1/r-1$, $r,p\in(0,\infty)$.

If a sequence $\{c_k\}_{k\in\mathbb{Z}_+^n}$ is a multiplier from $\mathcal{R}^sH^r$ to $A_\alpha^{\infty,p}$, then the function $g=\sum_{k\in\mathbb{Z}_+^n} c_k z^k$ satisfies condition $M_p^r(D^\beta g,\rho)(1-\rho)^{r\alpha+((\beta+1)r-1)n+sr}<\infty$, where $\beta>1/r-1$, $r,p\in(0,\infty)$.

We can collect information about multipliers of analytic spaces based on $\mathcal{R}^s$ operator from these estimates from [19], [20], which relate with classes with zero smoothness and spaces with more usual D^s differential operator.

Theorem 2. *(i) Let* $0 < p < \infty$, $\alpha > -1$, $s \in \mathbb{N}$, $f \in H(\mathbb{D}^n)$. *If* $\gamma > \frac{\alpha+2}{p} - 1$, *for* $p \leq 1$ *and* $\gamma > \frac{\alpha+1}{p} + \frac{1}{n}\left(1 - \frac{1}{p}\right)$ *for* $p > 1$, $v = sp + \alpha n - \gamma pn + n - 1$, *then*

$$\int_{\mathbb{D}^n} |D^\gamma f(z)|^p \, (1 - |z|^2)^\alpha dV(z) \leq C \int_0^1 \int_{\mathbb{T}^n} |\mathcal{R}^s f(w)|^p \, (1 - |w|^2)^v d\xi d|w|,$$

where $w = |w|\xi$.

(ii) Let $0 < q < \infty$, $\alpha \geq 0$, $s \in \mathbb{N}$, $1 < p < \infty$, $f \in H(\mathbb{D}^n)$. *Then*

$$\int_{\mathbb{T}^n} \left(\int_0^1 (1 - |z|)^\alpha |f(z)|^p |z|^{sp} d|z| \right)^{q/p} d\xi \leq C \int_{\mathbb{T}^n} \left(\int_0^1 |\mathcal{R}^s f(u\xi)|^p \, (1 - u)^{sp+\alpha} du \right)^{q/p} d\xi.$$

(iii) Let $0 < q < \infty$, $\alpha \geq 0$, $s \in \mathbb{N}$, $1 < p < \infty$, $\gamma \in \left(\frac{-1}{p}, \frac{1}{p'}\right)$, $\frac{1}{p'} + \frac{1}{p} = 1$, $f \in H(\mathbb{D}^n)$. *Then*

$$\int_{\mathbb{T}^n} \left(\int_0^1 (1 - r)^\gamma |f(r\xi)|^p r^p dr \right)^{q/p} d\xi \leq C \int_{\mathbb{T}^n} \left(\int_0^1 |\mathcal{R}^s f(r\xi)|^p \, (1 - r)^{sp+\gamma} dr \right)^{q/p} d\xi.$$

We can also easily collect information about multipliers of Bergman spaces on subframe and expanded disk from information about multipliers of Bergman spaces in polydisk using the following estimates from [19], [20].

Theorem 3. *(i) Let* $0 < p < \infty$, $\alpha > -1$, $n \in \mathbb{N}$, $f \in H(\mathbb{D}^n)$. *Then*

$$\int_0^1 \int_{\mathbb{T}^n} |f(z)|^p \, (1 - |z|^2)^\alpha d\xi d|z| \leq C \int_{\mathbb{D}^n} |f(z)|^p \prod_{k=1}^n (1 - |z_k|^2)^{\frac{\alpha}{n} - 1 + \frac{1}{n}} dV(z).$$

(ii) Let $0 < p < \infty$, $\alpha_j > -1$, $j = 1, \ldots, n$, $n \in \mathbb{N}$, $f \in H(\mathbb{D}^n)$. *Then*

$$\int_{\mathbb{T}} \int_{[0,1]^n} |f(|z_1|\xi, \ldots |z_n|\xi)|^p \prod_{k=1}^n (1 - |z_k|)^{\alpha_k + \frac{n-1}{n}} d|z_1| \cdots d|z_n| d\xi$$

$$\leq C \int_{\mathbb{D}^n} |f(z)|^p \prod_{k=1}^n (1 - |z_k|^2)^{\alpha_k} dV(z).$$

References

[1] A. B. Alexandrov, V. V. Peller, *Hankel operators and similarity to a contraction*, Intern. Math. Research Notices, No. 9, (1996), 263-275.

[2] O. Blasco, *Multipliers on spaces of analytic functions*, Canad. J. Math., Vol. 47(1), (1995), 44-64.

[3] O. Blasco, J. L. Arregui, *Multipliers on vector valued Bergman spaces*, Canad. J. Math. 54 (2002), no. 6, 1165–1186.

[4] O. Blasco, M. Pavlovic, *Coefficient multipliers in spaces of analytic functions*, preprint, 2011.

[5] M. Djrbashian, F. Shamoian, *Topics in the theory of A^p_α classes*, Teubner Texte zur Mathematik, 1988, v 105.

[6] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

[7] V. S. Guliev, P. I. Lizorkin, *Holomorphic and harmonic function classes in the polydisc and its boundary values*, in Russian, Trudi Math. Inst. RAN, Vol. 204, (1993), 137-159.

[8] M. Jevtić, M. Pavlović, *Coefficient multipliers on spaces of analytic functions*, Acta Sci. Math., Vol. 64 (1998), 531-545.

[9] M. Mateljević, M. Pavlović, *Multipliers of H^p and $BMOA$*, Pacific J. Math., Vol. 146, (1990), 71-89.

[10] M. Nawrocki, *Multipliers, linear functionals and the Frechet envelope of the Smirnov class* $N_*(U^n)$, Trans. Amer. Math. Soc., Vol 322, (1990), No. 2, 493-506.

[11] M. Nowak, *Coefficient multipliers of spaces of analytic functions*, Ann. Univ. Mar. Sklod. Lublin, Vol. 52, (1998), 108-119.

[12] J. Ortega, J. Fabrega, *Hardy's inequality and embeddings in holomorphic Lizorkin-Triebel spaces*, Illinois Journal of Mathematics, Vol. 43, 4, 1999, 733-751.

[13] J. Ortega, J. Fabrega, *Pointwise multipliers and corona type decomposition in BMOA*, Ann. Inst. Fourier (Grenoble), Vol. 46, (1996), 111-137.

[14] J. Ortega, J. Fabrega, *Holomoprhic Triebel-Lizorkin spaces*, J. Funct. Anal., Vol. 151, (1997), 177-212.

[15] R. F. Shamoyan, *Holomorphic Lizorkin-Triebel type spaces in the unit disk and polydisk*, Izvestia NAN Acad. Armenii, 37, 3 (2002), 57-78.

[16] R. F. Shamoyan, *On multipliers from Bergman type spaces into Hardy spaces in the polydisc*, in Russian, Ukrainian. Math. Journal., No. 10, (2000), 1405-1415.

[17] R. F. Shamoyan, *Continuous functionals and multipliers of a class of functions analytic in the polydisc*, in Russian, Izv. Vuzov Matematika., No. 7, (2000), 67-69.

[18] R. F. Shamoyan, *Multipliers of power series Toeplitz operators and space embeddings*, Izv. Nat. Acad. Nauk Armenii. Vol. 34, (1999), No. 4, 43-59.

[19] R. Shamoyan, O. Mihić, *Analytic classes on subframe and expanded disk and the R^s differential operator in polydisk*, Journal of Inequalities and Applications, Volume 2009, Article ID 353801, 22 pages, doi:10.1155/2009/353801.

[20] R. F. Shamoyan, S. Li, *On some properties of a differential operator in the polydisc*, Banach Journal of Mathematical Analysis, v.3, (2009), 68-84.

[21] S. V. Shvedenko, *Hardy classes and related spaces of analytic functions in the unit disc, polydisc and ball*, in Russian, Itogi nauki i tekhniki, VINITI, Ser. matematicheski analiz, Vol. 23, (1985), 2-124.

[22] R. M. Trigub, *Multipliers of the $H^p(\mathbb{D}^m)$ class for $p \in (0,1)$, approximation properties of summability methods for power series*, in Russian, Doklady. RAN, Vol. 335, (1994), No. 6, 697-699.

[23] H. Triebel, *Theory of Function Spaces*, Springer- Birkhauser, 1983.

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Received March 13, 2013