MATHEMATICA MONTISNIGRI
Vol XXXIV (2015)

GENERALIZATION OF A DISCRETE OPIAL TYPE INEQUALITY APPLIED TO THE EIGENVALUES OF GRAPH

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Summary. In this paper we present the modifications of a discrete Opial type inequality applied to the eigenvalues of the graph G. We are going to derive a new proof of generalized inequalitiy

$$-\cos\frac{\pi}{r+1}\sum_{k=1}^{n} x_k^2 \le \sum_{k=1}^{n-m} x_k x_{k+m} \le \cos\frac{\pi}{r+1}\sum_{k=1}^{n} x_k^2 ,$$

as well as its special case for $m=1$ and $m=2$, whereby use of well-known inequality for real eigenvalues of the graph $\sum_{k=1}^{n}\lambda_k^2 = 2m$, where n is number of vertexs, and m is number of edges of graph. This proofs are much simpler and shorter then the one given in [6]. In proofs starting from a classic inequality

$$\left(\sqrt{\frac{\alpha_k}{\beta_k}}\lambda_k \pm \sqrt{\frac{\beta_k}{\alpha_k}}\lambda_{k+m}\right)^2 \ge 0,$$

where $\vec{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and $\vec{\beta}=(\beta_1,\beta_2,\ldots,\beta_n)$ are arbitry n-tuples of real numbers, where $\alpha_k\beta_k>0$, for each $k=1,2,\ldots,n$ and $\vec{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ are eigenvalues of graph G. After transformation of this inequalities, depending on the appropriate choice of function α_k and β_k obtained above inequalities for different angles. Using trigonometric transformations is obtained their equivalent form

$$8m\sin^2\frac{\pi}{2(r+1)} \le \sum_{k=0}^{n}(\lambda_k-\lambda_{k+1})^2 \le 8m\cos^2\frac{\pi}{2(r+1)}.$$

It also enables to derive a number of other disrete inequalities related with it.

1 INTRODUCTION

In [6] Opial established the following integral inequality:

Theorem 1.1. (see [6]) Suppose $f\in C^1[0,h]$ satisfies $f(0)=f(h)=0$ and $f(x)>0$ for all $x\in(0,h)$. Then the following integral inequality holds

2010 Mathematics Subject Classification: 26D15, 26D10.
Key words and Phrases: Opial inequality, discrete Opial'type inequality.

$$\int_0^h |f(x)f'(x)|dx \le \frac{h}{4}\int_0^h (f')^2\,dx, \tag{1}$$

where the constant $\frac{h}{4}$ is best possible.

This inequality, its generalizations and modifications play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations, as well as difference equations. These inequalities are also important in solving the problems in polynomial theory, approximation theory, spectral graph theory, and technical applications.

2 MAIN RESULTS

Let us prove the following theorem:

Theorem 2.1. If $\lambda_0, \lambda_1, ..., \lambda_n, \lambda_{n+1}$, real eigenvalues of graph $G = (V, E)$, $|V| = n > 1$, $|E| = m > 1$, $\lambda_0 = \lambda_{n+1} = 0$, then inequality

$$-2m\cos\frac{\pi}{n+1} \le \sum_{k=1}^{n-1} \lambda_k \lambda_{k+1} \le 2m\cos\frac{\pi}{n+1} \tag{2}$$

holds. Equality in the left (right) side of inequality (3) holds if and only if $\lambda_k = (-1)^{k-1}\sqrt{2m}\sin\frac{k\pi}{n+1}$ ($\lambda_k = \sqrt{2m}\sin\frac{k\pi}{n+1}$) for $k = 1,...,n.$

Proof. Let $\vec{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n)$, $\vec{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\vec{\beta} = (\beta_1, \beta_2, ..., \beta_n)$ are arbitrary n-type of real numbers, whereby $\alpha_k \beta_k > 0$ for all $k = 1,...,n.$

From inequality

$$\left(\sqrt{\frac{\alpha_k}{\beta_k}}\lambda_k \pm \sqrt{\frac{\beta_k}{\alpha_k}}\lambda_{k+1}\right)^2 \ge 0, \tag{3}$$

where equality holds if and only if $\alpha_k \lambda_k \pm \beta_k \lambda_{k+1} = 0$, we get following inequality:

$$\frac{\alpha_k}{\beta_k}\lambda_k^2 + \frac{\beta_k}{\alpha_k}\lambda_{k+1}^2 \ge \pm 2\lambda_k \lambda_{k+1}$$

$$\sum_{k=1}^{n}\left(\frac{\alpha_k}{\beta_k}\lambda_k^2 + \frac{\beta_k}{\alpha_k}\lambda_{k+1}^2\right) \ge \pm 2\sum_{k=1}^{n}\lambda_k \lambda_{k+1}$$

$$\sum_{k=1}^{n}\left(\frac{\alpha_k}{\beta_k} + \frac{\beta_{k-1}}{\alpha_{k-1}}\right)\lambda_k^2 \ge \pm 2\sum_{k=1}^{n-1}\lambda_k \lambda_{k+1} + \frac{\beta_0}{\alpha_0}\lambda_1^2 + \frac{\alpha_n}{\beta_n}\lambda_n^2 \tag{4}$$

where $\lambda_{n+1} = 0.$

By introducing $\alpha_k = \sin(k+1)t$ and $\beta_k = \sin kt,$ inequality (4) becomes

$$\sum_{k=1}^{n}\left(\frac{\sin(k+1)t}{\sin kt}+\frac{\sin(k-1)t}{\sin kt}\right)\lambda_k^2 \geq \pm 2\sum_{k=1}^{n-1}\lambda_k\lambda_{k+1}+\frac{\sin(n+1)t}{\sin nt}\lambda_n^2$$

$$2\cos t\sum_{k=1}^{n}\lambda_k^2 \geq \pm 2\sum_{k=1}^{n-1}\lambda_k\lambda_{k+1}+\frac{\sin(n+1)t}{\sin nt}\lambda_n^2.$$

From $\sin(n+1)t \geq 0,$ it follows that $0<(n+1)t<\pi$ and $t<\frac{\pi}{n+1}$ and we take $t=\frac{\pi}{n+1},$ and the last inequality becomes

$$\cos\frac{\pi}{n+1}\sum_{k=1}^{n}\lambda_k^2 \geq \pm\sum_{k=1}^{n-1}\lambda_k\lambda_{k+1}.$$

Based on the known values of equality for their own graph $\sum_{k=1}^{n}\lambda_k^2 = 2m,$ followd by

$$-2m\cos\frac{\pi}{n+1} \leq \sum_{k=1}^{n-1}\lambda_k\lambda_{k+1} \leq 2m\cos\frac{\pi}{n+1}. \qquad \lozenge$$

Depending on the choice of values α_k and β_k, as well as parameter m in inequality (4), leads to discrete inequalities related inequality (1).

Corollary 2.2. For real eigenvalues $\lambda_0, \lambda_1, ..., \lambda_n, \lambda_{n+1},$ $\lambda_0 = \lambda_{n+1} = 0$ of graph G the imequalities

1. $\sum_{k=1}^{n}\left(k+m+\frac{1}{k+m+1}\right)\lambda_k^2 \geq \pm 2\sum_{k=1}^{n-1}\lambda_k\lambda_{k+1}+(m+1)\lambda_1^2+\frac{1}{n+m+1}\lambda_n^2;$
2. $(m+1)\lambda_1^2+\frac{1}{n+m+1}\lambda_n^2-\sum_{k=1}^{n}\left(\frac{1}{k+m+1}+k+m\right)\lambda_k^2 \leq \pm 2\sum_{k=1}^{n-1}\lambda_k\lambda_{k+1}$

hold. Equality in first inequality holds iff $\lambda_k = \frac{\sqrt{2m}}{(k+m)_{k-1}}$. Equality in second inequality holds iff $\lambda_k = \frac{(-1)^{k-1}\sqrt{2m}}{(k+m)_{k-1}}$.

Proof. If we take in inequality (4) $\alpha_k = ((k+m+1)!)^{-1}$ and $\beta_k = ((k+m)!)^{-1},$ $m \geq 0$, we obtain

$$\sum_{k=1}^{n}\left[\frac{((k+m+1)!)^{-1}}{((k+m)!)^{-1}}+\frac{((k+m-1)!)^{-1}}{((k+m)!)^{-1}}\right]\lambda_k^2 \ge \pm 2\sum_{k=1}^{n-1}\lambda_k\lambda_{k+1}+\frac{(m!)^{-1}}{((m+1)!)^{-1}}\lambda_1^2+\frac{((n+m+1)!)^{-1}}{((n+m)!)^{-1}}\lambda_n^2$$

$$\sum_{k=1}^{n}\left(\frac{1}{k+m+1}+k+m\right)\lambda_k^2 \ge \pm 2\sum_{k=1}^{n-1}\lambda_k\lambda_{k+1}+(m+1)\lambda_1^2+\frac{1}{n+m+1}\lambda_n^2$$

from which follows both inequalities. ◊

Theorem 2.3. For each sequence of the real eigenvalues $0=\lambda_0,\lambda_1,\ldots,\lambda_n,\lambda_{n+1}=0$, of a graph G, the following inequalities hold

$$8m\sin^2\frac{\pi}{2(n+1)} \le \sum_{k=0}^{n}(\lambda_k-\lambda_{k+1})^2 \le 8m\cos^2\frac{\pi}{2(n+1)}. \tag{5}$$

Proof. From inequality (2) it follows that

$$-\left(2\cos^2\frac{\pi}{2(n+1)}-1\right)\sum_{k=1}^{n}\lambda_k^2 \le \sum_{k=1}^{n-1}\lambda_k\lambda_{k+1} \le \left(1-2\sin^2\frac{\pi}{2(n+1)}\right)\sum_{k=1}^{n}\lambda_k^2$$

$$\left(2-4\cos^2\frac{\pi}{2(n+1)}\right)\sum_{k=1}^{n}\lambda_k^2 \le 2\sum_{k=1}^{n-1}\lambda_k\lambda_{k+1} \le \left(2-4\sin^2\frac{\pi}{2(n+1)}\right)\sum_{k=1}^{n}\lambda_k^2$$

$$\sum_{k=1}^{n}\lambda_k^2+\sum_{k=1}^{n}\lambda_k^2-4\cos^2\frac{\pi}{2(n+1)}\sum_{k=1}^{n}\lambda_k^2 \le 2\sum_{k=1}^{n-1}\lambda_k\lambda_{k+1} \le \sum_{k=1}^{n}\lambda_k^2+\sum_{k=1}^{n}\lambda_k^2-4\sin^2\frac{\pi}{2(n+1)}\sum_{k=1}^{n}\lambda_k^2$$

$$-4\cos^2\frac{\pi}{2(n+1)}\sum_{k=1}^{n}\lambda_k^2 \le -\left(\sum_{k=0}^{n}\lambda_k^2-2\sum_{k=0}^{n}\lambda_k\lambda_{k+1}-\sum_{k=0}^{n}\lambda_k^2\right) \le -4\sin^2\frac{\pi}{2(n+1)}\sum_{k=1}^{n}\lambda_k^2$$

$$4\sin^2\frac{\pi}{2(n+1)}\sum_{k=1}^{n}\lambda_k^2 \le \sum_{k=0}^{n}(\lambda_k-\lambda_{k+1})^2 \le 4\cos^2\frac{\pi}{2(n+1)}\sum_{k=1}^{n}\lambda_k^2$$

whence follows the desired inequality. ◊

Theorem 2.4. For each sequence of the real eigenvalues $\lambda_1,\ldots,\lambda_n,\lambda_{n+1}$, of graph G, the following inequalities hold

$$-2m\cos\frac{\pi}{r+1} \le \sum_{k=1}^{n-2}\lambda_k\lambda_{k+m} \le 2m\cos\frac{\pi}{r+1} \tag{6}$$

where $n=2\cdot r-q$, $0\le q\le 1$ and $r=\left[\frac{n-1}{2}\right]+1$. The equality in the left (right) side of this inequality exists if and only if $\lambda_{2k+1}=2\sin\frac{(k+1)\pi}{r+1}$ ($\lambda_{2k+i}=(-1)^k 2\sin\frac{(k+1)\pi}{r+1}$). The index i takes values $i=1,2$, $k=0,1,\ldots,r-2$ and $i=1,\ldots,2-q$, for $k=r-1$.

Proof. From

$$\left(\sqrt{\frac{\alpha_k}{\beta_k}}\lambda_k \pm \sqrt{\frac{\beta_k}{\alpha_k}}\lambda_{k+2}\right)^2 \geq 0, \tag{7}$$

where equality holds if and only if $\alpha_k \lambda_k \pm \beta_k \lambda_{k+2} = 0$, we get:

$$\frac{\alpha_k}{\beta_k}\lambda_k^2 + \frac{\beta_k}{\alpha_k}\lambda_{k+2}^2 \geq \pm 2\lambda_k \lambda_{k+2}$$

$$\sum_{k=1}^{n}\left(\frac{\alpha_k}{\beta_k} + \frac{\beta_{k-2}}{\alpha_{k-2}}\right)\lambda_k^2 \geq \sum_{k=1}^{2}\frac{\beta_{k-2}}{\alpha_{k-2}}\lambda_k^2 \pm 2\sum_{k=1}^{n-2}\lambda_k\lambda_{k+2}, \qquad n-2>0$$

where $\lambda_{n+1} = \lambda_{n+2} = 0.$

By substituted $\alpha_k = \sin(k+2)t$ and $\beta_k = \sin 2t,$ the last inequality becomes

$$\sum_{k=1}^{n}\left(\frac{\sin(k+2)t}{\sin t} + \frac{\sin(k-2)t}{\sin t}\right)\lambda_k^2 \geq \pm 2\sum_{k=1}^{n-2}\lambda_k\lambda_{k+2},$$

$$2\cos 2t\sum_{k=1}^{n}\lambda_k^2 \geq \pm\sum_{k=1}^{n-2}\lambda_k\lambda_{k+2}.$$

$$-2\cos 2t\sum_{k=1}^{n}\lambda_k^2\sum_{k=1}^{n-2}\lambda_k\lambda_{k+2}. \leq 2\cos 2t\sum_{k=1}^{n}\lambda_k^2$$

Since $\sin(n+2)t \geq 0,$ it follows that

$$0 < (n+2)t < \pi \Rightarrow 0 < t < \frac{\pi}{n+2} = \frac{\pi}{2\cdot r - q + 2} = \frac{\pi}{2(r+1)-q}, \tag{8}$$

where $n = 2\cdot r - q,\ 0 \leq q \leq 1,\ r = \left[\frac{n-1}{2}\right] + 1.$

$$\begin{cases} n = 2\cdot r \Rightarrow t \leq \dfrac{\pi}{2r+2} \Rightarrow \cos 2t = \cos\dfrac{\pi}{r+1} \\ n = 2\cdot r - 1 \Rightarrow t \leq \dfrac{\pi}{2r+1} \Rightarrow t = \dfrac{\pi}{2r+1} \Rightarrow \cos 2t = \cos\dfrac{\pi}{r+1}. \end{cases}$$

Substituting these values in the last inequality is obtained desired inequality. ◊

The following Theorem is a generalization of Theorem 1 and Theorem 4.

Theorem 2.5. If $\lambda_1,...,\lambda_n,\lambda_{n+1},$ are real eigenvalues of a graph $G,$ then

$$-2m\cos\frac{\pi}{r+1}\le\sum_{k=1}^{n-m}\lambda_k\lambda_{k+m}\le 2m\cos\frac{\pi}{r+1} \tag{9}$$

where $n=m\cdot r-q,\ r=\left[\frac{n-1}{m}\right]+1,\ 0\le q\le m-1.$ The equality in the right (left) side of this inequality exists if and only if $\lambda_{km+1}=(-1)^{k-1}\sqrt{2m}\sin\frac{(k+1)\pi}{r+1}$ $\lambda_{km+i}=(-1)^{k}\sqrt{2m}\sin\frac{(k+1)\pi}{r+1}$). The index i takes values $i=1,\dots,m,$ for $k=0,1,\dots,r-2$ and $i=1,\dots,m-q,$ where $k=r-1$.

Proof. From

$$\left(\sqrt{\frac{\alpha_k}{\beta_k}}\lambda_k\pm\sqrt{\frac{\beta_k}{\alpha_k}}\lambda_{k+m}\right)^2\ge 0,\qquad m\ge 1,$$

where equality holds if and only if $\alpha_k\lambda_k\pm\beta_k\lambda_{k+m}=0$, we get:

$$\frac{\alpha_k}{\beta_k}\lambda_k^2+\frac{\beta_k}{\alpha_k}\lambda_{k+m}^2\ge\pm 2\lambda_k\lambda_{k+m}$$

$$\sum_{k=1}^{n}\left(\frac{\alpha_k}{\beta_k}+\frac{\beta_{k-m}}{\alpha_{k-m}}\right)\lambda_k^2-\sum_{k=1}^{m}\frac{\beta_{k-m}}{\alpha_{k-m}}\lambda_k^2\ge\pm 2\sum_{k=1}^{n-m}\lambda_k\lambda_{k+m},$$

$$\sum_{k=1}^{n}\left(\frac{\alpha_k}{\beta_k}+\frac{\beta_{k-m}}{\alpha_{k-m}}\right)\lambda_k^2\ge\sum_{k=1}^{m}\frac{\beta_{k-m}}{\alpha_{k-m}}\lambda_k^2\pm 2\sum_{k=1}^{n-m}\lambda_k\lambda_{k+m},$$

Since $\alpha_k\beta_k>0$ the left side of upper inequality is always positive, so

$$\sum_{k=1}^{n}\left(\frac{\alpha_k}{\beta_k}+\frac{\beta_{k-m}}{\alpha_{k-m}}\right)\lambda_k^2\ge\pm 2\sum_{k=1}^{n-m}\lambda_k\lambda_{k+m},\quad n-m>0, \tag{10}$$

where $\lambda_{n+1}=\dots=\lambda_{n+m}=0$.

If α_k and β_k are substituted by $\alpha_k=\sin(k+m)t$ and $\beta_k=\sin kt,\ t\in\left(0,\frac{\pi}{4}\right]$, then (10) becomes

$$\sum_{k=1}^{n}\left(\frac{\sin(k+m)t}{\sin t}+\frac{\sin(k-m)t}{\sin t}\right)\lambda_k^2\ge\pm 2\sum_{k=1}^{n-m}\lambda_k\lambda_{k+m},$$

$$\cos mt\sum_{k=1}^{n}\lambda_k^2\ge\pm\sum_{k=1}^{n-m}\lambda_k\lambda_{k+m}. \tag{11}$$

In accordance with the requirement $\sin(n+m)t \ge 0$, following inequality holds

$$0,(n+m)t < \pi, \quad 0 < t < \frac{\pi}{n+m} = \frac{\pi}{m\cdot r - q + m} = \frac{\pi}{m(r+1)-q}, \tag{12}$$

where $n = m\cdot r - q,\ 0 \le q \le m-1$.

Based on the property $0 \le q \le m-1$, is valid $\frac{\pi}{m(r+1)} \le \frac{\pi}{m(r+1)-q}$, for each q. From (12) maximum value for t is $t = \frac{\pi}{m(r+1)}$. If we substitute t in (11) we obtain

$$\cos\frac{\pi}{r+1}\sum_{k=1}^{n}\lambda_k^2 \ge \pm\sum_{k=1}^{n-m}\lambda_k\lambda_{k+m},$$

i.e.

$$-\cos\frac{\pi}{r+1}\sum_{k=1}^{n}\lambda_k^2 \le \sum_{k=1}^{n-m}\lambda_k\lambda_{k+m} \le \cos\frac{\pi}{r+1}\sum_{k=1}^{n}\lambda_k^2, \tag{13}$$

whence follows the desired inequality. ◊

If inequality (13), when $m=1$ is multiplied $-2pq$ and term $p^2\sum_{k=1}^{n}\lambda_k^2 + q^2\sum_{k=1}^{n}\lambda_{k+1}^2,\ \lambda_{n+1}=0,$ is added to each part of the inequality, after arranging the following inequality

$$2m\left(p^2+q^2-2pq\cos\frac{\pi}{n}\right) \le \sum_{k=1}^{n-1}(p\lambda_k - q\lambda_{k+1})^2 \le 2m\left(p^2+q^2+2pq\cos\frac{\pi}{n}\right).$$

This inequality it is only equivalent form of the inequality (13) for the case $m=1$.

Theorem 2.6. Let $n=2m,\ n\ge 4$ and $\lambda_1,\ldots,\lambda_n$ are real eigenvalues of a graph G, when $\lambda_{n+k}=\lambda_k,\ k=1,\ldots,m$. Then

$$\sum_{k=1}^{n}(\lambda_k-\lambda_{k+1})^2 \ge \left(\sin^2\frac{2\pi}{n}-\sin^2\frac{\pi}{n}\right)\sum_{k=1}^{n}(\lambda_k+\lambda_{n+1-k})^2 + 8m\cdot\sin^2\frac{\pi}{2n}.$$

Equality holds if and only if $\lambda_k = \sqrt{2m}\left(\cos\frac{2k\pi}{n}+\sin\frac{2k\pi}{n}+\cos\frac{4k\pi}{n}+\sin\frac{4k\pi}{n}\right),\quad k=1,\ldots,m.$

Theorem 2.7. Let $\lambda_1,\ldots,\lambda_n$ are real eigenvalues of a graph $G,\ n\ge 2$. Then inequality

$$\sum_{k=1}^{n-1}(\lambda_k-\lambda_{k+1})^2 \ge \left(\sin^2\frac{\pi}{n}-\sin^2\frac{\pi}{2n}\right)\sum_{k=1}^{n}(\lambda_k+\lambda_{n+1-k})^2 + 8m\cdot\sin^2\frac{\pi}{2n}$$

holds. Equality holds if and only if $\lambda_k = \sqrt{2m}\left(\cos\frac{(2k-1)\pi}{2n} + \cos\frac{(2k-1)\pi}{2n}\right), \quad k = 1,\ldots,n.$

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Received Oktober, 5 2015.