ON SOME NEW EXTREMAL PROBLEMS IN ANALYTIC SPACES IN SIEGEL DOMAINS OF SECOND TYPE

R.F. SHAMOYAN[1], A.S. OSADCHYI[2]

Research laboratory of complex and function analysis, Bryansk State University, Russia
[1]e-mail: rsham@mail.ru
[2]e-mail: Osadchy-Artem93@yandex.ru

Summary. We provide some new far reaching sharp extensions of some our previously known results related with extremal problems in analytic Hardy and BMOA type function spaces in Siegel domains of second type putting an additional condition on Bergman representation formula. We extend some our less general sharp results obtained earlier in tube and pseudoconvex domains.

1. NOTATIONS, DEFINITIONS AND PRELIMINARIES.

This paper is continuation of our intensive research on extremal problems in analytic spaces in Siegel domains of second type (see for example [6], [24] and various references there).

We first recall some basic facts on Siegel domains of second type and establish basic notations to formulate our main theorems in Siegel domains of second type. All facts we indicate below can be seen in [1], [2], [3], [16] and [19]. Recall first the explicit formula for the Bergman kernel function is known for very few domains. The explicit forms and zeros of the Bergman kernel function for Hartogs domains and Hartogs type domains (Cartan-Hartogs domains) were found only recently. On the other hand in strictly pseudoconvex domains the principle part of the Bergman kernel can be expressed explicitly by kernels closely related to so-called Henkin-Ramirez kernel (see, for example, [3, 6, 20], [17] and references there). The Bergman kernel

$$b((\tau_1,\tau_2),(\tau_3,\tau_4))$$

for the Siegel domain of the second type was computed explicitly (see [1], [2], [3], [16], [19]). It is an integral via V^* a convex homogeneous open irreducible cone of rank l in R^n, a conjugate cone of V cone and which also contains no straight line and in that integral the fixed Hermitian form from definition of D Siegel domain(see below for definition) participates.(see for details of this [1], [2] and also an important paper [3]). This fact was heavily used in [1], [2] in solutions of several classical problems in Siegel domains of the second type . We will need now some short, but more concrete review of certain results from [1], [2] to make this exposition more complete. To be more precise the authors in [1], [2] showed that on homogeneous Siegel domain of type 2 under certain conditions on parameters the subspace of a weighted L^p space on D for all positive p consisting of holomorphic functions is reproduced by a concrete weighted Bergman kernel which we just mentioned. They also obtain some standard L^p estimates for weighted Bergman projections in this case. The proof relies on direct generalization of the Plancherel-Gindikin formula for the Bergman space A^2 (see [1],

2010 Mathematics Subject Classification: 30H20, 32A36.
Key words and Phrases: Extremal Problems, BMOA Spaces, Bergman Spaces, Siegel Domains

[2]). We remind the reader that the Siegel domain of type 2 associated with the open convex homogeneous irreducible cone V of rank l which contains no straight line, $V \subset R^n$, and a V-Hermitian homogeneous form F which act from product of two C^m into C^n is a set of points (w, τ) from C^{m+n} so that the difference D of $\Im w$ and the value of F on (τ, τ) is in V cone. This domain is affine homogeneous and we now should recall the following expression for the Bergman kernel of

$$D = D(V, F).$$

Let D be an affine-homogeneous Siegel domain of type 2. Let $dv(z)$ or $(dv(z))$ denote the Lebesgue measure on D domain and let $H(D)$ denote the space of all holomorphic functions on D. The Bergman kernel is given by the following formula (see [1]) for $(\tau_1, \tau_2) \in D$ and $(\tau_3, \tau_4) \in D$

$$b((\tau_1, \tau_2), (\tau_3, \tau_4)) = (\tfrac{\tau_1 - \bar{\tau}_3}{2i} - (F(\tau_2, \tau_4))^{2d-q},$$

and we put also for $m \in N$), $b^m(((\tau_1, \tau_2), (\tau_3, \tau_4)) = (\frac{\tau_1 - \bar{\tau}_3}{2i} - (F(\tau_2, \tau_4)^{(2d-q)m}$, (see [1], [2]), where two vectors $q = (q_i)$ and $d = (d_i)$ and in addition $n = (n_i)$, (here the i index is running from 1 to l) are specified via $n_{i,k}$, where these $n_{i,k}$ numbers are dimensions of certain $(R_{i,k})$ and $(C_{i,j})$ subspaces of the certain canonical decomposition of C^{m+n} and R^n via the V cone from definition of our D domain(see for some additional details about this [1], [2], [16]). We will call this family of triples parameters of a Siegel domain D of second type. They will constantly appear in all our main theorems. As usual $H(D)$ is endowed with the topology of uniform convergence on compact subsets of D.

The Bergman projection P of D is as usual the orthogonal projection of Hilbert space $L^2(D, dv)$ onto its subspace $A^2(D)$ consisting of holomorphic functions. Moreover it is known P is the integral operator defined on Hilbert space $L^2(D, dv)$ by the Bergman kernel $b(z, \zeta)$ which for our D domains was computed for example in [3].

Let r be a real number, for example. We fix it. Since D is homogeneous the $\zeta \to b(\zeta, \zeta)$ function does not vanish on D, we can set weighted L^p spaces as follows.

$$L^{p,r}(D) = L^p(D, b^{-r}(\zeta, \zeta) dv(\zeta)), 0 < p < \infty,$$

(see [1], [2]).

Let p be an arbitrary positive number. The weighted Bergman space will be denoted as usual by $A^{p,r}(D)$, it is the analytic part of $L^{p,r}(D)$, with usual modification for $p = \infty$ case (see [1],[2]). We put also $A^{p,0} = A^p(D)$. The so-called weighted Bergman projection P_ε is the orthogonal projection of Hilbert space $L^{2,\varepsilon}(D)$ onto $A^{2,\varepsilon}(D)$. These facts can be found in [1], [2]. It is proved in [1], [2] that there exist a real number $\varepsilon_D < 0$ such that $A^{2,\varepsilon}(D) = \{0\}$ if $\varepsilon \le \varepsilon_D$; and that for $\varepsilon > \varepsilon_D$, P_ε is the integral operator defined on $L^{2,\varepsilon}(D)$ by the weighted Bergman kernel $c_\varepsilon b^{1+\varepsilon}(\zeta, z)$. In all this our work we shall assume that $\varepsilon > \varepsilon_D$.

The norm $\| \|_{p,r}$ of $A^{p,r}(D)$ with $r > \varepsilon_D$ is defined by

$$\|f\|_{p,r} = \left(\int_D |f(z)|^p b^{-r}(z, z) dv(z)\right)^{\frac{1}{p}}, f \in A^{p,r}(D).$$

with usual modification for $p = \infty$ case. Let further $dv_\beta = b^{-\beta}(z, z) dv(z)$, $\beta \in R$, $z \in D$.

We need some assertions (see [1], [2], [16]), namely some basic facts on Bergman kernel and Bergman projection in Siegel domains of second type. They will be partially used by us below in proofs of our theorems. Some proofs of these preliminaries are rather intricate [1], [2], [16]. We indicate for readers in advance that some assertions below involving integrals can be easily extended to m-products of Siegel domains of second type by simple application of "one variable" result m-times by each variable separately. This procedure is well- known in much simpler case of polydisk (see, for example, [4], [5]).

Lemma A. Let $h \in L^{\infty}(D)$. Take $\rho > \rho_0$, for a large fixed ρ_0. Then the function

$$z \to G(z) = \int_D b^{1+\rho}(z,\zeta)h(\zeta)dv(\zeta)$$

satisfies the estimate $\sup_{z\in D}|G(z)|b^{-\rho}(z,z) \le c\| h\|_{\infty}$ and $G \in H(D)$.

The following lemma is complete analogue of so-called Forelli-Rudin type estimates for our Siegel domains of second type.(note these type estimates are well-known in simpler domains). This lemma is extremely vital for us.

We denote by c, c_1 various constants in estimates which depend on parameters.

Lemma B. Let α and ε be in R^l, $(\zeta, v) \in D$. Then for $\varepsilon_i > \left(\frac{n_i+2}{2(2d-q)_i}\right)$ and $(\alpha_i - \varepsilon_i) > \frac{n_i}{(-2)(2d-q)_i}$, $i = 1,\dots,l$

$$\int_D |b^{1+\alpha}((\zeta,v),(z,u))|b^{-\varepsilon}((z,u),(z,u))dv(z,u) = c_{\alpha,\varepsilon}b^{\alpha-\varepsilon}((\zeta,v),(\zeta,v)).$$

The following lemma is a version of classical reproducing Bergman formula for Bergman spaces in Siegel domains of second type.

Lemma C. Let r be a vector of R^l such that $r_i > (\frac{n_i+2}{2(2d-q)_i})$ for all $i = 1,\dots,l$ and a p is a real number such that $1 \le p < min\{\frac{n_i-2(2d-q)_i(1+r_i)}{n_i}\} = \tilde{q}$. Then for all $\varepsilon \in R^l$ such that $(\varepsilon_i) > \frac{n_i+2}{2(2d-q)_i}(\frac{p-1}{p}) + (\frac{r_i}{p})$, $i = 1\dots l$ the following equality holds $P_\varepsilon f = f$, $f \in A^{p,r}$.

We list in lemma E some other properties of Bergman kernel. The last estimate in assertion below is an embedding theorem. It relates the so-called growth spaces with Bergman spaces. (see also the complete analogue of this result in other simpler domains in [11, 12] and in [4, 5]).

Lemma D. Let $\alpha \in R^l$; $\alpha_j > 0$ or $\alpha_j = 0$ $,j = 1,\dots,l$. Then $|b^{\alpha}((\zeta,v),(z,u))| \le c_\alpha b^{\alpha}((\zeta,v),(\zeta,v))$ and $|b^{\alpha}((\zeta,v)+(\zeta',v');(z,u)+(z',u'))| \le c_\alpha b^{\alpha}((\zeta,v),(\zeta,v))$ for all $(\zeta,v),(\zeta',v'),(z,u),(z',u)$ in D. For all $f \in A^{p,r}(D)$, $p > 0$

$$|f(z,u)|^p \le cb^{1+r}((z,u),(z,u))\|f\|^p_{p,r},$$

for all (z,u) points taken from D.

The following result is crucial for various theorems. It concerns the boundedness of Bergman type projection in weighted Bergman spaces. Note this fact is classical in simpler domains and it has also many applications in analytic function theory (see [4, 5]).

Proposition A. Let k and r be in R^l such that $k_i > \frac{1}{(2d-q)_i}$ and $r_i > \frac{n_i+2}{2(2d-q)_i}$, $i = 1,\dots,l$. Then P_k is bounded from $L^{p,r}(D)$ into $A^{p,r}(D)$ if

$$\max_{i=1,\dots,l}\left\{1, \frac{2n_i+2-2(2d-q)_i r_i}{n_i+2-2(2d-q)_i k_i}\right\} < p < \min_{i=1,\dots,l}\frac{2n_i+2-2(2d-q)_i r_i}{n_i}$$

The following assertion is a base of proof of many theorems. It provides integral representation for so-called analytic "growth space" on Siegel domains of the second type.

Proposition B. Let r and ε be two vectors of R^l such that $\varepsilon_i > \frac{n_i}{(-2)(2d-q)_i}$; $r_i > \frac{n_i+2}{2(2d-q)_i} + \varepsilon_i$, $i = 1, \dots, l$. Let G be in $H(D)$ so that

$$\|G\|_{A_\varepsilon^\infty} = \sup_{z \in D}\{|G(z)| b^{-\varepsilon}(z,z)\} < \infty.$$

Then $P_r G = G$.

The following result explains the structure of functions from Bergman spaces on Siegel domains of second type. It is an extension of a classical theorem on atomic decomposition of Bergman spaces in the unit disk on a complex plane (see, for example, [19] and references there).

Proposition C. Let $D \subset C^n$ be a symmetric Siegel domain of type II, $p \in (\frac{2n}{2n+1}, 1)$, $r \in R^l$; $r_i > \frac{n_i+2}{2(2d-q)_i}$. Then there are two constants $c = c(p,r)$ and $c_1 = c_1(p,r)$ such that for every $f \in A^{p,r}(D)$ there exists an l^p sequence $\{\lambda_i\}$ such that

$$f(z) = \sum_{i=0}^{\infty} \lambda_i b^{\alpha/p}(z, z_i) b^{\frac{1+r+\alpha}{p}}(z_i, z_i)$$

where $\{z_i\}$ is a lattice in D and the following estimate holds $c\|f\|_{p,r}^p \le \sum_{i=1}^{\infty} |\lambda_i|^p \le c_1\|f\|_{p,r}^p$, where α is a special fixed vector depending on p, r and parameters of Siegel domain (see for this vector [19]).

We denote by (χ_M) as usual the characteristic function of M set.

2 SHARP THEOREMS FOR A DISTANCE FUNCTION IN BERGMAN SPACES IN SIEGEL DOMAINS OF THE SECOND TYPE

In this section we show our main theorems. Extending some previously obtained our results in tube and pseudoconvex domains to general Siegel domains.

A very important tool for us is the Forelli-Rudin estimates (see [1]). Let $\alpha, \varepsilon \in R^l$, $(\xi, v) \in D$. Then

$$\int_D \left|b^{1+\alpha}\big((\xi,v),(z,u)\big)\right| \times b^{-\varepsilon}\big((z,u),(z,u)\big) dv(z,u) = c_{\alpha,\varepsilon} b^{\alpha-\varepsilon}\big((\xi,v),(\xi,v)\big), \qquad (1)$$

$$\varepsilon_j > \varepsilon_0 = (\max_j)\left\{\frac{n_j+2}{2(2d-q)_i}\right\}, \alpha_j - \varepsilon_j > \tilde{\varepsilon}_0 = \left(\max_j\right)\left(\frac{n_j}{-2(2d-q)_i}\right), j = 1, \dots, l.$$

We well use also the following known estimates

$$\left(\int_{R^n} |f(x+i(y+\eta),u)|^p dx\right)^{\frac{1}{p}} \left(b^{-\frac{(1+r)}{p}}(\tilde{x}+iy,u),(\tilde{x}+iy,u)\right) \le$$

$$\le c\left(\int_D (|f(z)|^p) \times \big(b^{-r}(z,z)\big) dv(z)\right)^{\frac{1}{p}} = c\|f\|_{A_r^p}, \qquad (1')$$

(we put $\|f\|_{A_r^\infty} = \sup_{z \in D}|f(z)| b^{-r}(z,z) < \infty$) where $(y,u) \in V \times C^m$, $\tilde{x} \in R^n$, $\tilde{x} = i\eta$.

Taking sup on the left we define so-called weighted Hardy classes $(H_r^p)(D)$ in Siegel domain of second type for all $1 \le p < \infty$, $r \in R^l$, $r_j \ge -1$, $i = 1, \dots, l$. We refer reader for [1], [2] for these spaces.In case of simpler domains this weighted Hardy space is well studied by various authors during last several decades.

We will be interested in a special subspace of this space namely we consider all functions of this functional class for which Bergman representation formula with large index is valid. So $\widetilde{H}_r^p$ is a Bergman representation subspace of H_r^p.We assume in this paper that this subspace is not empty.In case of simpler domains as tube,ball or polydisk this assertion is known.

If $X \subset Y, f \in Y$ then the problem of finding estimates for $dist_Y(f, X)$ appears naturally where Y, X are two normed analytic function spaces. We hence based on (1') have a similar dist problem between these two function spaces in Sigel domains of second type. Let $D = D_1 \times D_2$. Let

$$(L_{\varepsilon,f}) =$$

$$\left\{(y,u) \in V \times D_1 : \left(\int_{R^n} |f(x + i(y + r), u)|^p dx\right)^{\frac{1}{p}} \left(b^{-\frac{(1+r)}{p}}(\tilde{x} + iy, u), (\tilde{x} + iy, u)\right) \ge \varepsilon\right\},$$

$$\tilde{x} \in R^n, r \in V.$$

Then we have .

Theorem 1. $D = D_1 \times D_2$, $D \subset \mathbb{C}^m \times \mathbb{C}^n$, $D_2 = R^n + iV$. Let $1 \le p < \infty$, $\vec{r} > (-1)$. Let $f \in (\widetilde{H}_r^p)$. Then for $p = 1$ for any Siegel domain of second type and for all $p \ge 1$ for all tube and pseudoconvex domains we have

$$\left(dist_{H_r^p(D)}\right)(f, A_r^p) \cong$$

$$\cong (Inf)\left\{\varepsilon > 0 : M(f) = \int_V \int_{D_1} \left[b\big((z,u),(z,u)\big)\right] \left(\chi_{L_{\varepsilon,f}}\big((z,u),(z,u)\big) dv\right) < \infty\right\},$$

$z = iy$.

Remark. Note this theorem for $p = 1$ and all $1 \le p < \infty$ in less general tubular domains over cones and bounded strongly pseudoconvex domains can be seen in recent papers [8], [17]. Note also proofs of all parallel assertions in various domains are similar.

We provide full sketch of proof for general Siegel domains of second type below. For $p = 1$ case.

Next we define $BMOA$ type space in Siegel domain of second type as follows. We denote it by $BMOA_{r,\alpha}$, $r \ge 0$, $\alpha \ge 0$.Let

$$BMOA_{r,\alpha} = \left\{f \in H(D) : \left(\sup_{(u,v)\in D}\right) \int_D |f(z,w)| \; \left|b\big((z,w),(u,v)\big)\right|^{\alpha} \times\right.$$

$$\left. \times\, b^{-\alpha^{+}1}\big((z,w),(z,w)\big) dv(z,w) \times b\big((u,v),(u,v)\big)^{-r} < \infty\right\},$$

(see [2], § 1). We below will prove another sharp theorem in this direction (namely similar type distance theorem).

First note that the following estimate holds for Bergman Kernel in Siegel domain.$\left|b^{\alpha}\big((z,u),(\tilde{z},\tilde{u})\big)\right| \le c\left|b^{\alpha}\big((z,u),(z,u)\big)\right|$, $(z,u),(\tilde{z},\tilde{u}) \in D$, $\alpha \ge 0$) (see [2], [1] for this estimate).

Hence based on this estimate we have the following inequality

$$\|f\|_{BMOA_{\alpha,r}} \le c\left(\int_D |f(z,w)| b^{-r+1}\big((z,w),(z,w)\big) dv(z,w)\right),$$

$\alpha > 1, \alpha > r + \varepsilon_0 - 1, r \ge 0.$

So based on this last estimate we can pose a distance problem for this $BMOA$ -- Bergman pair of spaces here again and then formulate the following sharp theorem on distances.

We put wave on BMOA for that subspace of BMOA space for which the Bergman representation with large index is valid.We assume this subspace is not empty in this paper.For simpler domains like tube,ball or polydisk this assertion is known.(see for example[24])

Theorem 2. Let $\alpha > r + \varepsilon_0 - 1$, $r \ge \tilde{\varepsilon}_0 + 1$, $f \in \widetilde{BMOA}_{\alpha,r}(D)$, then the following equivalence relation is valid.

$$\big(dist_{BMOA_{\alpha,r}}\big)(f, A^1_{r-1}) \cong$$

$$\cong (Inf)\left\{\varepsilon > 0: \int_D \left(\chi_{N_f}\big((z,w),(z,w)\big)\left[b\big((z,w),(z,w)\big)\right] dv(z,w)\right) < \infty\right\},$$

where

$$N_{f,\varepsilon} = \left\{(u,v) \in D: \int_D |f(z,w)| \; \left|b\big((z,w),(u,v)\big)^{\alpha}\right| \times\right.$$

$$\left. \times \left[b\big((z,w),(z,w)\big)^{-\alpha+1} dv(z,w)\right] b^{-r}\big((u,v),(u,v)\big) \ge \varepsilon\right\}.$$

Remark. For the unit ball, tube and pseudoconvex domains this theorem can be seen in recent paper [24] without condition on Bergman integral representation.Moreover the proof in case of all domains are similar.

The proof of theorem 1. Note one part of theorem is almost trivial (see also [8] for same type argument). Now we show the second part of our assertion.

Since $f \in \widetilde{H}^1_r$ we have the following intergal representation.

$$\big(f(z,u)\big) = \int_{D_1}\int_V\int_{R^n} \left(b^{1+\rho}\big((\xi,v),(z,u)\big)\right)\big(f(\xi,v)\big) \times b^{-\rho}\big((\xi,v),(\xi,v)\big) \; dv(\xi,v),$$

$\rho_i > r_0$ for large enough r_0, $j = 1,\dots,n$ for all $(z,u) \in D$.

$$= \left(\int_{\widetilde{D}_1}\int_{R^n} + \int_{\widetilde{D}_1^0}\int_{R^n}\right) = (T_1 + T_2) = (f_2 + f_1)(z,u), (z,u) \in D,$$

where $\widetilde{D}_1 = (V \times D_1)\backslash\big(L_{\varepsilon,f}\big)$, $\widetilde{D}_1^0 = L_{\varepsilon,f}$.

Then using Forelli-Rudin estimate (1) we have the following.
We first have $\|f_2\|_{H^1_r} \le c_1\varepsilon$, and then also

$$\|f_1\|_{A_r^1} \leq c_2 \|f\|_{H_r^1} \int_V \int_{\widetilde{D}_1^0} \left[b\big((z,u),(z,u)\big)\right] \left[\chi_{L_{\varepsilon,f}}\big((z,u),(z,u)\big) dv\right] < \infty,$$

We note these easy calculations are based only on Fubini"s theorem and Forelli-Rudin estimate. For general $1 < p < \infty$ case we refer to [8] for the complete proof of the most typical unbounded Siegel domain namely tubular domain case and to [17] for case of bounded strictly pseudoconvex domain with smooth boundary. The new technical feature for $p > 1$ case is the combination of applications of Young and Minkowski estimates. We refer to [8] for full proof for this case in tube domains over symmetric cones. The case $p = 1$ is simpler and is valid for every Siegel domain of second type as we just showed.

Indeed using the Forelli-Rudin estimate and Fubini's theorem we have

$$\left(\int_{R^n} |f_2(x+iy,w)dx|\right)\left(b^{-(1+r)}\big((iy,u),(iy,u)\big)\right) \leq$$

$$\leq c \int_{R^n} \int_{\widetilde{D}_1} \int_{R^n} \left|b^{1+\rho}\big((\xi,v),(z,u)\big)\right| \; |f(\xi,v)| b^{-\rho}\big((\xi,v),(\xi,v)\big)$$

$$d\tilde{V}(\xi,v) b^{-(1+r)}\big((iy,u),(iy,u)\big)$$

$$\leq (c_1\varepsilon) \int_D \left|b^{1+\rho}\big((\xi,v),(z,u)\big)\right| \; b^{-\rho+1+r}\big((\xi,v),(\xi,v)\big) d\tilde{V}(\xi,v) \left[b^{-(1+r)}\big((iy,u),(iy,u)\big)\right]$$

$$\leq c_2 \varepsilon,$$

$z = iy$, $y \in V$ for large enough $\rho_j > \rho_0, j = 1, \ldots . l$.

To show the second estimate note that (see [1], [2])

$$\left|b^\alpha\big((\xi,v)+(\xi',v')\big);(\xi,v)+(\xi'',v'')\right| \leq c_\alpha\big(b^\alpha(\xi,v),(\xi,v)\big),$$

$(\xi,v),(\xi',v'),(\xi'',v'') \in D$, $\alpha \geq 0$ and hence we have again based on Forelli-Rudin estimate the following (here we omit some simple and easy calculations).

$$\int_D |f_1(z,u)| \big(b(z,u),(z,u)\big)^{-r} dv(z,u) \leq c_2 \|f\|_{H_r^1} M(f).$$

Now using simple standard arguments(see, for example,[6,8,23,24]) we finish the proof of our theorem.

Theorem 1 is proved.

The proof of theorem 2. We follow arguments used in [24] for same type results in tubular domains over symmetric cones. Applying Bergman representation formula we have if $f \in \widetilde{BMOA}_{r,\alpha}$ then

$$f\big((z,u)\big) = c \int_D \big(f(\xi,v)\big) \left[b^{-\rho}\big((\xi,v),(\xi,v)\big)\right] \times \left[b^{-(1+\rho)}\big((\xi,v)(z,u)\big)\right] dv(\xi,v),$$

$$(z,u) \in D, \rho_j > \rho_0, j = 1, \ldots, n.$$

and where ρ_0 is large enough. Choose large $\tilde{\rho}_0$ and apply for $f(\xi,v)$ one more time Bergman formula then for f we have $f = f_1 + f_2$, where

$$(f_1)(z,u) = c\int_D\int_{D\setminus N_f}\Big[b^{-\rho}\big((\xi,v),(\xi,v)\big)\times\Big(f\big(\tilde{\xi},\tilde{v}\big)\Big)\times b^{-\tilde{\rho}}\Big(\big(\tilde{\xi},\tilde{v}\big),\big(\tilde{\xi},\tilde{v}\big)\Big)\times$$

$$\times\, b^{-(1+\rho)}\big((\xi,v),(z,u)\big)\times b^{-(1+\tilde{\rho})}\Big((\xi,v),\big(\tilde{\xi},\tilde{v}\big)\Big)\,dv(\xi,v)dv\big(\tilde{\xi},\tilde{v}\big),$$

$$(f_2)(z,u) = c\int_D\int_{N_f}\Big[b^{-\rho}\big((\xi,v),(\xi,v)\big)\times\Big(f\big(\tilde{\xi},\tilde{v}\big)\Big)\times b^{-\tilde{\rho}}\Big(\big(\tilde{\xi},\tilde{v}\big),\big(\tilde{\xi},\tilde{v}\big)\Big)\times$$

$$\times\, b^{-(1+\rho)}\big((\xi,v),(z,u)\big)\times b^{-(1+\tilde{\rho})}\Big((\xi,v),\big(\tilde{\xi},\tilde{v}\big)\Big)\,dv(\xi,v)dv\big(\tilde{\xi},\tilde{v}\big).$$

$\rho > \rho_0$, $\tilde{\rho} > \tilde{\rho}_0$, $\rho_0, \tilde{\rho}_0$ are large enough.

We will show now using Forelli-Rudin estimate that

$$\|f_1\|_{BMOA^{r,\alpha}} \le c\varepsilon,\ \|f_2\|_{A^1_{r-1}} \le c_1,$$

for some constant c_1, c.

This shows one part of our theorem. Indeed, we have the following estimates using the fact that $\rho, \tilde{\rho}$ can be large enough

$$\int_D |f_2(z,u)|\times\big[b^{-r+1}\big((z,u),(z,u)\big)\big]dv(z,u) \le$$

$$\le c_1\int_D\Big(b^{-r+1}\big((z,u),(z,u)\big)\Big)\int_D\int_{N_f}\big|f\big(\tilde{\xi},\tilde{v}\big)\big|\times b^{-\rho}\big((\xi,v),(\xi,v)\big)\times$$

$$\times\, b^{-\tilde{\rho}}\Big(\big(\tilde{\xi},\tilde{v}\big),\big(\tilde{\xi},\tilde{v}\big)\Big)\times$$

$$\times\, b^{-(1+\rho)}\big((\xi,v),(z,u)\big)\times b^{-(1+\tilde{\rho})}\Big((\xi,v),\big(\tilde{\xi},\tilde{v}\big)\Big)\,dv(z,u)dv(\xi,v)dv\big(\tilde{\xi},\tilde{v}\big) \le$$

$$\le c_2\left(\sup_{(\xi,v)}\right)\int_D\big|f\big(\tilde{\xi},\tilde{v}\big)\big|\ \times b^{(1+\tilde{\rho})}\Big((\xi,v),\big(\tilde{\xi},\tilde{v}\big)\Big)\times b^{-\tilde{\rho}}\Big(\big(\tilde{\xi},\tilde{v}\big),\big(\tilde{\xi},\tilde{v}\big)\Big)\,dv\big(\tilde{\xi},\tilde{v}\big)\times$$

$$\times\int_D\Big(\chi_{N_f}(z,u)\Big)\times\big[b^{+1}\big((z,u),(z,u)\big)\big]dv(z,u).$$

$$\|f_1\|_{BMOA_{r,\alpha}} = \sup_{(\tilde{u},\tilde{v})}\int_D |f_1(z,u)|\ \big|b\big((z,u),(\tilde{u},\tilde{v})\big)\big|^{\alpha}\times b^{-\alpha+1}\big((z,u),(z,u)\big)dv(z,u)\times$$

$$\times\big[b\big((\tilde{u},\tilde{v}),(\tilde{u},\tilde{v})\big)\big]^{-r} \le$$

$$\le \sup_{(\tilde{u},\tilde{v})} c_2 b\big((\tilde{u},\tilde{v}),(\tilde{u},\tilde{v})\big)^{-r}|b((z,u),(\tilde{u},\tilde{v}))|^{\alpha}\int_D\Big(b^{-\alpha+1}(z,u),(z,u)\Big)\times$$

$$\int_D\int_{D\setminus N_f}\big|f\big(\tilde{\xi},\tilde{v}\big)\big|\times b^{-\rho}\big((\xi,v),(\xi,v)\big)\times b^{-\rho}\Big(\big(\tilde{\xi},\tilde{v}\big),\big(\tilde{\xi},\tilde{v}\big)\Big)\times$$

$$\times\, b^{(1+\rho)}\big((\xi,v),(z,u)\big)\times b^{(1+\tilde{\rho})}\Big((\xi,v),\big(\tilde{\xi},\tilde{v}\big)\Big)\,dv(\xi,v)dv(z,w)dv\big(\tilde{\xi},\tilde{v}\big) \le c_1\varepsilon,$$

$\tilde{\rho} = (\alpha-1)$, ρ is large enough.

Since using Forelli-Rudin estimate we have

$$\|f_1\|_{BMOA_{r,\alpha}} \le (c\varepsilon)\int_D\int_D |b(z,u),(\tilde u,\tilde v)|^{\alpha} \times b\big((\xi,v),(\xi,v)\big)^{-\rho+1} \times \left(b^{-\alpha+1}\big((z,u),(z,u)\big)\right) \times$$

$$\times\, b^{-r}\big((\tilde u,\tilde v),(\tilde u,\tilde v)\big) \times b^{1+\rho}\big((\xi,v),(z,u)\big) dv(z,u)dv(\xi,v) \le c\varepsilon \int_D \left|b\big((z,u),(\tilde u,\tilde v)\big)\right|^{\alpha} \times$$

$$\times \left(b^{r+1-\alpha}(z,u),(z,u)\right)dv(z,u) \times \left(b^{-r}\big((\tilde u,\tilde v),(\tilde u,\tilde v)\big)\right) \le c\varepsilon.$$

Let us show the reverse assertion now. We will again use Forelli-Rudin estimate. First assume the reverse statement is valid. Then there are $\varepsilon,\varepsilon_1$, so that $\varepsilon,\varepsilon_1 > 0$, $f_{\varepsilon_1} \in A^1_{r-1}$, $\varepsilon > \varepsilon_1$, $\left\|f - f_{\varepsilon_1}\right\|_{BMOA_{r,\alpha}} \le \varepsilon_1$,

$$\int_D \left[\chi_{N_f}(z,w)\right]\left[b\big((z,w),(z,w)\big)\right]dv(z,w) = \infty.$$

We will arrive at contradiction. Indeed let $\tilde f = f - f_{\varepsilon_1}$. Then we have for $a = (u,v) \in D$

$$M(a) = \int_D \left(\left|f_{\varepsilon_1}(z,w)\right|\right)\left(b\big((z,w),(z,w)\big)^{-\alpha+1}\right)\left|b\big((z,w),(u,v)\big)\right|^{\alpha} dv(z,w) \times$$

$$\times \left(b\big((u,v),(u,v)\big)^{-r}\right) \ge b\big((u,v),(u,v)\big)^{-r}\int_D |f(z,w)|\ \ b\big((z,w),(z,w)\big)^{-\alpha+1} dv(z,w) =$$

$$= \left(\sup_{(u,v)\in D}\right)\int_D \left|f_{\varepsilon_1}(z,w) - f(z,w)\right|\ \ b\big((z,w),(z,w)\big)^{-\alpha+1} \times$$

$$\times \left|b\big((z,w),(u,v)\big)\right|^{\alpha} dv(z,w)(b(u,v),(u,v)^{-r});$$

Hence $M(a) \ge (\varepsilon - \varepsilon_1)\left(\chi_{N_f}(a)\right)$, $a \in D$, from here using again Forelli-Rudin estimate we have that

$$\left\|f_{\varepsilon_1}\right\|_{A^1_{r-1}} \ge (\varepsilon - \varepsilon_1)\int_D \chi_{N_f}(\tilde z,\tilde w)\left(b\big((\tilde z,\tilde w),(\tilde z,\tilde w)\big)\right)dv(\tilde z,\tilde w),$$

so we arrive at contradiction with our assumption.

Now using simple standard arguments(see, for example,[6,8,23,24]) we finish the proof of our theorem.

Theorem 2 is proved.

Remark. Finally in [23] we got sharp similar dist theorem for $(A^{\infty}_{\alpha}, A^{p}_{\alpha})$ pair in Siegel domains of second type. We note that for $(A^{\infty}_{\alpha}, A^{\infty}_{\beta})$ ($p = \infty$ case) pair such type general sharp result can be also obtained by similar proof as in [23] without any additional condition on Bergman representation formula since for any f, $f \in A^{\infty}_{\beta}$ in Siegel domains of second type such representation exists (see [1], [2], [24]).

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Received November, 16 2016